# 149.

## ON THE SYMMETRIC FUNCTIONS OF THE ROOTS OF CERTAIN SYSTEMS OF TWO EQUATIONS.

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SUPPOSE in general that $\phi = 0$, $\psi = 0$, &c. denote a system of $(n-1)$ equations between the $n$ variables $(x, y, z, \ldots)$, where the functions $\phi$, $\psi$, &c. are quantics (i.e. rational and integral homogeneous functions) of the variables. Any values $(x_1, y_1, z_1, \ldots)$ satisfying the equations, are said to constitute a set of roots of the system; the roots of the same set are, it is clear, only determinate to a common factor *près*, i.e. only the ratios *inter se* and not the absolute magnitudes of the roots of a set are determinate. The number of sets, or the degree of the system, is equal to the product of the degrees of the component equations. Imagine a function of the roots which remains unaltered when any two sets $(x_1, y_1, z_1, \ldots)$ and $(x_2, y_2, z_2, \ldots)$ are interchanged (that is, when $x_1$ and $x_2$, $y_1$ and $y_2$, &c. are simultaneously interchanged), and which is besides homogeneous of the same degree as regards each entire set of roots, although not of necessity homogeneous as regards the different roots of the same set; thus, for example, if the sets are $(x_1, y_1)$, $(x_2, y_2)$, then the functions $x_1x_2$, $x_1y_2 + x_2y_1$, $y_1y_2$ are each of them of the form in question; but the first and third of these functions, although homogeneous of the first degree in regard to each entire set, are not homogeneous as regards the two variables of each set. A function of the above-mentioned form may, for shortness, be termed a symmetric function of the roots; such function (disregarding an arbitrary factor depending on the common factors which enter implicitly into the different sets of roots) will be a rational and integral function of the coefficients of the equations, i.e. any symmetric function of the roots may be considered as a rational and integral function of the coefficients. The general process for the investigation of such expression for a symmetric function of the roots is indicated in Professor Schläfli's Memoir, "Ueber die Resultante eines Systemes mehrerer algebraischer

Gleichungen," *Vienna Transactions*, t. IV. (1852). The process is as follows:—Suppose that we know the resultant of a system of equations, one or more of them being linear; then if $\phi = 0$ be the linear equation or one of the linear equations of the system, the resultant will be of the form $\phi_1\phi_2 \ldots$, where $\phi_1$, $\phi_2$, &c. are what the function $\phi$ becomes upon substituting therein the different sets $(x_1, y_1, z_1 \ldots)$, $(x_2, y_2, z_2 \ldots)$ of the remaining $(n-1)$ equations $\psi = 0$, $\chi = 0$, &c.; comparing such expression with the given value of the resultant, we have expressed in terms of the coefficients of the functions $\psi$, $\chi$, &c., certain symmetric functions which may be called the fundamental symmetric functions of the roots of the system $\psi = 0$, $\chi = 0$, &c.; these are in fact the symmetric functions of the first degree in respect to each set of roots. By the aid of these fundamental symmetric functions, the other symmetric functions of the roots of the system $\psi = 0$, $\chi = 0$, &c. may be expressed in terms of the coefficients, and then combining with these equations a non-linear equation $\Phi = 0$, the resultant of the system $\Phi = 0$, $\psi = 0$, $\chi = 0$, &c. will be what the function $\Phi_1\Phi_2 \ldots$ becomes, upon substituting therein for the different symmetric functions of the roots of the system $\psi = 0$, $\chi = 0$, &c. the expressions for these functions in terms of the coefficients. We thus pass from the resultant of a system $\phi = 0$, $\psi = 0$, $\chi = 0$, &c., to that of a system $\Phi = 0$, $\psi = 0$, $\chi = 0$, &c., in which the linear function $\phi$ is replaced by the non-linear function $\Phi$. By what has preceded, the symmetric functions of the roots of a system of $(n-1)$ equations depend on the resultant of the system obtained by combining the $(n-1)$ equations with an arbitrary linear equation; and moreover, the resultant of any system of $n$ equations depends ultimately upon the resultant of a system of the same number of equations, all except one being linear; but in this case the linear equations determine the ratios of the variables or (disregarding a common factor) the values of the variables, and by substituting these values in the remaining equation we have the resultant of the system. The process leads, therefore, to the expressions for the symmetric functions of the roots of any system of $(n-1)$ equations, and also to the expression for the resultant of any system of $n$ equations. Professor Schläfli discusses in the general case the problem of showing how the expressions for the fundamental symmetric functions lead to those of the other symmetric functions, but it is not necessary to speak further of this portion of his investigations. The object of the present Memoir is to apply the process to two particular cases, viz. I propose to obtain thereby the expressions for the simplest symmetric functions (after the fundamental ones) of the following systems of two ternary equations; that is, first, a linear equation and a quadric equation; and secondly, a linear equation and a cubic equation.

First, consider the two equations

$$(a, b, c, f, g, h \between x, y, z)^2 = 0,$$

$$(\alpha, \beta, \gamma \between x, y, z) = 0,$$

and join to these the arbitrary linear equation

$$(\xi, \eta, \zeta \between x, y, z) = 0,$$

then the two linear equations give

$$x : y : z = \beta\zeta - \gamma\eta : \gamma\xi - \alpha\zeta : \alpha\eta - \beta\xi ;$$

and substituting in the quadratic equation, we have for the resultant of the three equations,

$$(a, b, c, f, g, h \between \beta\zeta - \gamma\eta,\ \gamma\xi - \alpha\zeta,\ \alpha\eta - \beta\xi)^2 = 0,$$

which may be represented by

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h} \between \xi, \eta, \zeta)^2 = 0,$$

where the coefficients are given by means of the Table.

| | $a$ | $b$ | $c$ | $f$ | $g$ | $h$ | |
|---|---|---|---|---|---|---|---|
| a = | | $+\gamma^2$ | $+\beta^2$ | $-2\beta\gamma$ | | | $(\xi^2)$ |
| b = | $+\gamma^2$ | | $+\alpha^2$ | | $-2\gamma\alpha$ | | $(\eta^2)$ |
| c = | $+\beta^2$ | $+\alpha^2$ | | | | $-2\alpha\beta$ | $(\zeta^2)$ |
| f = | $-\beta\gamma$ | | | $-\alpha^2$ | $+\alpha\beta$ | $+\gamma\alpha$ | $2(\eta\zeta)$ |
| g = | | $-\gamma\alpha$ | | $+\alpha\beta$ | $-\beta^2$ | $+\beta\gamma$ | $2(\zeta\xi)$ |
| h = | | | $-\alpha\beta$ | $+\gamma\alpha$ | $+\beta\gamma$ | $-\gamma^2$ | $2(\xi\eta)$ |

viz. $\mathrm{a} = b\gamma^2 + c\beta^2 - 2f\beta\gamma$, &c.

But if the roots of the given system are

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2),$$

then the resultant of the three equations will be

$$(x_1, y_1, z_1 \between \xi, \eta, \zeta) \cdot (x_2, y_2, z_2 \between \xi, \eta, \zeta) = 0 ;$$

and comparing the two expressions, we have

$$\mathrm{a} = x_1 x_2,$$

$$\mathrm{b} = y_1 y_2,$$

$$\mathrm{c} = z_1 z_2,$$

$$2\mathrm{f} = y_1 z_2 + y_2 z_1,$$

$$2\mathrm{g} = z_1 x_2 + z_2 x_1,$$

$$2\mathrm{h} = x_1 y_2 + x_2 y_1,$$

which are the expressions for the six fundamental symmetric functions, or symmetric functions of the first degree in each set, of the roots of the given system.

By forming the powers and products of the second order $\mathrm{a}^2$, ab, &c., we obtain linear relations between the symmetric functions of the second degree in respect to each set of roots. The number of equations is precisely equal to that of the symmetric functions of the form in question, and the solution of the linear equations gives—

$$\begin{aligned}
a^2 &= x_1^2 x_2^2, \\
b^2 &= y_1^2 y_2^2, \\
c^2 &= z_1^2 z_2^2, \\[6pt]
bc &= y_1 z_1 y_2 z_2, \\
ca &= z_1 x_1 z_2 x_2, \\
ab &= x_1 y_1 x_2 y_2, \\[6pt]
4f^2 - 2bc &= y_1^2 z_2^2 + y_2^2 z_1^2, \\
4g^2 - 2ca &= z_1^2 x_2^2 + z_2^2 x_1^2, \\
4h^2 - 2ab &= x_1^2 y_2^2 + x_2^2 y_1^2, \\[6pt]
2af &= x_1 y_1 z_2 x_2 + z_1 x_1 x_2 y_2, \\
2bg &= y_1 z_1 x_2 y_2 + x_1 y_1 y_2 z_2, \\
2ch &= z_1 x_1 y_2 z_2 + y_1 z_1 z_2 x_2, \\[6pt]
4gh - 2af &= x_1^2 y_2 z_2 + x_2^2 y_1 z_1, \\
4hf - 2bg &= y_1^2 z_2 x_2 + y_2^2 z_1 x_1, \\
4fg - 2ch &= z_1^2 x_2 y_2 + z_2^2 x_1 y_1, \\[6pt]
2bf &= y_1^2 y_2 z_2 + y_2^2 y_1 z_1, \\
2cg &= z_1^2 z_2 x_2 + z_2^2 z_1 x_1, \\
2ah &= x_1^2 x_2 y_2 + x_2^2 x_1 y_1, \\[6pt]
2cf &= z_1^2 y_2 z_2 + z_2^2 y_1 z_1, \\
2ag &= x_1^2 z_2 x_2 + x_2^2 z_1 x_1, \\
2bh &= y_1^2 x_2 y_2 + y_2^2 x_1 y_1.
\end{aligned}$$

Proceeding next to the powers and products of the third order $a^3$, $a^2b$, &c., the total number of linear relations between the symmetric functions of the third degree in respect to each set of roots exceeds by unity the number of the symmetric functions of the form in question; in fact the expressions for abc, $af^2$, $bg^2$, $ch^2$, fgh, contain, not five, but only four symmetric functions of the roots; for we have

$$\begin{aligned}
abc &= x_1 y_1 z_1 \,.\, x_2 y_2 z_2, \\
4af^2 &= (x_1 y_1^2 x_2 z_2^2 + x_2 y_2^2 x_1 z_1^2) + 2x_1 y_1 z_1 x_2 y_2 z_2, \\
4bg^2 &= (y_1 z_1^2 y_2 x_2^2 + y_2 z_2^2 y_1 x_1^2) + 2x_1 y_1 z_1 x_2 y_2 z_2, \\
4ch^2 &= (z_1 x_1^2 z_2 y_2^2 + z_2 x_2^2 z_1 y_1^2) + 2x_1 y_1 z_1 x_2 y_2 z_2, \\
8fgh &= \left.\begin{matrix} (x_1 y_1^2 x_2 z_2^2 + x_2 y_2^2 x_1 z_1^2) \\ + (y_1 z_1^2 y_2 z_2^2 + y_2^2 z_2^2 y_1 x_1^2) \\ + (z_1 x_1^2 z_2 x_2^2 + z_2 x_2^2 z_1 y_1^2) \end{matrix}\right\} + 2x_1 y_1 z_1 x_2 y_2 z_2,
\end{aligned}$$

and consequently the quantities a, b, c, f, g, h, are not independent, but are connected by the equation

$$abc - af^2 - bg^2 - ch^2 + 2fgh = 0,$$

an equation, which is in fact verified by the foregoing values of a, &c. in terms of the coefficients of the given system.

The expressions for the symmetric functions of the third degree considered as functions of a, b, c, f, g, h, are consequently not absolutely determinate, but they may be modified by the addition of the term $\lambda(abc - af^2 - bg^2 - ch^2 + 2fgh)$, where $\lambda$ is an indeterminate numerical coefficient.

The simplest expressions are those obtained by disregarding the preceding equation for fgh, and the entire system then becomes:

$$\begin{aligned}
a^3 &= x_1^3x_2^3,\\
b^3 &= y_1^3y_2^3,\\
c^3 &= z_1^3z_2^3,\\
b^2c &= y_1^2z_1y_2^2z_2,\\
c^2a &= z_1^2x_1z_2^2x_2,\\
a^2b &= x_1^2y_1x_2^2y_2,\\
bc^2 &= y_1z_1^2y_2z_2^2,\\
ca^2 &= z_1x_1^2z_2x_2^2,\\
ab^2 &= x_1y_1^2x_2y_2^2,\\
abc &= x_1y_1z_1x_2y_2z_2,\\
2a^2f &= x_1^2y_1z_2x_2^2 + x_2^2y_2z_1x_1^2,\\
2b^2g &= y_1^2z_1x_2y_2^2 + y_2^2z_2x_1y_1^2,\\
2c^2h &= z_1^2x_1y_2z_2^2 + z_2^2x_2y_1z_1^2,\\
2a^2g &= x_1^3z_2x_2^2 + x_2^3z_1x_1^2,\\
2b^2h &= y_1^3x_2y_2^2 + y_2^3x_1y_1^2,\\
2c^2f &= z_1^3y_2z_2^2 + z_2^3y_1z_1^2,\\
2a^2h &= x_1^3z_2^2x_2 + x_2^3z_1^2x_1,\\
2b^2f &= y_1^3x_2^2y_2 + y_2^3x_1^2y_1,\\
2c^2g &= z_1^3y_2^2z_2 + z_2^3y_1^2z_1,\\
2bcf &= y_1^2z_1y_2z_2^2 + y_2^2z_2y_1z_1^2,\\
2cag &= z_1^2x_1z_2x_2^2 + z_2^2x_2z_1x_1^2,\\
2abh &= x_1^2y_1x_2y_2^2 + x_2^2y_2x_1y_1^2,\\
2bcg &= y_1z_1^2x_2y_2z_2 + y_2z_2^2x_1y_1z_1,\\
2cah &= z_1x_1^2x_2y_2z_2 + z_2x_2^2x_1y_1z_1,\\
2abf &= x_1y_1^2x_2y_2z_2 + x_2y_2^2x_1y_1z_1,
\end{aligned}$$

$$\begin{aligned}
2\mathrm{bch} &= y_1^2z_1x_2y_2z_2 + y_2^2z_2x_1y_1z_1,\\
2\mathrm{caf} &= z_1^2x_1x_2y_2z_2 + z_2^2x_2x_1y_1z_1,\\
2\mathrm{abg} &= x_1^2y_1x_2y_2z_2 + x_2^2y_2x_1y_1z_1,\\[1ex]
4\mathrm{af}^2 - 2\mathrm{abc} &= x_1y_1^2z_2^2x_2 + x_2y_2^2z_1^2x_1,\\
4\mathrm{bg}^2 - 2\mathrm{abc} &= y_1z_1^2x_2^2y_2 + y^2z_2^2x_1^2y_1,\\
4\mathrm{ch}^2 - 2\mathrm{abc} &= z_1x_1^2y_2^2z_2 + z_2x_2^2y_1^2z_1,\\
4\mathrm{bf}^2 - 2\mathrm{b}^2\mathrm{c} &= y_1^3y_2z_2^2 + y_2^3y_1z_1^2,\\
4\mathrm{cg}^2 - 2\mathrm{c}^2\mathrm{a} &= z_1^3z_2x_2^2 + z_2^3z_1x_1^2,\\
4\mathrm{ah}^2 - 2\mathrm{a}^2\mathrm{b} &= x_1^3x_2y_2^2 + x_2^3x_1y_1^2,\\
4\mathrm{cf}^2 - 2\mathrm{bc}^2 &= z_1^3y_2^2z_2 + z_2^3y_1^2z_1,\\
4\mathrm{ag}^2 - 2\mathrm{ca}^2 &= x_1^3z_2^2x_2 + x_2^3z_1^2x_1,\\
4\mathrm{bh}^2 - 2\mathrm{ab}^2 &= y_1^3x_2^2y_2 + y_2^3x_1^2y_1,\\
4\mathrm{agh} - 2\mathrm{a}^2\mathrm{f} &= x_1^3x_2y_2z_2 + x_2^3x_1y_1z_1,\\
4\mathrm{bhf} - 2\mathrm{b}^2\mathrm{g} &= y_1^3x_2y_2z_2 + y_2^3x_1y_1z_1,\\
4\mathrm{cfg} - 2\mathrm{c}^2\mathrm{h} &= z_1^3x_2y_2z_2 + z_2^3x_1y_1z_1,\\
4\mathrm{bgh} - 2\mathrm{abf} &= y_1^2z_1x_2^2y_2 + y_2^2z_2x_1^2y_1,\\
4\mathrm{chf} - 2\mathrm{bcg} &= z_1^2x_1y_2^2z_2 + z_2^2x_2y_1^2z_1,\\
4\mathrm{afg} - 2\mathrm{cah} &= x_1^2y_1z_2^2x_2 + x_2^2y_2z_1^2x_1,\\
4\mathrm{cgh} - 2\mathrm{acf} &= y_1z_1^2x_2^2z_2 + y_2z_2^2x_1^2z_1,\\
4\mathrm{ahf} - 2\mathrm{bag} &= z_1x_1^2y_2^2x_2 + z_2x_2^2y_1^2x_1,\\
4\mathrm{bfg} - 2\mathrm{cbh} &= x_1y_1^2z_2^2y_2 + x_2y_2^2z_1^2y_1,\\[1ex]
8\mathrm{f}^2\mathrm{g} - 4\mathrm{chf} - 2\mathrm{bcg} &= z_1^3x_2y_2^2 + z_2^3x_1y_1^2,\\
8\mathrm{g}^2\mathrm{h} - 4\mathrm{afg} - 2\mathrm{cah} &= x_1^3y_2z_2^2 + x_2^3y_1z_1^2,\\
8\mathrm{h}^2\mathrm{f} - 4\mathrm{bgh} - 2\mathrm{abf} &= y_1^3z_2x_2^2 + y_2^3z_1x_1^2,\\[1ex]
8\mathrm{fg}^2 - 4\mathrm{chg} - 2\mathrm{acf} &= z_1^3x_2^2y_2 + z_2^3x_1^2y_1,\\
8\mathrm{gh}^2 - 4\mathrm{afh} - 2\mathrm{bag} &= x_1^3y_2^2z_2 + x_2^3y_1^2z_1,\\
8\mathrm{hf}^2 - 4\mathrm{bgf} - 2\mathrm{cbh} &= y_1^3z_2^2x_2 + y_2^3z_1^2x_1,\\
8\mathrm{f}^3 - 6\mathrm{bcf} &= y_1^3z_2^3 + y_2^3z_1^3,\\
8\mathrm{g}^3 - 6\mathrm{cag} &= z_1^3x_2^3 + z_2^3x_1^3,\\
8\mathrm{h}^3 - 6\mathrm{abh} &= x_1^3y_2^3 + x_2^3y_1^3.
\end{aligned}$$

Secondly, consider the system of equations

$$\begin{aligned}
(a, b, c, f, g, h, i, j, k, l \between x, y, z)^3 &= 0,\\
(\alpha, \beta, \gamma \between x, y, z) &= 0,
\end{aligned}$$

where the cubic function written at full length is

$$ax^3 + by^3 + cz^3 + 3fy^2z + 3gz^2x + 3hx^2y + 3iyz^2 + 3jzx^2 + 3kxy^2 + 6lxyz.$$

Joining to the system the linear equation

$$(\xi,\ \eta,\ \zeta \between x,\ y,\ z) = 0,$$

the linear equations give

$$x : y : z = \beta\zeta - \gamma\eta : \gamma\xi - \alpha\zeta : \alpha\eta - \beta\xi,$$

and the resultant is

$$(a,\ b,\ c,\ f,\ g,\ h,\ i,\ j,\ k,\ l \between \beta\zeta - \gamma\eta,\ \gamma\xi - \alpha\zeta,\ \alpha\eta - \beta\xi)^3 = 0,$$

which may be represented by

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h},\ \mathrm{i},\ \mathrm{j},\ \mathrm{k},\ \mathrm{l} \between \xi,\ \eta,\ \zeta)^3 = 0,$$

where the coefficients a, b, &c. are given by means of the Table :—

| | $a$ | $b$ | $c$ | $f$ | $g$ | $h$ | $i$ | $j$ | $k$ | $l$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| a = | | $+\gamma^3$ | $-\beta^3$ | $-3\beta\gamma^2$ | | | $+3\beta^2\gamma$ | | | | $\xi^3$ |
| b = | $-\gamma^3$ | | $+\alpha^3$ | | $-3\gamma\alpha^2$ | | | $+3\gamma^2\alpha$ | | | $\eta^3$ |
| c = | $+\beta^3$ | $-\alpha^3$ | | | | $-3\alpha\beta^2$ | | | $+3\alpha^2\beta$ | | $\zeta^3$ |
| f = | $+\beta\gamma^2$ | | | | $+\alpha^2\beta$ | $-\gamma^2\alpha$ | $-\alpha^3$ | $-2\alpha\beta\gamma$ | | $+2\gamma\alpha^2$ | $3\eta^2\zeta$ |
| g = | | $+\gamma\alpha^2$ | | $-\alpha^2\beta$ | | $+\beta^2\gamma$ | | $-\beta^3$ | $-2\alpha\beta\gamma$ | $+2\alpha\beta^2$ | $3\zeta^2\xi$ |
| h = | | | $+\alpha\beta^2$ | $+\gamma^2\alpha$ | $-\beta^2\gamma$ | | $-2\alpha\beta\gamma$ | | $-\gamma^3$ | $+2\beta\gamma^2$ | $3\xi^2\eta$ |
| i = | $-\beta^2\gamma$ | | | $+\alpha^3$ | | $+2\alpha\beta\gamma$ | | $+\alpha\beta^2$ | $-\gamma\alpha^2$ | $-2\alpha^2\beta$ | $3\eta\zeta^2$ |
| j = | | $-\gamma^2\alpha$ | | $+2\alpha\beta\gamma$ | $+\beta^3$ | | $-\alpha\beta^2$ | | $+\beta\gamma^2$ | $-2\beta^2\gamma$ | $3\zeta\xi^2$ |
| k = | | | $-\alpha^2b$ | | $+2\alpha\beta\gamma$ | $+\gamma^3$ | $+\gamma\alpha^2$ | $-\beta\gamma^2$ | | $-2\gamma^2\alpha$ | $3\xi^2\eta$ |
| l = | | | | $-\gamma\alpha^2$ | $-\alpha\beta^2$ | $-\beta\gamma^2$ | $+\alpha^2\beta$ | $+\beta^2\gamma$ | $+\gamma^2\alpha$ | | $6\xi\eta\zeta$ |

viz. $\mathrm{a} = b\gamma^3 - c\beta^3 - 3f\beta\gamma^2 + 3i\beta^2\gamma$, &c.

But if the roots of the given system are

$$(x_1,\ y_1,\ z_1),\quad (x_2,\ y_2,\ z_2),\quad (x_3,\ y_3,\ z_3),$$

then the resultant of the three equations may also be represented by

$$(x_1,\ y_1,\ z_1 \between \xi,\ \eta,\ \zeta)\cdot(x_2,\ y_2,\ z_2 \between \xi,\ \eta,\ \zeta)\cdot(x_3,\ y_3,\ z_3 \between \xi,\ \eta,\ \zeta);$$

and comparing with the former expression, we find :

$$\mathrm{a} = x_1x_2x_3,$$
$$\mathrm{b} = y_1y_2y_3,$$
$$\mathrm{c} = z_1z_2z_3,$$

$$3\mathrm{f} = y_1y_2z_3 + y_2y_3z_1 + y_3y_1z_2,$$
$$3\mathrm{g} = z_1z_2x_3 + z_2z_3x_1 + z_3z_1x_2,$$
$$3\mathrm{h} = x_1x_2y_3 + x_2x_3y_1 + x_3x_1y_2,$$
$$3\mathrm{i} = y_1z_2z_3 + y_2z_3z_1 + y_3z_1z_2,$$
$$3\mathrm{j} = z_1x_2x_3 + z_2x_3x_1 + z_3x_1x_2,$$
$$3\mathrm{k} = x_1y_2y_3 + x_2y_3y_1 + x_3y_1y_2,$$
$$6\mathrm{l} = x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 + x_1y_3z_2 + x_2y_1z_3 + x_3y_2z_1.$$

But there is in the present case a relation independent of the quantities a, &c., viz. we have $(\alpha, \beta, \gamma \between x_1, y_1, z_1) = 0$, $(\alpha, \beta, \gamma \between x_2, y_2, z_2) = 0$, $(\alpha, \beta, \gamma \between x_3, y_3, z_3) = 0$, and thence eliminating the coefficients $(\alpha, \beta, \gamma)$, we find

$$\nabla = x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1 = 0.$$

By forming the powers and products of the second degree $\mathrm{a}^2$, ab, &c., we obtain 55 equations between the symmetric functions of the second degree in each set of roots. But we have $\nabla^2 = 0 =$ a symmetric function of the roots, and thus the entire number of linear relations is 56, and this is in fact the number of the symmetric functions of the second degree in each set. I use for shortness the sign $S$ to denote the sum of the distinct terms obtained by permuting the different sets of roots, so that the equations for the fundamental symmetric functions are—

$$\mathrm{a} = x_1x_2x_3,$$
$$\mathrm{b} = y_1y_2y_3,$$
$$\mathrm{c} = z_1z_2z_3,$$
$$3\mathrm{f} = S\, y_1y_2z_3,$$
$$3\mathrm{g} = S\, z_1z_2x_3,$$
$$3\mathrm{h} = S\, x_1x_2y_3,$$
$$3\mathrm{i} = S\, y_1z_2z_3,$$
$$3\mathrm{j} = S\, z_1x_2x_3,$$
$$3\mathrm{k} = S\, x_1y_2y_3,$$
$$6\mathrm{l} = S\, x_1y_2z_3;$$

then the complete system of expressions for the symmetric functions of the second order is as follows, viz.

$$\mathrm{a}^2 = x_1^2x_2^2x_3^2,$$
$$\mathrm{b}^2 = y_1^2y_2^2y_3^2,$$
$$\mathrm{c}^2 = z_1^2z_2^2z_3^2,$$
$$\mathrm{bc} = y_1z_1y_2z_2y_3z_3,$$
$$\mathrm{ca} = z_1x_1z_2x_2z_3x_3,$$
$$\mathrm{ab} = x_1y_1x_2y_2x_3y_3,$$

$$3af = S\, x_1y_1x_2y_2z_3x_3,$$
$$3bg = S\, y_1z_1y_2z_2x_3y_3,$$
$$3ch = S\, z_1x_1z_2x_2y_3z_3,$$

$$3bf = S\, y_1{}^2y_2{}^2y_3z_3,$$
$$3cg = S\, z_1{}^2z_2{}^2z_3x_3,$$
$$3ah = S\, x_1{}^2x_2{}^2x_3y_3,$$

$$3cf = S\, y_1z_1y_2z_2z_3{}^2,$$
$$3ag = S\, z_1x_1z_2x_2x_3{}^2,$$
$$3bh = S\, x_1y_1x_2y_2y_3{}^2,$$

$$3ai = S\, x_1y_1z_2x_2z_3x_3,$$
$$3bj = S\, y_1z_1x_2y_2x_3y_3,$$
$$3ck = S\, z_1x_1y_2z_2y_3z_3,$$

$$3bi = S\, y_1{}^2y_2z_2y_3z_3,$$
$$3cj = S\, z_1{}^2z_2x_2z_3x_3,$$
$$3ak = S\, x_1{}^2x_2y_2x_3y_3,$$

$$3ci = S\, y_1z_1z_2{}^2z_3{}^2,$$
$$3aj = S\, z_1x_1x_2{}^2x_3{}^2,$$
$$3bk = S\, x_1y_1y_2{}^2y_3{}^2,$$

$$6al = S\, x_1{}^2x_2y_2z_3x_3,$$
$$6bl = S\, y_1{}^2y_2z_2x_3y_3,$$
$$6cl = S\, z_1{}^2z_2x_2y_3z_3,$$

$$9f^2 - 6bi = S\, y_1{}^2y_2{}^2z_3{}^2,$$
$$9g^2 - 6cj = S\, z_1{}^2z_2{}^2x_3{}^2,$$
$$9h^2 - 6ak = S\, x_1{}^2x_2{}^2y_3{}^2,$$

$$9i^2 - 6cf = S\, y_1{}^2z_2{}^2z_3{}^2,$$
$$9j^2 - 6ag = S\, z_1{}^2x_2{}^2x_3{}^2,$$
$$9k^2 - 6bh = S\, x_1{}^2y_2{}^2y_3{}^2,$$

$$9fg - 3ck = S\, x_1y_1y_2z_2z_3{}^2,$$
$$9gh - 3ai = S\, y_1z_1z_2x_2x_3{}^2,$$
$$9hf - 3bj = S\, z_1x_1x_2y_2y_3{}^2,$$

$$9\mathrm{jk} - 3\mathrm{af} = S\, x_1^2 x_2 y_2 y_3 z_3,$$

$$9\mathrm{ki} - 3\mathrm{bg} = S\, y_1^2 y_2 z_2 z_3 x_3,$$

$$9\mathrm{ij} - 3\mathrm{ch} = S\, z_1^2 z_2 x_2 x_3 y_3,$$

$$9\mathrm{fi} - 3\mathrm{bc} = S\, y_1^2 y_2 z_2 z_3^2,$$

$$9\mathrm{gj} - 3\mathrm{ca} = S\, z_1^2 z_2 x_2 x_3^2,$$

$$9\mathrm{hk} - 3\mathrm{ab} = S\, x_1^2 x_2 y_2 y_3^2,$$

$$3\,(\mathrm{fj} + \mathrm{gk} + \mathrm{hi} - \mathrm{l}^2) = S\, x_1 y_1 z_2 x_2 y_3 z_3,$$

$$3\,(2\mathrm{fj} - \mathrm{gk} - \mathrm{hi} + \mathrm{l}^2) = S\, x_1 y_1 x_2 y_2 z_3^2,$$

$$3\,(2\mathrm{gk} - \mathrm{hi} - \mathrm{fj} + \mathrm{l}^2) = S\, y_1 z_1 y_2 z_2 x_3^2,$$

$$3\,(2\mathrm{hi} - \mathrm{fj} - \mathrm{gk} + \mathrm{l}^2) = S\, z_1 x_1 z_2 x_2 y_3^2,$$

$$3\,(6\mathrm{fl} - 3\mathrm{ki} - \mathrm{bg}) = S\, x_1 y_1 y_2^2 z_3^2,$$

$$3\,(6\mathrm{gl} - 3\mathrm{ij} - \mathrm{ch}) = S\, y_1 z_1 z_2^2 x_3^2,$$

$$3\,(6\mathrm{hl} - 3\mathrm{jk} - \mathrm{af}) = S\, z_1 x_1 x_2^2 y_3^2,$$

$$3\,(6\mathrm{il} - 3\mathrm{fg} - \mathrm{ck}) = S\, z_1 x_1 y_2^2 z_3^2,$$

$$3\,(6\mathrm{jl} - 3\mathrm{gh} - \mathrm{ai}) = S\, x_1 y_1 z_2^2 x_3^2,$$

$$3\,(6\mathrm{kl} - 3\mathrm{hf} - \mathrm{bj}) = S\, y_1 z_1 x_2^2 y_3^2,$$

$$6\,(-\mathrm{fj} - \mathrm{gk} - \mathrm{hi} + 4\mathrm{l}^2) = S\, x_1^2 y_2^2 z_3^2.$$

As an instance of the application of the formulæ, let it be required to eliminate the variables from the three equations,

$$(a,\ b,\ c,\ f,\ g,\ h,\ i,\ j,\ k,\ l \between x,\ y,\ z)^3 = 0,$$

$$(a',\ b',\ c',\ f',\ g',\ h' \between x,\ y,\ z)^2 = 0,$$

$$(\alpha,\ \beta,\ \gamma \between x,\ y,\ z) = 0.$$

This may be done in two different ways; first, representing the roots of the linear equation and the quadric equation by $(x_1,\ y_1,\ z_1)$, $(x_2,\ y_2,\ z_2)$, the resultant will be

$$(a, \ldots \between x_1,\ y_1,\ z_1)^3 \,.\, (a, \ldots \between x_2,\ y_2,\ z_2)^3,$$

which is equal to

$$a^2\, x_1^3 x_2^3 + \&\mathrm{c.},$$

where the symmetric functions $x_1^3 x_2^3$, &c. are given by the formulæ $\mathrm{a}'^3 = x_1^3 x_2^3$, &c., in which, since the coefficients of the quadratic equation are $(a',\ b',\ c',\ f',\ g',\ h')$, I have written a′ instead of a. Next, if the roots of the linear equation and the cubic equation are represented by $(x_1,\ y_1,\ z_1)$, $(x_2,\ y_2,\ z_2)$, $(x_3,\ y_3,\ z_3)$, then the resultant will be

$$(a', \ldots \between x_1,\ y_1,\ z_1)^2 \,.\, (a', \ldots \between x_2,\ y_2,\ z_2)^2\, (a', \ldots \between x_3,\ y_3,\ z_3)^2,$$

which is equal to

$$a'^3\, x_1^2 x_2^2 x_3^2 + \&c.,$$

the symmetric functions $x_1^2 x_2^2 x_3^2$, &c. being given by the formulæ $\mathrm{a}^2 = x_1^2 x_2^2 x_3^2$, &c. The expression for the Resultant is in each case of the right degree, viz. of the degrees 6, 3, 2, in the coefficients of the linear, the quadric, and the cubic equations respectively: the two expressions, therefore, can only differ by a numerical factor, which might be determined without difficulty. The third expression for the resultant, viz.

$$(\alpha,\ \beta,\ \gamma \between x_1,\ y_1,\ z_1) \,.\, (\alpha,\ \beta,\ \gamma \between x_2,\ y_2,\ z_2) \ldots (\alpha,\ \beta,\ \gamma \between x_6,\ y_6,\ z_6),$$

(where $(x_1,\ y_1,\ z_1), \ldots (x_6,\ y_6,\ z_6)$ are the roots of the cubic and quadratic equations) compared with the foregoing value, leads to expressions for the fundamental symmetric functions of the cubic and quadratic equations, and thence to expressions for the other symmetric functions of these two equations; but it would be difficult to obtain the actually developed values even of the fundamental symmetric functions. I hope to return to the subject, and consider in a general point of view the question of the formation of the expressions for the other symmetric functions by means of the expressions for the fundamental symmetric functions.