144.

A THIRD MEMOIR UPON QUANTICS.

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MY object in the present memoir is chiefly to collect together and put upon record various results useful in the theories of the particular quantics to which they relate. The tables at the commencement relate to binary quantics, and are a direct sequel to the tables in my Second Memoir upon Quantics, vol. CXLVI. (1856), [141]. The definitions and explanations in the next part of the present memoir are given here for the sake of convenience, the further development of the subjects to which they relate being reserved for another occasion. The remainder of the memoir consists of tables and explanations relating to ternary quadrics and cubics.

Covariant and other Tables, Nos. 27 to 50 (Nos. 1 to 50 binary quantics)[1].

Nos. 27 to 29 are a continuation of the tables relating to the quintic

$$(a, b, c, d, e, f \between x, y)^5.$$

No. 27 gives the values of the different determinants of the matrix

$$\begin{array}{cccccc} (& a, & 4b, & 6c, & 4d, & e\ ) \end{array}$$
$$\left|\begin{array}{cccccc} a, & 4b, & 6c, & 4d, & e & \\ & b, & 4c, & 6d, & 4e, & f \\ b, & 4c, & 6d, & 4e, & f & \end{array}\right|$$

determinants which are represented by 1234, 1235, &c., where the numbers refer to

[1] The Tables 49 and 50 were inserted October 6, 1856.—A. C.

the different columns of the matrix. No. 28 gives the values of certain linear functions of these determinants, viz.

$$
\begin{aligned}
L &= 1256 + 2345 - 2\,.\,1346,\\
L' &= 3\,.\,1256 - 1346,\\
8M &= -1345 + 2\,.\,1246,\\
8M' &= -2346 + 2\,.\,1356,\\
8N &= -1245 + 3\,.\,1236,\\
8N' &= -2356 + 3\,.\,1456,\\
80P &= L' - 3L = 5\,.\,1346 - 3\,.\,2345,\\
16P' &= -5L' - L = -16\,.\,1256 - 3\,.\,1346 - 2356.
\end{aligned}
$$

At the end of the two tables there are given certain relations which exist between the terms of Tables 14, 16, 25, 26, 27 and 28.

No. 27.

1234.	1235.	1236.	1245.	1246.	1345.	1256.	2345.
a^2bf ...	$a^2cf\ -\ 4$	$a^2df\ +\ 6$	$a^2df\ -\ 6$	$a^2ef\ +\ 4$	a^2ef ...	$a^2f^2\ +\ 1$	a^2f^2 ...
$a^2ce\ -\ 16$	$a^2de\ +\ 24$	a^2e^2 ...	$a^2e^2\ +\ 16$	$abdf\ -\ 4$	$abdf\ -\ 24$	$abef\ -\ 2$	$abef$ ...
$a^2d^2\ +\ 36$	$ab^2f\ +\ 4$	$abcf\ -\ 22$	$abcf\ +\ 6$	$abe^2\ -\ 4$	$abe^2\ +\ 64$	$acdf\ -\ 16$	$acdf\ +\ 20$
$ab^2e\ +\ 16$	$abce\ -\ 84$	$abde\ -\ 6$	$abde\ -\ 26$	$ac^2f\ -\ 24$	$ac^2f\ +\ 24$	$ace^2\ +\ 16$	$ace^2\ -\ 80$
$abcd\ -\ 152$	$abd^2\ -\ 24$	$ac^2e\ +\ 16$	$ac^2e\ -\ 96$	$acde\ +\ 24$	$acde\ -\ 208$	$ad^2e\ +\ 16$	$ad^2e\ +\ 60$
$ac^3\ +\ 96$	$ac^2d\ +\ 64$	acd^2 ...	$acd^2\ +\ 96$	ad^3 ...	$ad^3\ +\ 144$	$b^2df\ -\ 15$	$b^2df\ -\ 80$
$b^3d\ +\ 80$	$b^3e\ +\ 60$	$b^3f\ +\ 16$	b^3f ...	$b^2cf\ +\ 24$	b^2cf ...	b^2e^2 ...	$b^2e^2\ +\ 240$
$b^2c^2\ -\ 60$	$b^2cd\ -\ 40$	$b^2ce\ -\ 10$	$b^2ce\ +\ 90$	$b^2de\ -\ 20$	$b^2de\ -\ 40$	bc^2f ...	$bc^2f\ +\ 60$
	bc^3 ...	b^2d^2 ...	$b^2d^2\ -\ 80$	bc^2e ...	$bc^2e\ +\ 60$	$bcde$ ...	$bcde\ -\ 860$
		bc^2d ...	bc^2d ...	bcd^2 ...	$bcd^2\ -\ 40$	bd^3 ...	$bd^3\ +\ 960$
		c^4 ...	c^4 ...	c^3d ...	c^3d ...	c^3e ...	$c^3e\ +\ 960$
						c^2d^2 ...	$c^2d^2\ -\ 320$

1346.	2346.	1356.	2356.	1456.	2456.	3456.
a^2f^2 ...	abf^2 ...	$abf^2\ +\ 4$	$acf^2\ -\ 6$	$acf^2\ +\ 6$	$adf^2\ -\ 4$	aef^2 ...
$abef\ +\ 16$	$acef\ -\ 24$	$acef\ -\ 4$	$adef\ +\ 6$	$adef\ -\ 22$	$ae^2f\ +\ 4$	$bdf^2\ -\ 16$
$acdf\ -\ 36$	$ad^2f\ +\ 24$	$ad^2f\ -\ 24$	ae^3 ...	$ae^3\ +\ 16$	$bcf^2\ +\ 24$	$be^2f\ +\ 16$
$ace^2\ -\ 16$	ade^2 ...	$ade^2\ +\ 24$	$b^2f^2\ +\ 16$	b^2f^2 ...	$bdef\ -\ 84$	$c^2f^2\ +\ 36$
$ad^2e\ +\ 36$	$b^2ef\ +\ 64$	$b^2ef\ -\ 4$	$bcef\ -\ 26$	$bcef\ -\ 6$	$be^3\ +\ 60$	$cdef\ -\ 152$
$b^2df\ -\ 16$	$bcdf\ -\ 208$	$bcdf\ +\ 24$	$b^2df\ -\ 96$	$bd^2f\ +\ 16$	$c^2ef\ -\ 24$	$ce^3\ +\ 80$
b^2e^2 ...	$bce^2\ -\ 40$	$bce^2\ -\ 20$	$bde^2\ +\ 90$	$bde^2\ -\ 10$	$cd^2f\ +\ 64$	$d^3f\ +\ 96$
$bc^2f\ +\ 36$	$bd^2e\ +\ 60$	bd^2e ...	$c^2df\ +\ 96$	c^2df ...	$cde^2\ -\ 40$	$d^2e^2\ -\ 60$
$bcde\ -\ 20$	$c^3f\ +\ 144$	c^3f ...	$c^2e^2\ -\ 80$	c^2e^2 ...	d^3e ...	
bd^3 ...	$c^2de\ -\ 40$	c^2de ...	cd^2e ...	cd^2e ...		
c^3e ...	cd^3 ...	cd^3 ...	d^4 ...	d^4 ...		
c^2d^2 ...						

No. 28.

N.	M.	L.	L′.	P.	P′.	M′.	N′.
$a^2df + 3$	$a^2ef + 1$	$a^2f^2 + 1$	$a^2f^2 + 3$	$a^2f^2 \;\ldots$	$a^2f^2 - 1$	$abf^2 + 1$	$acf^2 + 3$
$a^2e^2 - 2$	$abdf + 2$	$abef - 34$	$abef - 22$	$abef + 1$	$abef + 9$	$acef + 2$	$adef - 9$
$abcf - 9$	$abe^2 - 9$	$acdf + 76$	$acdf - 12$	$acdf - 3$	$acdf - 1$	$ad^2f - 9$	$ae^3 + 6$
$abde + 1$	$ac^2f - 9$	$ace^2 - 32$	$ace^2 + 64$	$ace^2 + 2$	$ace^2 - 18$	$ade^2 + 6$	$b^2f^2 - 2$
$ac^2e + 18$	$acde + 32$	$ad^2e - 12$	$ad^2e - 36$	$ad^2e \;\ldots$	$ad^2e + 12$	$b^2ef - 9$	$bcef + 1$
$acd^2 - 12$	$ad^3 - 18$	$b^2df - 32$	$b^2df + 64$	$b^2df + 2$	$b^2df - 18$	$bcdf + 32$	$bd^2f + 18$
$b^3f + 6$	$b^2cf + 6$	$b^2e^2 + 225$	$b^2e^2 - 45$	$b^2e^2 - 9$	$b^2e^2 \;\ldots$	$bce^2 \;\ldots$	$bde^2 - 15$
$b^2ce - 15$	$b^2de \;\ldots$	$bc^2f - 12$	$bc^2f - 36$	$bc^2f \;\ldots$	$bc^2f + 12$	$bd^2e - 15$	$c^2df - 12$
$b^2d^2 + 10$	$bc^2e - 15$	$bcde - 820$	$bcde + 20$	$bcde + 31$	$bcde + 45$	$c^3f - 18$	$c^2e^2 + 10$
$bc^2d \;\ldots$	$bcd^2 + 10$	$bd^3 + 480$	$bd^3 \;\ldots$	$bd^3 - 18$	$bd^3 - 30$	$c^2de + 10$	$cd^2e \;\ldots$
$c^4 \;\ldots$	$c^3d \;\ldots$	$c^3e + 480$	$c^3e \;\ldots$	$c^3e - 18$	$c^3e - 30$	$cd^3 \;\ldots$	$d^4 \;\ldots$
		$c^2d^2 - 320$	$c^2d^2 \;\ldots$	$c^2d^2 + 12$	$c^2d^2 + 20$		

If the coefficients of the table 14 are represented by $\frac{1}{2}A$, B, $\frac{1}{2}C$, viz. writing

$$A = 2\,(ae - 4bd + 3c^2),$$
$$B = \quad af - 3be + 2cd,$$
$$C = 2\,(bf - 4ce + 3d^2),$$

then we have the following relations between 1234, &c. and A, B, C, viz.

	$C\times$	$+B\times$	$+A\times$
$1234 =$	$+\ 6\,a^2$	$-\,12\,ab$	$+\,16\,ac - 10\,b^2$
$1235 =$	$+\ 6\,ab$	$-\ 2\,ac - 10\,b^2$	$+\ 6\,ad$
$1236 =$	$-\ 2\,ac + 8\,b^2$	$+\ 6\,ad - 18\,bc$	$-\ 2\,df + 8\,e^2$
$1245 =$	$+\,18\,ac$	$-\ 6\,ad - 30\,bc$	$+\ 8\,ae + 10\,bd$
$1246 =$	$+\,12\,bc$	$+\ 4\,ae - 4\,bd - 24\,c^2$	$+\ 4\,be + 8\,cd$
$1345 =$	$+\,24\,ad$	$-\ 8\,ae - 40\,bd$	$+\ 4\,af + 20\,be$
$1256 =$	$-\ 1\,ae + 4\,bd + 3\,c^2$	$+\ 1\,af + 5\,be - 18\,cd$	$-\ 1\,bf + 4\,ce + 3\,d^2$
$2345 =$	$+\,20\,ae + 40\,bd - 30\,c^2$	$-\,80\,be + 20\,cd$	$+\,20\,bf + 40\,ce - 30\,d^2$
$1346 =$	$+\ 4\,ae + 8\,bd + 6\,c^2$	$-\,36\,cd$	$+\ 4\,bf + 8\,ce + 6d^2$
$2346 =$	$+\ 4\,af + 20\,be$	$-\ 8\,bf - 4\,ce$	$+\,24\,cf$
$1356 =$	$+\ 4\,be + 8\,cd$	$+\ 4\,bf - 4\,ce - 24\,d^2$	$+\,12\,de$
$2356 =$	$+\ 8\,bf + 10\,ce$	$-\ 6\,cf - 30\,de$	$+\,18\,df$
$1456 =$	$+\ 6\,ce$	$+\ 6\,cf - 18\,de$	$-\ 2\,df + 8\,e^2$
$2456 =$	$+\ 6\,cf$	$-\ 2\,df - 10\,e^2$	$+\ 6\,ef$
$3456 =$	$+\,16\,df - 10\,e^2$	$-\,12\,ef$	$+\ 6\,f^2$

and the following relations between L, L', &c. and A, B, C, viz.

	$C\times$	$+B\times$	$+A\times$
$N =$	$-\ 3\,ac + 3\,b^2$	$+\ 3\,ad - 3\,bc$	$-\ 1\,ae + 1\,bd$
$M =$	$-\ 3\,ad + 3\,bc$	$+\ 3\,ae - 3\,c^2$	$-\ 1\,af + 1\,cd$
$L =$	$+\,11\,ae + 28\,bd - 39\,c^2$	$+\ 1\,af - 75\,be + 74\,cd$	$+\,11\,bf + 28\,ce - 39\,d^2$
$L' =$	$-\ 7\,ae + 4\,bd + 3\,c^2$	$+\ 3\,af + 15\,be - 18\,cd$	$-\ 7\,bf + 4\,ce + 3\,d^2$
$2P =$	$-\ 1\,ae - 2\,bd + 3\,c^2$	$+\ 3\,be - 3\,cd$	$+\ 1\,bf + 2\,ce - 3\,d^2$
$P' =$	$+\ 3\,ae - 6\,bd + 3\,c^2$	$-\ 1\,af + 1\,cd$	$+\ 3\,bf - 6\,ce + 3\,d^2$
$M' =$	$-\ 1\,af + 1\,cd$	$+\ 3\,bf - 3\,d^2$	$-\ 3\,cf + 3\,de$
$N' =$	$-\ 1\,bf + 1\,ce$	$+\ 3\,cf - 3\,de$	$-\ 3\,df + 3\,e^2$

We have also the following relations between L, L', &c. and a, b, c, d, e, f, viz.

$$\begin{aligned}
aP - bM + cN &= 0,\\
aM' + bP' - 2cM + 3dN &= 0,\\
aN' + 2bM' - cL' \quad . \quad + 3eN &= 0,\\
3bN' \quad . \quad - dL' + 2eM + fN &= 0,\\
3cN' - 2dM' + eP' + fM &= 0,\\
dN' - eM' + fP &= 0.
\end{aligned}$$

The quartinvariant No. 19 [G] is equal to

$$-AC + B^2,$$

i.e. it is in fact equal to -4 into the discriminant of the quintic No. 14, [A].

The octinvariant No. 25 [Q] is expressible in terms of the coefficients of Nos. 14 and 16, viz. A, B, C, as before, and $\frac{1}{3}\alpha$, β, γ, $\frac{1}{3}\delta$ the coefficients of No. 16, [D], i.e.

$$\begin{aligned}
\alpha &= 3\,(ace - ad^2 - b^2e + 2bcd - c^3),\\
\beta &= acf - ade - b^2f + bd^2 + bce - c^2d,\\
\gamma &= adf - ae^2 - bcf + bde + c^2e - cd^2,\\
\delta &= 3\,(bdf - be^2 + 2cde - c^2f - d^3),
\end{aligned}$$

then No. 25 is equal to

$$\begin{vmatrix} A, & B, & C \\ \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix}.$$

The value of the discriminant No. 26, [Q′], is

$$(\text{No. } 19)^2 - 128 \text{ No. } 25. \quad [\text{that is } Q' = G^2 - 128Q.]$$

We have also an expression for the discriminant in terms of L, L', &c., viz. three times the discriminant No. 26 is equal to

$$[\text{or say } 3Q' =]\; LL' + 64MM' - 64NN',$$

a remarkable formula, the discovery of which is due to Mr Salmon.

It may be noticed, that in the particular case in which the quintic has two square factors, if we write

$$(a, b, c, d, e, f \between x, y)^5 = 5\,\{(p, q, r \between x, y)^2\}^2 \,.\, (\lambda, \mu \between x, y),$$

then

$$a = 5\lambda p^2,\qquad b = 4pq\lambda + p^2\mu,\qquad c = (2q^2+pr)\lambda + 2pq\mu,$$

$$f = 5r^2\mu,\qquad e = r^2\lambda + 4qr\mu,\qquad d = 2qr\lambda + (2q^2+pr)\mu\,;$$

and these values give

$$P = K(6q^2 - pr),\qquad P' = K(10q^2 - 15pr),$$

$$M = K\,.\,10pq,\qquad M' = K\,.\,10qr,$$

$$N = K\,.\,5p^2,\qquad N' = K\,.\,5r^2,$$

where the value of K is

$$8\,(p\mu^2 - 2q\mu\lambda + r\lambda^2)^2\,(pr - q^2)^2.$$

The table No. 29 is the invariant of the twelfth degree of the quintic, given in its simplest form, i.e. in a form not containing any power higher than the fourth of the leading coefficient a: this invariant was first calculated by M. Faa de Bruno.

No. 29. [See U. No. 29, p. 294.]

The tables Nos. 30 to 35 relate to a sextic. No. 30 is the sextic itself; No. 31 the quadrinvariant; Nos. 32 and 33 the quadricovariants (the latter of them the Hessian); No. 34 is the quartinvariant or catalecticant; and No. 35 is the sextinvariant in its best form, i.e. a form not containing any power higher than the second of the leading coefficient a.

No. 30.

(	$a+1$	$b+6$	$c+15$	$d+20$	$e+15$	$f+6$	$g+1$	$)(x, y)^6$

No. 31.

ag	$+\ 1$
bf	$-\ 6$
ce	$+\ 15$
d^2	$-\ 10$
± 16	

No. 32.

(	$ae+1$ $bd-4$ c^2+3	$af+2$ $be-6$ $cd+4$	$ag+1$ $ce-9$ d^2+8	$bg+2$ $cf-6$ $de+4$	$cg+1$ $df-4$ e^2+3	$)(x, y)^4$
	± 4	± 6	± 9	± 6	± 4	

No. 33.

(	$ac+1$ b^2-1	$ad+4$ $bc-4$	$ae+6$ $bd+4$ c^2-10	$af+4$ $be+16$ $cd-20$	$ag+1$ $bf+14$ $ce+5$ d^2-20	$bg+4$ $cf+16$ $de-20$	$cg+6$ $df+4$ e^2-10	$dg+4$ $ef-4$	$eg+1$ f^2-1	$)(x, y)^8$
	± 1	± 4	± 10	± 20	± 20	± 20	± 10	± 4	± 1	

No. 34.

$aceg$	$+1$
acf^2	-1
ad^2g	-1
$adef$	$+2$
ae^3	-1
b^2eg	-1
b^2f^2	$+1$
$bcdg$	$+2$
$bcef$	-2
bd^2f	-2
bde^2	$+2$
c^3g	-1
c^2df	$+2$
c^2e^2	$+1$
cd^2e	-3
d^4	$+1$

± 12

No. 35.

$a^2d^2g^2$	$+1$	$acde^2f$	-42	bc^2df^2	$+60$
a^2defg	-6	ace^4	$+12$	bc^2e^2f	-30
a^2df^3	$+4$	ad^4g	-20	bcd^3g	$+24$
a^2e^3g	$+4$	ad^3ef	$+24$	bcd^2ef	-84
$a^2e^2f^2$	-3	ad^2e^3	-8	$bcde^3$	$+66$
$abcdg^2$	-6	b^3dg^2	$+4$	bd^4f	$+24$
$abcefg$	$+18$	b^3efg	-12	bd^3e^2	-24
$abcf^3$	-12	b^3f^3	$+8$	c^4eg	$+12$
abd^2fg	$+12$	$b^2c^2g^2$	-3	c^4f^2	-27
$abde^2g$	-18	b^2ce^2g	$+30$	c^3d^2g	-8
abe^3f	$+6$	b^2cef^2	-24	c^3def	$+66$
ac^3g^2	$+4$	b^2d^2eg	-12	c^3e^3	-8
ac^2e^2g	-24	$b^2d^2f^2$	-24	c^2d^3f	-24
ac^2dfg	-18	b^2de^2f	$+60$	$c^2d^2e^2$	-39
ac^2ef^2	$+30$	b^2e^4	-27	cd^4e	$+36$
acd^2eg	$+54$	bc^3fg	$+6$	d^6	-8
acd^2f^2	-12	bc^2deg	-42		

± 565

The sextinvariant may be thus represented by means of a determinant of the sixth order and of the quadrinvariant and quartinvariant.

$$5 \times \text{No. } 35 = \left|\begin{array}{cccccc} & a, & 2b, & 3c, & 4d, & e \\ & b, & 2c, & 3d, & 4e, & f \\ & c, & 2d, & 3e, & 4f, & g \\ a, & 4b, & 3c, & 2d, & e & \\ b, & 4c, & 3d, & 2e, & f & \\ c, & 4d, & 3e, & 2f, & g & \end{array}\right|$$

$$+\,4\,(ag-6bf+15ce-10d^2)\left|\begin{array}{cccc} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{array}\right|$$

The tables Nos. 36 and 37 relate to a septimic. No. 36 is the septimic itself; No. 37 the quartinvariant.

No. 36.

$$(\,a+1 \mid b+7 \mid c+21 \mid d+35 \mid e+35 \mid f+21 \mid g+7 \mid h+1\,)(x, y)^7$$

No. 37.

a^2h^2	$-\ 1$	bd^2h	$-\ 40$
$abgh$	$+\ 14$	$bdeg$	$-\ 50$
$acfh$	$-\ 18$	bdf^2	$-\ 360$
acg^2	$-\ 24$	be^2f	$+\ 240$
$adeh$	$+\ 10$	c^2eg	$-\ 360$
$adfg$	$+\ 60$	c^2f^2	$-\ 81$
ae^2g	$-\ 40$	cd^2g	$+\ 240$
b^2fh	$-\ 24$	$cdef$	$+\ 990$
b^2g^2	$-\ 25$	ce^3	$-\ 600$
$bcfg$	$+\ 234$	d^3f	$-\ 600$
$bceh$	$+\ 60$	d^2e^2	$+\ 375$
			± 2223

The tables Nos. 38 to 45 relate to the octavic. No. 38 is the octavic itself; No. 39 the quadrinvariant; Nos. 40, 41 and 42 are the quadricovariants, the last of them being the Hessian; No. 43 is the cubinvariant; No. 44 the quartinvariant, and No. 45 the quintinvariant, which is also the catalecticant.

No. 38.

$$(a+1 \mid b+8 \mid c+28 \mid d+56 \mid e+70 \mid f+56 \mid g+28 \mid h+8 \mid i+1 \between x, y)^8$$

No. 39.

ai	$+\ 1$
bh	$-\ 8$
cg	$+\ 28$
df	$-\ 56$
e^2	$+\ 35$
	± 64

No. 40.

($ag+1$ $bf-6$ $ce+15$ d^2-10	$ah+2$ $bg-10$ $cf+18$ $de-10$	$ai+1$ $bh-2$ $cg-8$ $df+34$ e^2-25	$bi+2$ $ch-10$ $dg+18$ $ef-10$	$ci+1$ $dh-6$ $eg+15$ f^2-10	$\between x, y)^4$.
± 16	± 20	± 35	± 20	± 16	

No. 41.

($ae+1$ $bd-4$ c^2+3	$af+4$ $be-12$ $cd+8$	$ag+6$ $bf-8$ $ce-22$ d^2+24	$ah+4$ $bg+8$ $cf-48$ $de+36$	$ai+1$ $bh+12$ $cg-22$ $df-36$ e^2+45	$bi+4$ $ch+8$ $dg-48$ $ef+36$	$ci+6$ $dh-8$ $eg-22$ f^2+24	$di+4$ $eh-12$ $fg+8$	$ei+1$ $fh-4$ g^2+3	$\between x, y)^8$
± 4	± 12	± 30	± 48	± 58	± 48	± 30	± 12	± 4	

No. 42.

($ac+1$ b^2-1	$ad+6$ $bc-6$	$ae+15$ $bd+6$ c^2-21	$af+20$ $be+50$ $cd-70$	$ag+15$ $bf+90$ d^2-105	$ah+6$ $bg+78$ $cf+126$ $de-210$	$ai+1$ $bh+34$ $cg+154$ $df-14$ e^2-175	$bi+6$ $ch+78$ $dg+126$ $ef-210$	$ci+15$ $dh+90$ f^2-105	$di+20$ $eh+50$ $fg-70$	$ei+15$ $fh+6$ g^2-21	$fi+6$ $gh-6$	$gi+1$ h^2-1	$\between x, y)^{12}$
± 1	± 6	± 21	± 70	± 105	± 210	± 189	± 210	± 105	± 70	± 21	± 6	± 1	

No. 43.

aei	$+\ 1$				
afh	$-\ 4$				
ag^2	$+\ 3$				
bdi	$-\ 4$				
beh	$+\ 12$				
bfg	$-\ 8$				
c^2i	$+\ 3$				
cdh	$-\ 8$				
ceg	$-\ 22$				
cf^2	$+\ 24$				
d^2g	$+\ 24$				
def	$-\ 36$				
e^3	$+\ 15$				

± 82

No. 44.

$acgi$	$-\ 1$	$bcgh$	$+\ 3$	cd^2i	$-\ 2$
ach^2	$+\ 1$	$bdei$	$+\ 1$	$cdeh$	$-\ 23$
$adfi$	$+\ 3$	$bdfh$	$-\ 10$	$cdfg$	$+\ 27$
$adgh$	$-\ 3$	bdg^2	$+\ 9$	ce^3g	$+\ 19$
ae^2i	$-\ 2$	be^2h	$+\ 11$	cef^2	$-\ 21$
$aefh$	$+\ 1$	$befg$	$-\ 23$	d^3h	$+\ 12$
aeg^2	$+\ 3$	bf^3	$+\ 12$	d^2eg	$-\ 21$
af^2g	$-\ 2$	c^2ei	$+\ 3$	d^2f^2	$-\ 13$
b^2gi	$+\ 1$	c^2fh	$+\ 9$	de^2f	$+\ 32$
b^2h^2	$-\ 1$	c^2g^2	$-\ 12$	e^4	$-\ 10$
$bcfi$	$-\ 3$				

± 147

No. 45.

$acegi$	$+\ 1$	af^4	$+\ 1$	$bdeg^2$	$-\ 4$	cd^2g^2	$+\ 1$
$aceh^2$	$-\ 1$	b^2egi	$-\ 1$	bdf^2g	$+\ 2$	$cdefg$	$-\ 2$
acf^2i	$-\ 1$	b^2eh^2	$+\ 1$	be^3h	$-\ 2$	cdf^3	$-\ 2$
$acfgh$	$+\ 2$	b^2fgh	$-\ 2$	be^2fg	$+\ 4$	ce^3g	$-\ 3$
acg^3	$-\ 1$	b^2fi^2	$+\ 1$	bef^3	$-\ 2$	ce^2dh	$+\ 4$
ad^2gi	$-\ 1$	b^2g^3	$+\ 1$	c^3gi	$-\ 1$	ce^2f^2	$+\ 3$
ad^2h^2	$+\ 1$	$bcdgi$	$+\ 2$	c^3h^2	$+\ 1$	d^4i	$+\ 1$
$adefi$	$+\ 2$	$bcdh^2$	$-\ 2$	c^2dfi	$+\ 2$	d^3eh	$-\ 2$
$adegh$	$-\ 2$	$bcefi$	$-\ 2$	c^2dgh	$-\ 2$	d^3fg	$-\ 2$
adf^2h	$-\ 2$	$bcegh$	$+\ 2$	c^2e^2i	$+\ 1$	d^2e^2g	$+\ 3$
$adfg^2$	$+\ 2$	bcf^2h	$+\ 2$	c^2efh	$-\ 4$	d^2ef^2	$+\ 3$
ae^3i	$-\ 1$	$bcfg^2$	$-\ 2$	c^2eg^2	$+\ 2$	de^3f	$-\ 4$
ae^2fh	$+\ 2$	bd^2fi	$-\ 2$	c^2f^2g	$+\ 1$	e^5	$+\ 1$
ae^2g^2	$+\ 1$	bd^2gh	$+\ 2$	cd^2ei	$-\ 3$		
aef^2g	$-\ 3$	bde^2i	$+\ 2$	cd^2fh	$+\ 2$		

± 56

If we write

$$\text{No. } 39 = I,$$

$$\text{No. } 43 = J,$$

$$\text{No. } 44 = K,$$

$$\text{No. } 45 = L,$$

then the determinant called the lambdaic, viz.

$$\left|\begin{array}{lllll} a\ , & b\ , & c\ , & d\ , & e-12\lambda \\ b\ , & c\ , & d\ , & e+3\lambda, & f \\ c\ , & d\ , & e-2\lambda, & f\ , & g \\ d\ , & e+3\lambda, & f\ , & g\ , & h \\ e-12\lambda, & f\ , & g\ , & h\ , & i \end{array}\right|$$

is equal to

$$L + 2\lambda K + 3\lambda^2 J + 18\lambda^3 I - 2592\lambda^5.$$

Nos. 46 to 48 relate to the nonic. No. 46 is the nonic itself; Nos. 47 and 48 are the two quartinvariants, each of them in its best form, viz. No. 48 does not contain a^2, and No. 47 does not contain aci^2, the leading term of No. 48. The nonic is the lowest quantic with two quartinvariants.

No. 46.

($a+1$	$b+9$	$c+36$	$d+84$	$e+126$	$f+126$	$g+84$	$h+36$	$i+9$	$j+1$	$\between(x, y)^9$.

No. 47.

a^2j^2	$-\ \ 1$	bf^2h	...
$abij$	$+\ \ 18$	bfg^2	$-\ \ 720$
aci^2	...	c^2fj	$+\ \ 432$
$achj$	$-\ \ 72$	c^2gi	$-\ 1728$
$adgj$	$+\ 168$	c^2h^2	...
$adhi$	...	$cdej$	$-\ \ 720$
$aefj$	$-\ 108$	$cdfi$	$+\ 2160$
$aegi$	$-\ 576$	$cdgh$	$+\ 4608$
aeh^2	$+\ 432$	ce^2i	...
af^2i	$+\ 540$	$cefh$	$-\ 2592$
$afgh$	$-\ 720$	ceg^2	$-\ 5760$
ag^3	$+\ 320$	cf^2g	$+\ 4320$
b^2hj	...	d^3j	$+\ \ 320$
b^2i^2	$-\ \ 81$	d^2ei	$-\ \ 720$
$bcgj$	...	d^2fh	$-\ 5760$
$bchi$	$+\ 648$	d^2g^2	$-\ 1536$
$bdfj$	$-\ 576$	$defg$	$+\ 14688$
$bdgi$	$+\ 792$	de^2h	$+\ 4320$
bdh^2	$-\ 1728$	df^3	$-\ 8640$
$begh$	$+\ 2160$	e^3g	$-\ 8640$
be^2j	$+\ 540$	e^2f^2	$+\ 5184$
$befi$	$-\ 972$		
			± 41650

No. 48.

a^2j^2	...	bf^2h	$+\ 70$
$abij$	...	bfg^2	$-\ 45$
aci^2	$+\ 2$	c^2fj	$+\ 27$
$achj$	$-\ 2$	c^2gi	$-\ 52$
$adgj$	$+\ 7$	c^2h^2	$+\ 25$
$adhi$	$-\ 7$	$cdej$	$-\ 45$
$aefj$	$-\ 5$	$cdfi$	$+\ 23$
$aegi$	$-\ 22$	$cdgh$	$+\ 22$
aeh^2	$+\ 27$	ce^3i	$+\ 70$
af^2i	$+\ 25$	$cefh$	$-\ 127$
$afgh$	$-\ 45$	ceg^2	$+\ 32$
ag^3	$+\ 20$	cf^2g	$+\ 25$
b^2hj	$+\ 2$	d^3j	$+\ 20$
b^2i^2	$-\ 2$	d^2ei	$-\ 45$
$bcgj$	$-\ 7$	d^2fh	$+\ 32$
$bchi$	$+\ 7$	d^2g^2	$+\ 47$
$bdfj$	$-\ 22$	$defg$	$+\ 85$
$bdgi$	$+\ 74$	de^2h	$+\ 25$
bdh^2	$-\ 52$	df^3	$-\ 50$
$begh$	$+\ 23$	e^3g	$-\ 50$
be^2j	$+\ 25$	e^2f^2	$+\ 30$
$befi$	$-\ 73$		
			± 698

Nos 49, [49 A] and 50 relate to the dodecadic. No. 49 is the dodecadic itself; [No. 49 A, inserted in this place, but originally printed in the Fifth Memoir on Quantics, is the dodecadic quadricovariant], No. 50 is the cubinvariant. [The numerical coefficients in this last table as originally printed in the Third Memoir were altogether erroneous, and the table as here printed is in fact the table No. 50 *bis*, of the Fifth Memoir on Quantics.]

No. 49.

$a+1$	$b+12$	$c+66$	$d+220$	$e+495$	$f+792$	$g+924$	$h+792$	$i+495$	$j+220$	$k+66$	$l+12$	$m+1$	$\between(x, y$

No. 49 A.

$ag + 1$	$ah + 6$	$ai + 15$	$aj + 20$	$ak + 15$	$al + 6$	$am + 1$
$bf - 6$	$bg - 30$	$bh - 54$	$bi - 30$	$bj + 30$	$bk + 54$	$bl + 30$
$ce + 15$	$cf + 54$	$cg + 24$	$ch - 150$	$ci - 270$	$cj - 150$	$ck + 24$
($d^2 - 10$	$de - 30$	$df + 150$	$dg + 430$	$dh + 270$	$di - 270$	$dj - 430$
		$e^2 - 135$	$ef - 270$	$eg + 495$	$eh + 1080$	$ei + 495$
				$f^2 - 540$	$fg - 720$	$fh + 720$
						$g^2 - 840$
± 16	± 60	± 189	± 450	± 810	± 1140	± 1270

$bm + 6$	$cm + 15$	$dm + 20$	$em + 15$	$fm + 6$	$gm + 1$	
$cl + 54$	$dl + 30$	$el - 30$	$fl - 54$	$gl - 30$	$hl - 6$	
$dk - 150$	$ek - 270$	$fk - 150$	$gk + 24$	$hk + 54$	$ik + 15$	
$ej - 270$	$fj + 270$	$gj + 430$	$hj + 150$	$ij - 30$	$j^2 - 10$	$\big)(x, y)^{12}$.
$fi + 1080$	$gi + 495$	$hi - 270$	$i^2 - 135$			
$gh - 720$	$h^2 - 540$					
± 1140	± 810	± 450	± 189	± 60	± 16	

No. 50.

$agm + 1$	$cfl - 54$	$dhi + 270$
$ahl - 6$	$cgk + 24$	$e^2k - 135$
$aik + 15$	$chj + 150$	$efj + 270$
$aj^2 - 10$	$ci^2 - 135$	$egi + 495$
$bfm - 6$	$d^2m - 10$	$eh^2 - 540$
$bgl + 30$	$del + 30$	$f^2i - 540$
$bhk - 54$	$dfk + 150$	$fgh + 720$
$bij + 30$	$dgj - 430$	$g^3 - 280$
$cem + 15$		
		± 2200

Resuming now the general subject,—

54. The simplest covariant of a system of quantics of the form

$$(* \between x, y, \ldots)^m$$

(where the number of quantics is equal to the number of the facients of each quantic) is the functional determinant or *Jacobian*, viz. the determinant formed with the differential coefficients or derived functions of the quantics with respect to the several facients.

55. In the particular case in which the quantics are the differential coefficients or derived functions of a single quantic, we have a corresponding covariant of the single quantic, which covariant is termed the *Hessian;* in other words, the Hessian is the determinant formed with the second differential coefficients or derived functions of the quantic with respect to the several facients.

56. The expression, an *adjoint linear form,* is used to denote a linear function $\xi x + \eta y + \ldots$, or in the notation of quantics $(\xi, \eta, \ldots \between x, y, \ldots)$, having the same facients as

the quantic or quantics to which it belongs, and with indeterminate coefficients $(\xi, \eta,\ldots)$. The invariants of a quantic or quantics, and of an adjoint linear form, may be considered as quantics having $(\xi, \eta,\ldots)$ for facients, and of which the coefficients are of course functions of the coefficients of the given quantic or quantics. An invariant of the class in question is termed a contravariant of the quantic or quantics. The idea of a contravariant is due to Mr Sylvester.

In the theory of binary quantics, it is hardly necessary to consider the contravariants; for any contravariant is at once turned into an invariant by writing $(y, -x)$ for (ξ, η).

57. If we imagine, as before, a system of quantics of the form

$$(*\between x, y, \ldots)^m,$$

where the number of quantics is equal to the number of the facients in each quantic, the function of the coefficients, which, equated to zero, expresses the result of the elimination of the facients from the equations obtained by putting each of the quantics equal to zero, is said to be the *Resultant* of the system of quantics. The resultant is an invariant of the system of quantics.

And in the particular case in which the quantics are the differential coefficients, or derived functions of a single quantic with respect to the several facients, the resultant in question is termed the *Discriminant* of the single quantic; the discriminant is of course an invariant of the single quantic.

58. Imagine two quantics, and form the equations which express that the differential coefficients, or derived functions of the one quantic with respect to the several facients, are proportional to those of the other quantic. Join to these the equations obtained by equating each of the quantics to zero; we have a system of equations, one of which is contained in the others, and from which therefore the facients may be eliminated. The function which, equated to zero, expresses the result of the elimination is an invariant which (from its geometrical signification) might be termed the *Tactinvariant* of the two quantics, but I do not at present propose to consider this invariant except in the particular case where the system consists of a given quantic and of an adjoint linear form. In this case the tactinvariant is a contravariant of the given quantic, viz. the contravariant termed the *Reciprocant*.

59. Consider now a quantic

$$(*\between x, y, \ldots)^m,$$

and let the facients $x, y, \ldots$ be replaced by $\lambda x+\mu X, \lambda y+\mu Y, \ldots$ the resulting function may, it is clear, be considered as a quantic with the facients (λ, μ) and of the form

$$\left(\begin{array}{l} (*\between x, y, \ldots)^m \\ (*\between x, y, \ldots)^{m-1}(X, Y, \ldots) \\ \vdots \\ (*\between X, Y, \ldots)^m \end{array}\right\between \lambda, \mu)^m.$$

The coefficients of this quantic are termed *Emanants*, viz., excluding the first coefficient, which is the quantic itself (but which might be termed the 0-th emanant), the other coefficients are the first, second, and last or ultimate emanants. The ultimate emanant is, it is clear, nothing else than the quantic itself, with $(X, Y, \ldots)$ instead of $(x, y, \ldots)$ for facients: the penultimate emanant is, in like manner, obtained from the first emanant by interchanging $(x, y, \ldots)$ with $(X, Y, \ldots)$, and similarly for the other emanants. The facients $(X, Y, \ldots)$ may be termed the *facients of emanation*, or simply the *new facients*. The theory of emanation might be presented in a more general form by employing two or more sets of emanating facients; we might, for example, write $\lambda x + \mu X + \nu X'$, $\lambda y + \mu Y + \nu Y', \ldots$ for $x, y, \ldots$, but it is not necessary to dwell upon this at present.

The invariants, in respect to the new facients, of any emanant or emanants of a quantic (i.e. the invariants of the emanant or emanants, considered as a function or functions of the new facients), are, it is easy to see, covariants of the original quantic, and it is in many cases convenient to define a covariant in this manner; thus the Hessian is the discriminant of the second or quadric emanant of the quantic.

60. If we consider a quantic

$$(a, b, \ldots \between x, y, \ldots)^m,$$

and an adjoint linear form, the operative quantic

$$(\partial_a, \partial_b, \ldots \between \xi, \eta, \ldots)^m$$

(which is, so to speak, a contravariant operator) is termed the *Evector*. The properties of the evector have been considered in the introductory memoir, and it has been in effect shown that the evector operating upon an invariant, or more generally upon a contravariant, gives rise to a contravariant. Any such contravariant, or rather such contravariant considered as so generated, is termed an *Evectant*. In the case of a binary quantic,

$$(a, b, \ldots \between x, y)^m,$$

the covariant operator

$$(\partial_a, \partial_b, \ldots \between y, -x)^m$$

may, if not with perfect accuracy, yet without risk of ambiguity, be termed the *Evector*, and a covariant obtained by operating with it upon an invariant or covariant, or rather such covariant considered as so generated, may in like manner be termed an *Evectant*.

61. Imagine two or more quantics of the same order,

$$(a, b, \ldots \between x, y)^m,$$
$$(\alpha, \beta, \ldots \between x, y)^m,$$
$$\vdots$$

we may have covariants such that for the coefficients of each pair of quantics the covariant is reduced to zero by the operators

$$a\partial_\alpha + b\partial_\beta + \ldots,$$
$$\alpha\partial_a + \beta\partial_b + \ldots.$$

Such covariants are called *Combinants*, and they possess the property of being invariantive, quoad the system, i.e. the covariant remains unaltered to a factor *près*, when each quantic is replaced by a linear function of all the quantics. This extremely important theory is due to Mr Sylvester.

Proceeding now to the theory of ternary quadrics and cubics,—

First for a ternary quadric, we have the following tables:—

Covariant and other Tables, Nos. 51 to 56 (a ternary quadric).

No. 51.

The quadric is represented by

$$(a, b, c, f, g, h \between x, y, z)^2,$$

which means

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

No. 52.

The first derived functions (omitting the factor 2) are—

$$\begin{array}{l}(a, \quad h, \quad g \between x, \quad y, \quad z),\\(h, \quad b, \quad f \between x, \quad y, \quad z),\\(g, \quad f, \quad c \between x, \quad y, \quad z).\end{array}$$

No. 53.

The operators which reduce a covariant to zero are

$$\begin{array}{l}(\ h, \quad b, \quad 2f \between \partial_g, \quad \partial_f, \quad \partial_c) - z\partial_y,\\(2g, \quad f, \quad c \between \partial_a, \quad \partial_h, \quad \partial_g) - x\partial_z,\\(\ a, \quad 2h, \quad g \between \partial_h, \quad \partial_b, \quad \partial_f) - y\partial_x,\\(\ g, \quad 2f, \quad c \between \partial_h, \quad \partial_b, \quad \partial_f) - y\partial_z,\\(\ a, \quad h, \quad 2g \between \partial_g, \quad \partial_f, \quad \partial_c) - z\partial_x,\\(2h, \quad b, \quad f \between \partial_a, \quad \partial_h, \quad \partial_g) - x\partial_y.\end{array}$$

No. 54.

The evector is

$$(\partial_a, \quad \partial_b, \quad \partial_c, \quad \partial_f, \quad \partial_g, \quad \partial_h \between \xi, \quad \eta, \quad \zeta)^2.$$

No. 55.

The discriminant is

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

which is equal to

$$abc - af^2 - bg^2 - ch^2 + 2fgh.$$

No. 56.

The reciprocant is

$$-\begin{vmatrix} & \xi, & \eta, & \zeta \\ \xi, & a, & h, & g \\ \eta, & h, & b, & f \\ \zeta, & g, & f, & c \end{vmatrix}$$

which is equal to

$$(bc - f^2, \quad ca - g^2, \quad ab - h^2, \quad gh - af, \quad hf - bg, \quad fg - ch \between \xi, \eta, \zeta)^2.$$

The discriminant is, it will be noticed, the same function as the Hessian. The reciprocant is the evectant of the discriminant. The covariants are the quadric itself and the discriminant; the reciprocant is the only contravariant.

Next, for a ternary cubic, we have the following Tables:

Covariant and other Tables, Nos. 57 to 70 (a ternary cubic).

No. 57.

The cubic is $U =$

$$(a, b, c, f, g, h, i, j, k, l \between x, y, z)^3,$$

which means—

$$ax^3 + by^3 + cz^3 + 3fy^2z + 3gz^2x + 3hx^2y + 3iyz^2 + 3jzx^2 + 3kxy^2 + 6lxyz.$$

No. 58.

The first derived functions (omitting the factor 3) are

$$(a, \ k, \ g, \ l, \ j, \ h \between x, \ y, \ z)^2,$$
$$(h, \ b, \ i, \ f, \ l, \ k \between x, \ y, \ z)^2,$$
$$(j, \ f, \ c, \ i, \ g, \ l \between x, \ y, \ z)^2.$$

The second derived functions (omitting the factor 6) are

$$
\begin{aligned}
&(a,\ h,\ j \between x,\ y,\ z),\\
&(k,\ b,\ f \between x,\ y,\ z),\\
&(g,\ i,\ c \between x,\ y,\ z),\\
&(l,\ f,\ i \between x,\ y,\ z),\\
&(j,\ l,\ g \between x,\ y,\ z),\\
&(h,\ k,\ l \between x,\ y,\ z).
\end{aligned}
$$

No. 59.

The operators which reduce a covariant to zero are

$$
\begin{aligned}
&(\ j,\ 3f,\ c,\ 2i,\ g,\ 2l \between \partial_h,\ \partial_b,\ \partial_i,\ \partial_f,\ \partial_l,\ \partial_k) - y\partial_z,\\
&(\ a,\ k,\ 3g,\ 2l,\ 2j,\ h \between \partial_j,\ \partial_f,\ \partial_c,\ \partial_i,\ \partial_g,\ \partial_l) - z\partial_z,\\
&(3h,\ b,\ i,\ f,\ 2l,\ 2k \between \partial_a,\ \partial_k,\ \partial_g,\ \partial_l,\ \partial_j,\ \partial_h) - x\partial_y,\\
&(\ h,\ b,\ 3i,\ 2f,\ 2l,\ k \between \partial_j,\ \partial_f,\ \partial_c,\ \partial_i,\ \partial_g,\ \partial_l) - z\partial_y,\\
&(3j,\ f,\ c,\ i,\ 2g,\ 2l \between \partial_a,\ \partial_k,\ \partial_g,\ \partial_l,\ \partial_j,\ \partial_h) - x\partial_z,\\
&(\ a,\ 3k,\ g,\ 2l,\ j,\ 2h \between \partial_h,\ \partial_b,\ \partial_i,\ \partial_f,\ \partial_l,\ \partial_k) - y\partial_x.
\end{aligned}
$$

No. 60.

The evector is

$$(\partial_a,\ \partial_b,\ \partial_c,\ \partial_f,\ \partial_g,\ \partial_h,\ \partial_i,\ \partial_j,\ \partial_k,\ \partial_l \between \xi,\ \eta,\ \zeta)^3.$$

No. 61.

The Hessian is $HU =$

$$
- \begin{vmatrix}
(a,\ h,\ j \between x,\ y,\ z), & (h,\ k,\ l \between x,\ y,\ z), & (j,\ l,\ g \between x,\ y,\ z)\\
(h,\ k,\ l \between x,\ y,\ z), & (k,\ b,\ f \between x,\ y,\ z), & (l,\ f,\ i \between x,\ y,\ z)\\
(j,\ l,\ g \between x,\ y,\ z), & (l,\ f,\ i \between x,\ y,\ z), & (g,\ i,\ c \between x,\ y,\ z)
\end{vmatrix}
$$

which is equal to

agk -1	bhi -1	cij -1	bch -1	acf -1	abg -1	bcj -1	ach -1	abi -1	abc -1
al^2 $+1$	bl^2 $+1$	cl^2 $+1$	bgl $+2$	ai^2 $+1$	afl $+2$	bg^2 $+1$	afg -1	af^2 $+1$	afi $+1$
gh^2 $+1$	f^2h $+1$	fg^2 $+1$	bij -1	chl $+2$	aik -1	cfh -1	ail $+2$	bgh -1	bgj $+1$
hjl -2	fkl -2	fkl -2	ck^2 $+1$	cjk -1	bj^2 $+1$	ckl $+2$	ch^2 $+1$	bjl $+2$	chk $+1$
j^2k $+1$	ik^2 $+1$	i^2j $+1$	f^2j $+1$	fgj $+1$	fhj -2	fij $+1$	fj^2 $+1$	fjk -2	fgh -3
			fgk -2	g^2k $+1$	ghk $+1$	gik -2	gjk $+1$	gk^2 $+1$	fjl $+2$
			fhi $+1$	ghi -2	h^2i $+1$	hi^2 $+1$	hij -2	hki $+1$	gkl $+2$
			fl^2 -1	gl^2 -1	hl^2 -1	il^2 -1	jl^2 -1	kl^2 -1	hil $+2$
									ijk -3
									l^3 -2
± 3	± 3	± 3	± 5	± 5	± 5	± 5	± 5	± 5	± 9

$\between (x,\ y,\ z)^2.$

No. 62.

The quartinvariant is $S=$

$abcl$	-1	f^2j^2	$+1$
$abgi$	$+1$	$fghl$	$+3$
$acfk$	$+1$	$fgjk$	-1
af^2g	-1	$fhij$	-1
$afil$	$+1$	fjl^2	-2
ai^2k	-1	g^2k^2	$+1$
$bchj$	$+1$	$ghki$	-1
bg^2h	-1	gkl^2	-2
$bgjl$	$+1$	h^2i^2	$+1$
bij^2	-1	hil^2	-2
cf^2h	-1	$ijkl$	$+3$
$chkl$	$+1$	l^4	$+1$
cjk^2	-1		
			± 16

No. 63.

The sextinvariant is $T=$

$a^2b^2c^2$	$+1$	acf^2hl	-24	$bcfhj^2$	-12	cfh^3i	-12	$fgjkl^2$	-12
a^2bcfi	-6	acf^2jk	-12	$bcgh^2l$	-24	cfh^2l^2	$+12$	fh^2i^2j	-12
a^2bi^3	$+4$	$acfgk^2$	-12	$bcghjk$	$+18$	$cfhjkl$	-60	$fhijl^2$	-12
a^2cf^3	$+4$	$acfhki$	$+18$	bch^2ij	-12	cfj^2k^2	$+24$	fij^2kl	$+36$
$a^2f^2i^2$	-3	$acfkl^2$	$+36$	$bchjl^2$	$+36$	$cghk^2l$	$+12$	fil^4	$+24$
ab^2cgj	-6	$acik^2l$	-24	bcj^2kl	-24	$cgjk^3$	-12	g^3k^3	$+8$
ab^2g^3	$+4$	af^3gj	-12	bfg^2hj	$+6$	ch^2ikl	$+12$	g^2hk^2i	-12
abc^2hk	-6	af^2g^2k	$+24$	$bfgj^2l$	$+12$	$chijh^2$	$+6$	$g^2k^2l^2$	-24
$abcfgh$	$+6$	af^2ghi	$+6$	$bfij^3$	-12	$chkl^3$	-12	gh^2ki^2	-12
$abcfjl$	$+12$	af^2gl^2	$+12$	bg^3hk	-12	cjk^2l^2	$+12$	$ghkil^2$	-12
$abcgkl$	$+12$	af^2ijl	$+12$	bg^2h^2i	$+24$	f^3j^3	$+8$	$gijk^2l$	$+36$
$abchil$	$+12$	$afgkil$	-60	bg^2hl^2	$+12$	$f^2g^2h^2$	-27	gkl^4	$+24$
$abcijk$	$+6$	$afhi^2l$	$+12$	bg^2jkl	$+12$	f^2gj^2k	-12	h^3i^3	$+8$
$abcl^3$	-20	afi^2jk	$+6$	$bghijl$	-60	f^2ghjl	$+36$	$h^2i^2l^2$	-24
$abfgij$	$+18$	$afil^3$	-12	$bgij^2k$	$+6$	f^2hij^2	-12	hil^4	$+24$
$abfg^2l$	-24	agk^2i^2	$+24$	$bgjl^3$	-12	$f^2j^2l^2$	-24	hi^2jkl	$+36$
abg^2ki	-12	ahi^3k	-12	bhi^2j^2	$+24$	fg^2hkl	$+36$	$i^2j^2k^2$	-27
$abghi^2$	-12	ai^2kl^2	$+12$	bij^2l^2	$+12$	fg^2jk^2	-12	$ijkl^3$	-36
$abgil^2$	$+36$	b^2cj^3	$+4$	$c^2h^2k^2$	-3	fgh^2il	$+36$	l^6	-8
abi^2jl	-24	$b^2g^2j^2$	-3	cf^2h^2j	$+24$	$fghijk$	-6		
ac^2k^3	$+4$	bc^2h^3	$+4$	$cfgh^2k$	$+6$	$fghl^3$	-36		
									± 871

The discovery of the invariants S and T is due to Aronhold, the developed expressions were first obtained by Mr Salmon.

No. 64.

There is an octicovariant for which we may take

$$\Theta U = \left|\begin{array}{cccc} & \partial_x HU, & \partial_y HU, & \partial_z HU \\ \partial_x HU, & \tfrac{1}{6}\partial_x^2\ U, & \tfrac{1}{6}\partial_x\partial_y U, & \tfrac{1}{6}\partial_x\partial_z U \\ \partial_y HU, & \tfrac{1}{6}\partial_y\partial_x U, & \tfrac{1}{6}\partial_y^2\ U, & \tfrac{1}{6}\partial_y\partial_z U \\ \partial\ HU, & \tfrac{1}{6}\partial_z\partial_x U, & \tfrac{1}{6}\partial_y\partial_z U, & \tfrac{1}{6}\partial_z^2\ U \end{array}\right| -,$$

or else

$$\Theta_{,}U = \begin{vmatrix} & \tfrac{1}{3}\partial_x \quad U, & \tfrac{1}{3}\partial_y \quad U, & \tfrac{1}{3}\partial_z \quad U \\ \tfrac{1}{3}\partial_x U, & \partial_x{}^2 \quad HU, & \partial_x\partial_y HU, & \partial_x\partial_z HU \\ \tfrac{1}{3}\partial_y U, & \partial_y\partial_x HU, & \partial_y{}^2 \quad HU, & \partial_y\partial_z HU \\ \tfrac{1}{3}\partial_z U, & \partial_z\partial_x HU, & \partial_z\partial_y HU, & \partial_z{}^2 \quad HU \end{vmatrix}$$

or else, what I believe is more simple, a function $\Theta_{,,}U$, which is a linear function of the last-mentioned two functions.

The relations between ΘU, $\Theta_{,}U$, $\Theta_{,,}U$ are

$$-\Theta_{,}U + 4\Theta U = T \,.\, U^2 - 24S \,.\, U \,.\, HU,$$
$$\Theta_{,,}U + 2\Theta U = T \,.\, U^2 - 10S \,.\, U \,.\, HU.$$

I have not worked out the developed expressions.

No. 65.

The cubicontravariant is $PU =$

bcl -1	acl -1	abl -1	ack $+1$	abi $+1$	bcj $+1$	abg $+1$	bch $+1$	acf $+1$	abc -1
bgi $+1$	agi $+1$	afk $+1$	afg -2	af^2 -1	bg^2 -1	afl $+1$	bgl $+1$	ai^2 -1	afi $+1$
cfk $+1$	chj $+1$	bhj $+1$	ail $+1$	bgh -2	cfh -2	aik -2	bij -2	chl $+1$	bgj $+1$
f^2g -1	g^2h -1	fh^2 -1	ch^2 -1	bjl $+1$	ckl $+1$	bj^2 -1	ck^2 -1	cjk -2	chk $+1$
fil $+1$	gjl $+1$	ckl $+1$	fj^2 $+2$	fhl $+3$	fgl $+3$	fhj -1	f^2j $+2$	fgj -1	fgh $+3$
i^2k -1	ij^2 -1	jk^2 -1	fhl $+3$	fjk -1	fij -1	ghk -1	fgk -1	g^2k $+2$	fjl -4
			gjk -1	gk^2 $+2$	gki -1	h^2i $+2$	fhi -1	ghi -1	gkl -4
			hij -1	hki -1	hi^2 $+2$	hl^2 -2	fl^2 -2	gl^2 -2	hil -4
			jl^2 -2	kl^2 -2	il^2 -2	jkl $+3$	ikl $+3$	ijl $+3$	ijk $+3$
									l^3 $+4$
± 3	± 3	± 3	± 7	± 7	± 7	± 7	± 7	± 7	± 13

$(\xi, \eta, \zeta)^3.$

No. 66.

The quintic contravariant is $QU=$

ab^2c^2 + 1	a^2bc^2 + 1	a^2b^2c + 1	a^2bci − 3	ab^2cj − 3	abc^2k − 3	a^2bcf − 3	ab^2cg − 3	abc^2h − 3	$abcfj$ + 6
$abcfi$ − 6	a^2cfi − 3	a^2bfi − 3	a^2cf^2 + 6	ab^2g^2 + 6	$abcfg$ + 3	a^2bi^2 + 6	$abcfl$ + 6	$abcgl$ + 6	$abcgk$ + 6
abi^3 + 4	a^2i^3 + 2	a^2f^3 + 2	a^2fi^2 − 3	$abcfh$ + 3	$abcil$ + 6	a^2f^2i − 3	$abcik$ + 3	$abcij$ + 3	$abchi$ + 6
acf^3 + 4	$abcgj$ − 6	ab^2gj − 3	$abcgh$ + 3	$abckl$ + 6	$abgi^2$ − 6	$abchl$ + 6	$abfgi$ + 9	abg^2i − 6	$abcl^2$ − 30
af^2i^2 − 3	abg^3 + 4	$abchk$ − 6	$abcjl$ + 6	$abfij$ + 9	acf^2l − 12	$abcjk$ + 3	abi^2l − 12	ac^2k^2 + 6	$abfg^2$ − 12
b^2cgj − 3	ac^2hk − 3	$abfgh$ + 3	$abgij$ + 9	$abfgl$ − 24	$acfki$ + 9	$abfgj$ + 9	acf^2k − 6	acf^2j − 6	$abgil$ + 36
b^2g^3 + 2	$acfgh$ + 3	$abfjl$ + 6	abg^2l − 12	$abgki$ − 12	af^2gi + 3	abg^2k − 6	af^3g − 6	$acfgk$ − 12	abi^2j − 12
bc^2hk − 3	$acfjl$ + 6	$abgkl$ + 6	$acfhl$ − 24	$abhi^2$ − 6	afi^2l + 6	$abghi$ − 12	af^2il + 6	$acfhi$ + 9	acf^2h − 12
$bcfgh$ + 3	$acgkl$ + 6	$abhil$ + 6	$acfjk$ − 12	$abil^2$ + 18	ai^3k − 6	$abgl^2$ + 18	afi^2k + 3	$acfl^2$ + 18	$acfkl$ + 36
$bcfjl$ + 6	$achil$ + 6	$abijk$ + 3	$acgk^2$ − 6	$acfk^2$ − 6	bc^2h^2 + 6	$abijl$ − 24	b^2cj^2 + 6	$acikl$ − 24	$acik^2$ − 12
$bcgkl$ + 6	$acijk$ + 3	abl^3 − 10	$achki$ + 9	af^3j − 6	$bcfj^2$ − 6	$acfhk$ + 9	b^2g^2j − 3	af^2g^2 + 12	af^2gl + 12
$bchil$ + 6	acl^3 − 10	ack^3 + 4	$ackl^2$ + 18	af^2gk + 24	$bcghl$ − 24	ack^2l − 12	$bcfhj$ − 12	$afgil$ − 30	af^2ij + 6
$bcijk$ + 3	$afgij$ + 9	af^2hl − 12	af^2gj − 18	af^2hi + 3	$bcgjk$ + 9	af^2gh + 3	$bcghk$ + 9	afi^2j + 3	$afgki$ − 30
bcl^3 − 10	afg^2l − 12	af^2jk − 6	afg^2k + 24	af^2l^2 + 6	$bchij$ − 12	af^2jl + 6	bch^2i − 6	$agki^2$ + 24	$afhi^2$ + 6
$bfgij$ + 9	ag^2ki − 6	$afgk^2$ − 6	$afghi$ + 6	$afkil$ − 30	$bcjl^2$ + 18	$afgkl$ − 30	$bchl^2$ + 18	ahi^3 − 6	$afil^2$ − 18
bfg^2l − 12	$aghi^2$ − 6	$afhki$ + 9	$afgl^2$ + 12	ak^2i^2 + 12	bfg^2j + 3	$afhil$ + 12	$bcjkl$ − 24	ai^2l^2 + 6	ai^2kl + 12
bg^2ki − 6	$agil^2$ + 18	$ajkl^2$ + 18	$afijl$ + 12	b^2gj^2 − 3	bg^3k − 6	$afijk$ + 6	bfg^2h + 3	$bcghj$ + 9	$bcgh^2$ − 12
$bghi^2$ − 6	ai^2jl − 12	aik^2l − 12	$agkil$ − 30	bch^2l − 12	bg^2hi + 24	afl^3 − 6	$bfgjl$ + 12	bcj^2l − 12	$bchjl$ + 36
$bgil^2$ + 18	bcj^3 + 4	b^2j^3 + 2	ahi^2l + 6	$bchjk$ + 9	bg^2l^2 + 6	agk^2i + 24	$bfij^2$ − 18	bg^3h − 6	bcj^2k − 12
bi^2jl − 12	bg^2j^2 − 3	bch^3 + 4	ai^2jk + 3	$bfghj$ + 6	$bgijl$ − 30	ahi^2k − 18	bg^2kl + 6	bg^2jl + 6	$bfgj^2$ + 6
c^2k^3 + 2	c^2h^3 + 2	$bfhj^2$ − 6	ail^3 − 6	bfj^2l + 6	bi^2j^2 + 12	$aikl^2$ + 12	$bghil$ − 30	$bgij^2$ + 3	bg^2hl + 12
cf^2hl − 12	$cfhj^2$ − 6	bgh^2l − 12	$bchj^2$ − 6	bg^2hk − 18	c^2hk^2 − 3	bch^2j − 6	$bgijk$ + 6	c^2h^2k − 3	bg^2jk + 6
cf^2jk − 6	cgh^2l − 12	$bghjk$ + 9	bg^2hj + 3	bgh^2i + 24	cf^2hj + 24	bfj^3 − 6	bgl^3 − 6	$cfgh^2$ + 3	$bghij$ − 30
$cfgk^2$ − 6	$cghjk$ + 9	bh^2ij − 6	bgj^2l + 6	$bghl^2$ + 12	$cfghk$ + 6	bg^2h^2 + 12	bhi^2j + 24	$cfhjl$ − 30	$bgjl^2$ − 18
$cfhki$ + 9	ch^2ij − 6	$bhjl^2$ + 18	bij^3 − 6	$bgjkl$ + 12	cfh^2i − 18	$bghjl$ − 30	$bijl^2$ + 12	cfj^2k + 24	bij^2l + 12
$cfkl^2$ + 18	$chjl^2$ + 18	bj^2kl − 12	cfh^2j + 24	$bhijl$ − 30	$cfhl^2$ + 12	bgj^2k + 3	cf^2h^2 + 12	$cghkl$ + 12	cfh^2l + 12
cik^2l − 12	cj^2kl − 12	ch^2k^2 − 3	cgh^2k + 3	bij^2k + 3	$cfjkl$ − 30	$bhij^2$ + 24	$cfhkl$ − 30	$cgjk^2$ − 18	$cfhjk$ − 30
f^3gj − 6	fg^2hj + 3	f^2h^2j + 12	ch^3i − 6	bjl^3 − 6	cgk^2l + 6	bj^2l^2 + 6	$cfjk^2$ + 24	ch^2il + 6	$cghk^2$ + 6
f^2g^2k + 12	fgj^2l + 6	fgh^2k + 3	ch^2l^2 + 6	cfh^2k + 3	$chikl$ + 12	cfh^3 − 6	cgk^3 − 6	$chijk$ + 6	ch^2ik + 6
f^2ghi + 3	fij^3 − 6	fh^3i − 6	$chjkl$ − 30	chk^2l + 6	$cijk^2$ + 3	ch^2kl + 6	$chik^2$ + 3	chl^3 − 6	$chkl^2$ − 18
f^2gl^2 + 6	g^3hk − 6	fh^2l^2 + 6	cj^2k^2 + 12	cjk^3 − 6	ckl^3 − 6	$chjk^2$ + 3	ck^2l^2 + 6	$cjkl^2$ + 12	cjk^2l + 12
f^2ijl + 6	g^2h^2i + 12	$fhjkl$ − 30	f^2j^3 + 12	f^2gh^2 − 27	f^2g^2h − 27	f^2hj^2 − 6	f^3j^2 + 12	f^2gj^2 − 6	f^2ghj + 18
$fgkil$ − 30	g^2hl^2 + 6	fj^2k^2 + 12	fg^2h^2 − 27	f^2j^2k − 6	f^2gjl + 18	fgh^2l + 18	f^2gjk − 12	fg^2hl + 18	f^2j^2l − 24
fhi^2l + 6	g^2jkl + 6	ghk^2l + 6	fgj^2k − 12	f^2hjl + 18	f^2ij^2 − 6	$fghjk$ − 3	f^2ghl + 18	fg^2jk − 12	fg^2hk + 18
fi^2jk + 3	$ghijl$ − 30	gjk^3 − 6	$fghjl$ + 36	$fghkl$ + 36	fg^2kl + 18	fh^2ij − 12	f^2hij − 12	$fghij$ − 3	fgh^2i + 18
fil^3 − 6	gij^2k + 3	h^2ikl + 6	$fhij^2$ − 12	$fgjk^2$ − 12	$fghil$ + 36	$fhjl^2$ − 6	f^2jl^2 − 24	$fgjl^2$ − 6	$fghl^2$ − 54
gk^2i^2 + 12	gjl^3 − 6	$hijk^2$ + 3	fj^2l^2 − 24	fh^2il + 18	$fgijk$ − 3	fj^2kl + 18	fg^2k^2 − 6	fij^2l + 18	$fgjkl$ − 12
hi^3k − 6	hi^2j^2 + 12	hkl^3 − 6	g^2hkl + 18	$fhijk$ − 3	fgl^3 − 18	g^2hk^2 − 6	$fghik$ − 3	g^3k^2 + 12	$fhijl$ − 12
i^2kl^2 + 6	ij^2l^2 + 6	jk^2l^2 + 6	g^2jk^2 − 6	fhl^3 − 18	fhi^2j − 12	gh^2ki − 12	$fgkl^2$ − 6	g^2hki − 12	fij^2k + 18
			gh^2il + 18	$fjkl^2$ − 6	$fijl^2$ − 6	$ghkl^2$ − 6	fh^2i^2 − 6	g^2kl^2 − 24	fjl^3 + 48
			$ghijk$ − 3	g^2k^3 + 12	g^2k^2i − 6	gjk^2l + 18	$fhil^2$ − 6	gh^2i^2 − 6	g^2k^2l − 24
			ghl^3 − 18	ghk^2i − 12	$ghki^2$ − 12	h^3i^2 + 12	$fijkl$ + 36	$ghil^2$ − 6	$ghkil$ − 12
			$gjki^2$ − 6	gk^2l^2 − 24	$gkil^2$ − 6	h^2il^2 − 24	fl^4 + 12	$gijkl$ + 36	$gijk^2$ + 18
			h^2i^2j − 6	h^2ki^2 − 6	h^2i^3 + 12	hl^4 + 12	gik^2l + 18	gl^4 + 12	gkl^3 + 48
			$hijl^2$ − 6	$hkil^2$ − 6	hi^2l^2 − 24	$hijkl$ + 36	hi^2kl + 18	hi^2jl + 18	h^2i^2l − 24
			ij^2kl + 18	ijk^2l + 18	i^2jkl + 18	ij^2k^2 − 27	i^2jk^2 − 27	i^2j^2k − 27	hil^3 + 48
			jl^4 + 12	kl^4 + 12	il^4 + 12	jkl^3 − 18	ikl^3 − 18	jkl^3 − 18	hi^2jk + 18
									$ijkl^2$ − 54
									l^5 − 24
±145	±145	±145	±282	±282	±282	±282	±282	±282	±486

$(\xi, \eta\ \zeta$

No. 67.

The reciprocant is $FU = (*\between \xi, \eta, \zeta)^6 =$

ξ^6.	η^6.	ζ^6.	$\eta^5\zeta$.	$\zeta^5\xi$.	$\xi^5\eta$.	$\eta\zeta^5$.	$\zeta\xi^5$.	$\xi\eta^5$.
b^2c^2 + 1	a^2c^2 + 1	a^2b^2 + 1	a^2ci − 6	ab^2j − 6	bc^2k − 6	a^2bf − 6	b^2cg − 6	ac^2h − 6
$bcfi$ − 6	$acgj$ − 6	$abhk$ − 6	$acgh$ + 6	$abfh$ + 6	$bcfg$ + 6	$abhl$ + 12	$bcfl$ + 12	$acgl$ + 12
bi^3 + 4	ag^3 + 4	ak^3 + 4	$acjl$ + 12	$abkl$ + 12	$bcil$ + 12	$abjk$ + 6	$bcik$ + 6	$acij$ + 6
cf^3 + 4	cj^3 + 4	bh^3 + 4	$agij$ + 18	afk^2 − 12	bgi^2 − 12	$afhk$ + 18	$bfgi$ + 18	ag^2i − 12
f^2i^2 − 3	g^2j^2 − 3	h^2k^2 − 3	ag^2l − 24	bh^2l − 24	cf^2l − 24	ak^2l − 24	bi^2l − 24	$cghj$ + 18
			chj^2 − 12	$bhjk$ + 18	$cfki$ + 18	bh^2j − 12	cf^2k − 12	cj^2l − 24
			g^2hj + 6	fh^2k + 6	f^2gi + 6	fh^3 − 12	f^3g − 12	g^3h − 12
			gj^2l + 12	hk^2l + 12	fi^2l + 12	h^2kl + 12	f^2il + 12	g^2jl + 12
			ij^3 − 12	jk^3 − 12	i^3k − 12	hjk^2 + 6	fi^2k + 6	gij^2 + 6
±9	±9	±9	±54	±54	±54	±54	±54	±54

$\eta^4\zeta^2$.	$\zeta^4\xi^2$.	$\xi^4\eta^2$.	$\eta^2\zeta^4$.	$\zeta^2\xi^4$.	$\xi^2\eta^4$.	$\eta^3\zeta^3$.	$\zeta^3\xi^3$.	$\xi^3\eta^3$.
a^2cf + 6	ab^2g + 6	bc^2h + 6	a^2bi + 6	b^2cj + 6	ac^2k + 6	a^2bc − 2	ab^2c − 2	abc^2 − 2
a^2i^2 + 9	$abfl$ − 12	$bcgl$ − 12	a^2f^2 + 9	b^2g^2 + 9	$acfg$ − 6	a^2fi − 18	$abfi$ + 6	$acfi$ + 6
$achl$ − 12	$abik$ − 6	$bcij$ − 6	$abgh$ − 6	$bcfh$ − 6	$acil$ − 12	$abgj$ + 6	af^3 − 4	ai^3 − 4
$acjk$ − 6	af^2k + 12	bg^2i + 12	$abjl$ − 12	$bckl$ − 12	agi^2 + 12	$achk$ + 6	b^2gj − 18	$bcgj$ + 6
$afgj$ − 18	b^2j^2 + 9	c^2k^2 + 9	$afhl$ − 36	$bfgl$ − 36	c^2h^2 + 9	$afgh$ + 18	$bchk$ + 6	bg^3 − 4
ag^2k + 12	$bfhj$ − 18	cf^2j + 12	$afjk$ − 18	$bfij$ − 18	cfj^2 + 12	$afjl$ + 36	$bfgh$ + 18	c^2hk − 18
$aghi$ − 18	$bghk$ − 18	$cfgk$ − 18	agk^2 + 12	$bgik$ − 18	$cghl$ − 36	$agkl$ − 48	$bfjl$ + 36	$cfgh$ + 18
agl^2 + 48	bh^2i + 12	$cfhi$ − 18	$aihk$ − 18	bhi^2 + 12	$cgjk$ − 18	$ahil$ + 36	$bgkl$ + 36	$cfjl$ − 48
$aijl$ − 36	bhl^2 + 48	cfl^2 + 48	akl^2 + 48	bil^2 + 48	$chij$ − 18	$aijk$ + 18	$bhil$ − 48	$cgkl$ + 36
ch^2j + 12	$bjkl$ − 36	$cikl$ − 36	bhj^2 + 12	cfk^2 + 12	cjl^2 + 48	al^3 − 32	$bijk$ + 18	$cijk$ + 18
fj^3 + 12	f^2h^2 − 3	f^2g^2 − 3	gh^2k − 6	f^3j + 12	fg^2j − 6	bj^3 − 4	bl^3 − 32	$chil$ + 36
g^2h^2 − 3	$fhkl$ − 24	$fgil$ − 24	fh^2j + 36	f^2gk + 36	g^3k + 12	ch^3 − 4	ck^3 − 4	cl^3 − 32
$ghjl$ − 24	fjk^2 + 36	fi^2j − 6	h^3i + 12	f^2hi − 6	g^2hi + 36	fhj^2 − 36	f^2hl + 12	fg^2l + 12
gj^2k − 6	gk^3 + 12	gi^2k + 36	h^2l^2 − 12	f^2l^2 − 12	g^2l^2 − 12	gh^2l + 12	f^2jk − 36	$fgij$ + 12
hij^2 + 36	hik^2 − 6	hi^3 + 12	$hjkl$ − 24	$fikl$ − 24	$gijl$ − 24	$ghjk$ + 12	fgk^2 − 36	g^2ki − 36
j^2l^2 − 12	k^2l^2 − 12	i^2l^2 − 12	j^2k^2 − 3	i^2k^2 − 3	i^2j^2 − 3	h^2ij − 36	$fhki$ + 12	ghi^2 − 36
						hjl^2 + 24	fkl^2 + 24	gil^2 + 24
						j^2kl + 12	ik^2l + 12	i^2jl + 12
±135	±135	±135	±135	±135	±135	±180	±180	±180

$\xi^4\eta\zeta$.	$\eta^4\zeta\xi$.	$\zeta^4\xi\eta$.	$\xi^3\eta^2\zeta$.	$\eta^3\zeta^2\xi$.	$\zeta^3\xi\eta^2$.	$\xi^3\eta\zeta^2$.	$\eta^3\zeta\xi^2$.	$\zeta^3\xi^2\eta$.	$\xi^2\eta^2\zeta^2$.
$bcfj$ -12	$acfj$ -12	$abgk$ -12	$abci$ $+6$	$abcj$ $+6$	$abck$ $+6$	$abcf$ $+6$	$abcg$ $+6$	$abch$ $+6$	$abcl$ -24
$bcgk$ $+30$	$acgk$ -12	$abhi$ -12	acf^2 -12	abg^2 -12	$abfg$ -24	abi^2 -12	$acfl$ $+36$	$abgl$ $+36$	$abgi$ $+6$
$bchi$ -12	$achi$ $+30$	$abij$ $+30$	afi^2 $+6$	$acfh$ -24	$abil$ $+36$	af^2i $+6$	$acik$ -24	$abij$ -24	$acfk$ $+6$
bcl^2 -24	acl^2 -24	abl^2 -24	$bcgh$ -24	$ackl$ $+36$	af^2l $+12$	$bchl$ $+36$	$afgi$ -30	ack^2 -12	af^2g $+30$
bfg^2 -18	afg^2 $+24$	af^2h -18	$bcjl$ $+36$	$afgl$ -60	$afik$ -30	$bcjk$ -24	ai^2l $+12$	af^2j -36	$afil$ -48
$bgil$ $+12$	$agil$ $+12$	$afkl$ $+12$	bg^2l $+12$	$afij$ $+54$	bch^2 -12	$bfgj$ $+54$	bcj^2 -12	$afgk$ $+12$	ai^2k $+30$
bi^2j $+24$	ai^2j -18	aik^2 $+24$	$bgij$ -30	ahi^2 -36	bfj^2 -36	bg^2k -36	bg^2j $+6$	$afhi$ $+54$	$bchj$ $+6$
cf^2h $+24$	cgh^2 -18	bgh^2 $+24$	$cfhl$ -60	$agik$ $+12$	$bghl$ -60	$bghi$ $+12$	$cfhj$ $+12$	afl^2 $+24$	bg^2h $+30$
$cfkl$ $+12$	$chjl$ $+12$	$bhjl$ $+12$	$cfjk$ $+12$	ail^2 $+24$	$bgjk$ $+54$	bgl^2 $+24$	$cghk$ $+54$	$aikl$ -60	$bgjl$ -48
cik^2 -18	cj^2k $+24$	bj^2k -18	cgk^2 -36	bgj^2 $+6$	$bhij$ $+12$	$bijl$ -60	ch^2i -36	$bghj$ -30	bij^2 $+30$
f^2gl $+60$	fgj^2 -12	fh^2l $+60$	$chik$ $+54$	ch^2l $+12$	bjl^2 $+24$	$cfhk$ -30	chl^2 $+24$	bj^2l $+12$	cfh^2 $+30$
f^2ij -12	g^2hl $+60$	$fhjk$ -66	ckl^2 $+24$	$chjk$ -30	chk^2 $+6$	ck^2l $+12$	$cjkl$ -60	ch^2k $+6$	$chkl$ -48
$fgik$ -66	g^2jk -12	ghk^2 -12	f^2gj -24	$fghj$ $+78$	f^2hj $+60$	f^2gh -66	fg^2h -66	fgh^2 -66	cjk^2 $+30$
fhi^2 -12	$ghij$ -66	h^2ki -12	fg^2k $+60$	fj^2l -48	$fghk$ $+78$	f^2jl -48	$fgjl$ $+72$	$fhjl$ -12	f^2j^2 $+24$
fil^2 -48	gjl^2 -48	hkl^2 -48	$fghi$ $+78$	g^2hk -24	fh^2i -24	$fgkl$ -12	fij^2 -24	fj^2k $+60$	$fghl$ $+108$
i^2kl $+60$	j^2il $+60$	jk^2l $+60$	fgl^2 -96	gh^2i $+60$	fhl^2 -96	$fhil$ $+72$	g^2kl -48	$ghkl$ $+72$	$fgjk$ -114
			$fijl$ $+72$	ghl^2 -96	$fjkl$ -12	$fijk$ $+78$	$ghil$ -12	gjk^2 -24	$fhij$ -114
			$gikl$ -12	$gjkl$ $+72$	gk^2l -48	fl^3 $+48$	$gijk$ $+78$	h^2il -48	fjl^2 $+24$
			hi^2l -48	$hijl$ -12	$hikl$ $+72$	gik^2 $+60$	gl^3 $+48$	$hijk$ $+78$	g^2k^2 $+24$
			i^2jk -66	ij^2k -66	ijk^2 -66	hi^2k -24	hi^2j $+60$	hl^3 $+48$	$ghki$ -114
			il^3 $+48$	jl^3 $+48$	kl^3 $+48$	ikl^2 -96	ijl^2 -96	jkl^2 -96	gkl^2 $+24$
									h^2i^2 $+24$
									hil^2 $+24$
									$ijkl$ $+108$
									l^4 -48
± 222	± 222	± 222	± 408	± 408	± 408	± 408	± 408	± 408	± 558

The preceding Tables contain the complete system [not so] of the covariants and contravariants of the ternary cubic, i.e. the covariants are the cubic itself U, the quartinvariant S, the sextinvariant T, the Hessian HU, and an octicovariant, say ΘU; the contravariants are the cubicontravariant PU, the quinticontravariant QU, and the reciprocant FU.

The contravariants are all of them evectants, viz. PU is the evectant of S, QU is the evectant of T, and the reciprocant FU is the evectant of QU, or what is the same thing, the second evectant of T.

The discriminant is a rational and integral function of the two invariants; representing it by R, we have $R = 64\,S^3 - T^2$.

If we combine U and HU by arbitrary multipliers, say α and 6β, so as to form the sum $\alpha U + 6\beta HU$, this is a cubic, and the question arises, to find the covariants and contravariants of this cubic: the results are given in the following Table:

No. 68.

$$\alpha U + 6\beta HU \quad = \alpha U + 6\beta HU.$$

$$H(\alpha U + 6\beta HU) = \quad (0,\ 2S,\quad T,\quad 8S^2 \between \alpha,\ \beta)^3\, U$$
$$\qquad\qquad + (1,\quad 0, -12S, -2T \between \alpha,\ \beta)^3\, HU.$$

$$P(\alpha U+6\beta HU)=\ (1,\ 0,\ 12S,\ 4T \between \alpha,\ \beta)^3\, PU$$
$$+(0,\ 1,\ 0,\ -4S \between \alpha,\ \beta)^3\, QU.$$
$$Q(\alpha U+6\beta HU)=\ (0,\ 60S,\ 30T,\ 0,\ -120TS,\ -24T^2+576S^3 \between \alpha,\ \beta)^5\, PU$$
$$+(1,\ 0,\ 0,\ 10T,\ 240S^2,\ 24TS \between \alpha,\ \beta)^5\, QU.$$
$$S(\alpha U+6\beta HU)=\ (S,\ T,\ 24S^2,\ 4TS,\ T^2-48S^3 \between \alpha,\ \beta)^4.$$
$$T(\alpha U+6\beta HU)=(T,\ 96S^2,\ 60TS,\ 20T^2,\ 240TS^2,\ -48T^2S+4608S^4,\ -8T^3+576TS^3 \between \alpha,\ \beta)^6.$$
$$R(\alpha U+6\beta HU)=[(1,\ 0,\ -24S,\ -8T,\ -48S^2 \between \alpha,\ \beta)^4]^3 R.$$
$$F(\alpha U+6\beta HU)=\ (1,\ 0,\ -24S,\ -8T,\ -48S^2 \between \alpha,\ \beta)^4\, FU$$
$$+(0,\ 24,\ 0,\ 0,\ -48T \between \alpha,\ \beta)^4\,(PU)^2$$
$$+(0,\ 0,\ 24,\ 0,\ 96S \between \alpha,\ \beta)^4\, PU.QU$$
$$+(0,\ 0,\ 0,\ 8,\ 0 \between \alpha,\ \beta)^4.(QU)^2.$$

We have, in like manner, for the covariants and contravariants of the cubic $6\alpha PU+\beta QU$, the following Table:

No. 69.

$$6\alpha PU+\beta QU \quad =6\alpha PU+\beta QU.$$
$$H(6\alpha PU+\beta QU)=(-2T,\ 48S^2,\ 18TS,\ T^2+16S^3 \between \alpha,\ \beta)^3\, PU$$
$$+(8S,\ T,\ -8S^2,\ -TS \between \alpha,\ \beta)^3\, QU.$$
$$P(6\alpha PU+\beta QU)=(32S^2,\ 12TS,\ T^2+32S^3,\ 4TS^2 \between \alpha,\ \beta)^3\, U$$
$$+(4T,\ 96S^2,\ 12TS,\ T^2-32S^3 \between \alpha,\ \beta)^3\, HU.$$

$$Q(6\alpha PU+\beta QU)=\left(\begin{aligned}&+384T\,S^2,\\ &+120T^2S+7680\ \ S^4,\\ &+\ \ 10T^3\ \ +3200T\,S^3,\\ &+480T^2S^2,\\ &+\ \ 30T^3S,\\ &+\ \ 1T^4\ -\ \ 24T^2S^3+512S^6\end{aligned}\ \between \alpha,\ \beta\right)^5 U$$

$$+\left(\begin{aligned}&-\ \ \ 24T^2\ \ +4608\ S^3,\\ &+1920T\,S^2,\\ &+\ \ 480T^2S,\\ &+\ \ \ 30T^3\ \ +1920TS^3,\\ &+\ 120T^2S^2+7680\ \ S^5,\\ &-\ \ \ \ 6T^3S\ +\ 768TS^4\end{aligned}\ \between \alpha,\ \beta\right)^5 HU.$$

$$S(6\alpha PU+\beta QU)=\left(\begin{array}{l} +\ 1T^2 + 192\ S^3, \\ +128TS^2, \\ +\ 18T^2S+384\ S^4, \\ +\ 1T^3+\ 64TS^3, \\ +\ 5T^2S^2-\ 64\ S^5 \end{array}\right)(\alpha,\ \beta)^4.$$

$$T(6\alpha PU+\beta QU)=\left(\begin{array}{l} -\ 8T^3 + 4608TS^3, \\ +1920T^2S^2+73728\ S^5, \\ +\ 360T^3S+38400TS^4, \\ +\ 20T^4+\ 8960T^2S^3, \\ +\ 840T^3S^2+\ 7680TS^5, \\ +\ 36T^4S+\ 384T^2S^4+24576\ S^7, \\ +\ 1T^5-\ 40T^3S^2+\ 2560TS^6 \end{array}\right)(\alpha,\ \beta)^6.$$

$$R(6\alpha PU+\beta QU)=[(48S,\ 8T,\ -96S^2,\ -24TS,\ -T^2-16S^3)(\alpha,\ \beta)^4]^3\ R^2.$$

$$\begin{aligned} F(6\alpha PU+\beta QU)=&(\ 192S,\ 32T,\ -384\ S^2,\ -\ 96TS,\ -\ 4T^2-64S^3)(\alpha,\ \beta)^4\ \Theta U \\ &+(\ 0,\ 512\ S^3,\ 192TS^2,\ 24T^2S,\ T^3)(\alpha,\ \beta)^4 .\ U^2 \\ &+(1344S^2,\ 352TS,\ 24T^2-1152S^3,\ -288TS^2,\ -\ 20T^2S+64S^4)(\alpha,\ \beta)^4\ U.HU \\ &+(\ 48T,\ 0,\ 288TS,\ 24T^2+1536S^3,\ 144TS^3)(\alpha,\ \beta)^4\ (HU)^2. \end{aligned}$$

The tables for the ternary cubic become much more simple if we suppose that the cubic is expressed in Hesse's canonical form; we have then the following table:

No. 70.

$$\begin{aligned} U &= x^3+y^3+z^3+6lxyz. \\ S &= -l+l^4. \\ T &= 1-20l^3-8l^6. \\ R &= -(1+8l^3)^3. \\ HU &= l^2(x^3+y^3+z^3)-(1+2l^3)\,xyz. \\ \Theta U &= (1+8l^3)^2\,(y^3z^3+z^3x^3+x^3y^3) \\ &\quad +(-9l^6)\,(x^3+y^3+z^3)^2 \\ &\quad +(-2l-5l^4-20l^7)\,(x^3+y^3+z^3)\,xyz \\ &\quad +(-15l^2-78l^5+12l^8)\,x^2y^2z^2. \\ \Theta_{\prime} U &= 4\,(1+8l^3)^2\,(y^3z^3+z^3x^3+x^3y^3) \\ &\quad +(-1-4l^3-4l^6)\,(x^3+y^3+z^3)^2 \\ &\quad +(4l+100l^4+112l^7)\,(x^3+y^3+z^3)\,xyz \\ &\quad +(48l^2+552l^5+48l^8)\,x^2y^2z^2. \end{aligned}$$

$$\begin{aligned}
\Theta_{\prime\prime}U &= -2\,(1+8l^3)^2\,(y^3z^3+z^3x^3+x^3y^3)\\
&\quad +(1-10l^3)\,(x^3+y^3+z^3)^2\\
&\quad +(6l-180l^4+96l^7)\,(x^3+y^3+z^3)\,xyz\\
&\quad +(6l^2-624l^5-192l^8)\,x^2y^2z^2.\\
PU &= -l\,(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\,\xi\eta\zeta.\\
QU &= (1-10l^3)\,(\xi^3+\eta^3+\zeta^3)-6l^2\,(5+4l^3)\,\xi\eta\zeta.\\
FU &= -4\,(1+8l^3)\,(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3)\\
&\quad +(\xi^3+\eta^3+\zeta^3)^2\\
&\quad -24l^2\,(\xi^3+\eta^3+\zeta^3)\,\xi\eta\zeta\\
&\quad -24l\,(1+2l^3)\,\xi^2\eta^2\zeta^2,
\end{aligned}$$

to which it is proper to join the following transformed expressions for ΘU, $\Theta_{\prime}U$, $\Theta_{\prime\prime}U$, viz.

$$\begin{aligned}
\Theta U &= (1+8l^3)^2\,(y^3z^3+z^3x^3+x^3y^3)\\
&\quad +(-2l^3-l^6)\,U^2\\
&\quad +(2l-5l^4)\,U\,.\,HU\\
&\quad +(-3l^2\quad)\,(HU).\\
\Theta_{\prime}U &= 4\,(1+8l^3)^2\,(y^3z^3+z^3x^3+x^3y^3)\\
&\quad +\ (-1+12l^3+4l^6)\,U^2\\
&\quad +\ (-16l+4l^4\qquad)\,U\,.\,HU\\
&\quad +\ (-12l^2\qquad)\,(HU)^2.\\
\Theta_{\prime\prime}U &= -2\,(1+8l^3)^2\,(y^3z^3+z^3x^3+x^3y^3)\\
&\quad +\ (1-16l^3-6l^6)\,U^2\\
&\quad +\ (6l\qquad)\,U\,.\,HU\\
&\quad +\ (6l^2\qquad)\,(HU)^2.
\end{aligned}$$

The last preceding table affords a complete solution of the problem to reduce a ternary cubic to its canonical form.

[I add to the present Memoir, in the notation hereof $(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3$ for the ternary cubic, some formulæ originally contained in the paper "On Homogeneous Functions of the third order with three variables," (1846), but which on account of the difference of notation were omitted from the reprint, 35, of that paper.

Representing the determinant

$$-\begin{vmatrix} ax+hy+jz, & hx+ky+bz, & jx+ly+gz, & \xi\\ hx+ky+bz, & kx+by+fz, & lx+fy+iz, & \eta\\ jx+ly+gz, & lx+fy+iz, & gx+iy+cz, & \zeta\\ \xi\quad, & \eta\quad, & \zeta & \end{vmatrix}$$

by
$$(A, B, C, F, G, H \between x, y, z)^2$$

the values of A, B, C, F, G, H (equations (10) of 35) are

	A	B	C	F	G	H
ξ^2	$gk - l^2$	$bi - f^2$	$cf - i^2$	$bc - fi$	$fg + ck - 2il$	$ki + bg - 2fl$
η^2	$ag - j^2$	$hi - l^2$	$cj - g^2$	$ij + ch - 2gl$	$ca - gj$	$gh + ai - 2jl$
ζ^2	$ak - h^2$	$bh - k^2$	$fj - l^2$	$hf + bj - 2kl$	$jk + af - 2hl$	$ab - hk$
$\eta\zeta$	$2hj - 2al$	$2kl - 2hf$	$2gl - 2ij$	$l^2 + gk - hi - fj$	$gh - ai$	$jk - af$
$\zeta\xi$	$2hl - 2jk$	$2fk - 2bl$	$2il - 2fg$	$ki - bg$	$l^2 + hi - fj - gk$	$hf - bj$
$\xi\eta$	$2jl - 2gh$	$2fl - 2ki$	$2gi - 2cl$	$fg - ck$	$ij - ch$	$l^2 + fj - gk - hi$

Moreover writing

$$FU = \left|\begin{array}{ccccccc} a, & k, & g, & l, & j, & h, & \xi \\ h, & b, & i, & f, & l, & k, & \eta \\ j, & f, & c, & i, & g, & l, & \zeta \\ 2\xi & . & . & . & \zeta & \eta & . \\ . & 2\eta & . & \zeta & . & \xi & . \\ . & . & 2\zeta & \eta & \xi & . & . \\ A, & B, & C, & F, & G, & H & . \end{array}\right|,$$

so that

$$FU = A\mathrm{a} + B\mathrm{b} + C\mathrm{c} + 2F\mathrm{f} + 2G\mathrm{g} + 2H\mathrm{h},$$

then the values of a, b, c, f, g, h (equations (13) of 35) are

	a	b	c	f	g	h
ξ^4	0	$2cf - 2i^2$	$2bi - 2f^2$	$fi - bc$	0	0
η^4	$2cj - 2g^2$		$2ag - 2j^2$	0	$gj - ca$	0
ζ^4	$2bh - 2k^2$	$2ak - 2h^2$		0	0	$hk - ab$
$\eta^3\zeta$	$8gl - 6ij - 2ch$	0	$4hj - 4al$	$2j^2 - 2ag$	$3ai - 2jl - gh$	$ca - gj$
$\zeta^3\xi$	$4fk - 4bl$	$8hl - 6jk - 2af$	0	$ab - hk$	$2k^2 - 2bh$	$3bj - 2kl - hf$
$\xi\eta^3$	0	$4gi - 4cl$	$8fl - 6ki - 2bg$	$3ck - 2il - fg$	$bc - fi$	$2i^2 - 2cf$
$\eta\zeta^3$	$8kl - 6hf - 2bj$	$4hj - 4al$	0	$2h^2 - 2ak$	$ab - hk$	$3af - 2hl - jk$
$\zeta\xi^3$	0	$8il - 6fg - 2ck$	$4fk - 4bl$	$3bg - 2fl - ki$	$2f^2 - 2bi$	$bc - fi$
$\xi\eta^3$	$4gi - 4cl$	0	$8jl - 6gh - 2ai$	$ca - gj$	$3ch - 2gl - ij$	$2g^2 - 2cj$
$\eta^2\zeta^2$	$6hi + 6fj - 4gk - 8l^2$	$2ag - 2j^2$	$2ak - 2h^2$	$4al - 4hj$	$jk + 2hl - 3af$	$gh + 2jl - 3ai$
$\zeta^2\xi^2$	$2bi - 2f^2$	$6fj + 6gk - 4hi - 8l^2$	$2bh - 2k^2$	$hf + 2kl - 3bj$	$4bl - 4fk$	$ki + 2fl - 3bg$
$\xi^2\eta^2$	$2cf - 2i^2$	$2cj - 2g^2$	$6gk + 6hi - 4fj - 8l^2$	$ij + 2gl - 3ch$	$fg + 2il - 3ck$	$4cl - 4gi$
$\xi^2\eta\zeta$	$2fi - 2bc$	$4ch + 4gl - 8ij$	$4bj + 4kl - 8hf$	$4l^2 + 2hi + 2fj - 8gk$	$7ki - 6fl - bg$	$7fg - 6il - ck$
$\xi\eta^2\zeta$	$4ck + 4il - 8fg$	$2gj - 2ca$	$4af + 4hl - 8jk$	$7gh - 6jl - ai$	$4l^2 + 2fj + 2gk - 8hi$	$7ij - 6gl - ch$
$\xi\eta\zeta^2$	$4bg + 4fl - 8ki$	$4ai + 4jl + 8gh$	$2hk - 2ab$	$7jk - 6hl - af$	$7hf - 6kl - bj$	$4l^2 + 2gk + 2hi - 8fj$

Also if the discriminant be written

$$K(U)=\left|\begin{matrix} a & k & g & l & j & h \\ h & b & i & f & l & k \\ j & i & c & i & g & l \\ \mathfrak{A} & \mathfrak{K} & \mathfrak{G} & \mathfrak{L} & \mathfrak{J} & \mathfrak{H} \\ \mathfrak{H} & \mathfrak{B} & \mathfrak{I} & \mathfrak{F} & \mathfrak{L} & \mathfrak{K} \\ \mathfrak{J} & \mathfrak{I} & \mathfrak{C} & \mathfrak{I} & \mathfrak{G} & \mathfrak{L} \end{matrix}\right|,$$

then the values of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{I}$, $\mathfrak{J}$, $\mathfrak{K}$, $\mathfrak{L}$ (equations (20) of 35) are

$$\begin{aligned}
\mathfrak{A} &= akg + 2hjl - al^2 - gh^2 - j^2k,\\
\mathfrak{B} &= bih + 2fkl - bl^2 - hf^2 - k^2i,\\
\mathfrak{C} &= cjf + 2gil - cl^2 - fg^2 - i^2j,\\
3\mathfrak{F} &= bch + bij - ck^2 + 2gfk - 2bgl + fl^2 - f^2j - fih,\\
3\mathfrak{G} &= caf + cjk - ai^2 + 2hgi - 2chl + gl^2 - g^2k - gjf,\\
3\mathfrak{H} &= abg + aki - bj^2 + 2fhj - 2afl + hl^2 - h^2i - hkg,\\
3\mathfrak{I} &= bcj + cfh - bg^2 + 2kig - 2ckl + jl^2 - i^2h - fij,\\
3\mathfrak{J} &= cak + agf - ch^2 + 2ijh - 2ail + kl^2 - j^2f - gjk,\\
3\mathfrak{K} &= abi + bhg - af^2 + 2jkf - 2bjl + il^2 - k^2g - hki,\\
6\mathfrak{L} &= abc + 3fgh + 3ijk + 2l^3 - afi - bgj - chk - 2lgk - 2lhi - 2lfj.
\end{aligned}$$

The equation $K(U)=R=64S^3-T^2$ would however afford a perhaps easier formula for the calculation of the discriminant.]