# 49.

## ON THE SO-CALLED TSCHIRNHAUSEN TRANSFORMATION.

[*Crelle's Journal*, C. (1887), pp. 465—486.]

EXACTLY one hundred years ago, E. S. Bring (Dissertation, University of Lund, 1786. Meletemata quaedam mathematica circa transformationem aequationum algebraicarum) gave the method to which the name of Tschirnhausen by a common consent in error is now usually attached*. Sometimes but more rarely the method is attributed to Jerrard who came much later into the field. This is especially the case in England; Hamilton for instance in his "Report on Jerrard's method" published exactly 50 years ago in the

* The expression $P_\theta - L_{n-1} Q_{n-1-\theta} + M_n R_{n-2-\theta}$

where $L$, $M$ are given entire functions in $x$ of degrees $n-1$, $n$,

$P$, $Q$, $R$ ,, disposable ,, ,, ,, ,, $\theta$, $n-1-\theta$, $n-2-\theta$,

may be made identically zero by solving $2n-1-\theta$ homogeneous linear equations between the $2n-\theta$ disposable constants contained collectively in $P$, $Q$, $R$, and when this is done we have

$$\frac{P_\theta}{Q_{n-1-\theta}} \equiv L_{n-1} \text{ [mod. } M_n].$$

Hence it follows that the Tschirnhausen substitution has a one-to-one correspondence with any fractional substitution containing the requisite number of disposable constants: so for instance in the case of a *quintic* the Bring substitution

$$lx^4 + mx^3 + nx^2 + px + q$$

is only *another name* for the general *quadratic substitution* $\dfrac{ax^2+bx+c}{dx^2+ex+f}$.

This change of form in the substitution, supposed to be generalised, is interesting for the reason that it completes the analogy between the Tschirnhausen method of simplifying an algebraical equation and Combescure's method of simplifying a linear differential equation. Sir James Cockle appears to have arrived at the same result as M. Combescure in a paper on Linear Differential Equations. (*Quarterly Journal of Mathematics*, Aug. 1864.)

This method involves two quadratures, the integration of a differential equation of the second order, and substitutions impressed simultaneously upon *the two variables*.

The quadratures and solution of an equation of the second order are, of course, analogous to the solution of two simple and one quadratic algebraical equation; the substitutions impressed on the two variables run parallel to the two integral substitutions to be performed upon the two variables of the algebraical equation put under the form of a quantic which are equivalent to a fractional substitution performed upon the single variable of a non-homogeneous form.

*Reports of the British Association* makes hardly any mention of any other author but Jerrard in connexion with the subject.

In the following memoir I propose to present Hamilton's process under what appears to me to be a clearer and more easily intelligible form, to extend his numerical results and to establish the principles of a more general method than that to which he has confined himself.

But previously to entering upon this part of my work I think it may be well to call attention to a circumstance connected with the so-called Tschirnhausen transformation, as bearing upon the character of the transformed equation to which it leads, which hitherto appears to have escaped observation, and which is of particular interest as regards the application of the method to the equation of the 5th degree when it is reduced to the form

$$y^5 + By + C = 0,$$

for I shall be able to show in that case that in general the coefficients which remain (notwithstanding the large element of indeterminateness of which the method admits) cannot be made real when more than one of the roots of the original equation is real; this remark will be found to apply whether the method be used under its original form or under the modified form employed so advantageously by Hermite.

In order to make out this proposition it will be useful to give a somewhat more extended statement of the Law of Inertia (Trägheitsgesetz) for quadratic forms than that originally presented by me in the memoir: "On a theory of the syzygetic relations of two rational integral functions comprising an application to the theory of Sturm's functions and that of the greatest algebraical common measure" (*Phil. Trans.* for 1853)*.

Let us suppose a quadratic function of $m+n$ letters, either independent or connected by linear relations which in the latter case reduce the number of independent quantities to $\mu+\nu$.

Let the function be supposed to be expressed

(1) by the sum of $m$ positive and $n$ negative squares,

(2) by the sum of $\mu$ positive and $\nu$ negative squares

of *real* linear functions of the variables.

Then I affirm the impossibility of either of the two inequalities

$$\mu > m; \quad \nu > n.$$

(1) I say that the conjunction of the inequalities $m > \mu$, $\nu > n$ is impossible.

For suppose the two expressions of the same quadratic function to be

$$a_1^2 + a_2^2 + \ldots + a_m^2 - b_1^2 - b_2^2 - \ldots - b_n^2$$

and $$\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_\mu^2 - \beta_1^2 - \beta_2^2 - \ldots - \beta_\nu^2.$$

[* Vol. I. of this Reprint, p. 511.]

Then
$$a_1^2 + a_2^2 + \dots + a_m^2 + \beta_1^2 + \beta_2^2 + \dots + \beta_\nu^2$$
$$= b_1^2 + b_2^2 + \dots + b_n^2 + \alpha_1^2 + \alpha_2^2 + \dots + \alpha_\mu^2.$$

By hypothesis
$$\mu + n < \mu + \nu,$$
$$\mu + n < m + \nu.$$

By virtue of the first inequality it must be possible to establish $\mu + n$ relations between the $\mu + \nu$ independent variables.

Consequently we may equate each square on the right-hand side of the equation to some distinct square on the other side, and then by virtue of the second inequality some squares will remain over on the left-hand side of the equation whose sum will be identically zero. Which is impossible. Hence the inequalities $m > \mu$, $\nu > n$ cannot exist simultaneously. In like manner it follows that $n > \nu$, $\mu > m$ cannot exist simultaneously.

Now the only suppositions of combined relations of greater and less that can connect $m, n$; $\mu, \nu$ are the following:

$$m < \mu,\ n < \nu;\quad m < \mu,\ n = \nu;\quad m < \mu,\ n > \nu;$$
$$m = \mu,\ n < \nu;\quad m = \mu,\ n = \nu;\quad m = \mu,\ n > \nu;$$
$$m > \mu,\ n < \nu;\quad m > \mu,\ n = \nu;\quad m > \mu,\ n > \nu.$$

Of these 9 suppositions the 1st, 2nd, and 4th are excluded by the condition $m + n =$ or $> \mu + \nu$, and the 3rd and 7th by virtue of what has just been proved. Hence the only hypotheses admissible are the four contained in the negative statements:

$\mu$ not $> m$ and $\nu$ not $> n$. Q.E.D

Although the only application which I shall have to make of this Lemma is to the case where $m + n = \mu + \nu + 1$, I have thought that it is of sufficient interest in itself and collaterally in the logical process of its proof to deserve setting out in full.

Suppose now that we have the equation $f(x) = (x, 1)^n = 0$ where all the coefficients in $f$ are supposed to be real, and that we write in conformity with the ordinary so-called Tschirnhausen process:
$$y = u_1 x + u_2 x^2 + \dots + u_{n-1} x^{n-1} - S,$$
where
$$nS = u_1 \Sigma x + u_2 \Sigma x^2 + \dots + u_{n-1} \Sigma x^{n-1}$$
so that the transformed equation will be of the form:
$$y^n + B_2 y^{n-2} + B_3 y^{n-3} + \dots + B_n = 0,$$
where $B_i$ is a quantic of degree $i$ in the letters $u_1, u_2, \dots u_{n-1}$. Let us consider the projective character of the quadratic function $B_2$. This character is determined by the nature of the succession of algebraical signs in the sum of positive and negative squares to which $B_2$ regarded as a function of the $n - 1$ letters $u$ may be reduced by *real* linear transformations.

Since

$$y_1 + y_2 + \dots + y_n = 0,$$

$$-2B_2 = -\Sigma 2y_1y_2 = \Sigma y^2,$$

so that it is the character of $\Sigma y^2$ which determines the projective character of $B_2$. The number of real values of $y$ is the same as of $x$. Hence if $f$ has $i$ pairs of imaginary roots, $\Sigma y^2$ will be the sum of $n - i$ positive and $i$ negative squares of real linear functions of $u_1, u_2, \dots u_{n-1}$.

Consequently, by virtue of the lemma above proved, there is only one element of uncertainty as to the character of $\Sigma y^2$, that is, it must we know *à priori*, when reduced to a sum of $n-1$ positive and negative squares of linear functions of $u_1, u_2, \dots u_{n-1}$, contain either $i$ or $i-1$ negative squares. This uncertainty may be removed by means of a second lemma, namely, that the discriminant of $B_2$ is a numerical multiplier of the discriminant of $f$.

When two of the roots of $f$ are equal, two of the values of $y$ become equal so that $\Sigma y^2$ becomes reducible to a sum of $n-2$ instead of a sum of $n-1$ squares.

Hence the former contains the latter as a factor: moreover it is obvious from the form of each value of $y$ that its discriminant regarded as a function of the $n$ roots of $f$ will be of the degree $2\{1 + 2 + \dots + (n-1)\}$, that is, $n(n-1)$ which is the same as that of the squared product of the differences of the roots of $f$. Hence $B_2$ is a *numerical* multiplier of such squared product. To find the value of the multiplier, I observe that in general it follows from known algebraical principles that if $F$ is a sum of the squares of $n$ linear functions of $n-1$ variables the discriminant of $F$ may be found as follows. Form an oblong matrix with the coefficients of the several linear functions. The determinant represented by what Cauchy would have called the square of this matrix, but which is more correctly to be called the product of this matrix by its transverse, will be the discriminant in question, or which is the same thing this discriminant is the sum of the squares of all the *complete* minors that are contained in the oblong matrix.

In the case before us if we make $f = x^n - 1$* it will easily be seen that

* When $f = x^n - 1$ the value of $S$ (the mean of the values of $y$) is obviously zero. Suppose now by way of illustration that $n=5$, then calling the imaginary 5th roots of unity $\rho_1, \rho_2, \rho_3, \rho_4$, one of the complete minors referred to in the text will be the determinant of the matrix

$$\begin{matrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ \rho_1^2 & \rho_2^2 & \rho_3^2 & \rho_4^2 \\ \rho_1^3 & \rho_2^3 & \rho_3^3 & \rho_4^3 \\ \rho_1^4 & \rho_2^4 & \rho_3^4 & \rho_4^4, \end{matrix}$$

and when the columns of this matrix are divided respectively by $\rho_1^\theta, \rho_2^\theta, \rho_3^\theta, \rho_4^\theta$, $[\theta = 1, 2, 3, 4]$, which will leave the value of the determinant unaltered, the determinant of the matrix so modified will represent in succession each of the other 4 minors.

The value of the one above written, paying no attention to the algebraical sign, is by a well known theorem the product of the differences of $\rho_1, \rho_2, \rho_3, \rho_4$, that is, inasmuch as

$$(1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4) = 5$$

the $n$ minors in question, paying no regard to algebraical sign, become all equal, and each will be the product of the differences of the roots of $x^n - 1$ when the root 1 is excluded, or which is the same thing will be the product of the differences of all the roots (not excluding 1) divided by $n$.

Hence the sum of the $n$ squared minors will be the $n$th part of the square of the products of the differences of the roots of $x^n - 1$. Consequently in general the discriminant of $\Sigma y^2$ is the $n$th part of the product of the squares of the differences of the roots of the function $f$, and therefore by the process of reduction of $-\Sigma y^2$ to a sum of $n-1$ squares it is the *positive* sign always which will undergo the diminution of a unit, the number of negative signs remaining unaltered.

Hence when there are no imaginary roots in $f$, $-B_2$ will have all its signs positive; but when there are $i$ pairs of imaginary roots in $f$, $i$ of the signs in $-B_2$ will be negative, and thus the character of $B_2$, or of the quadratic *contour* (that is, curve, surface, hypersurface, etc.) represented by $B_2 = 0$ is completely determined when the number of real and imaginary roots in $f$ is given.

If we suppose $n = 5$ we see that according as the number of real roots in $f$ is 5, 3, or 1, the signs of $-B_2$ regarded as a sum of positive and negative squares of real linear functions of 4 letters will be:

$$+ \quad + \quad + \quad +$$
$$+ \quad + \quad + \quad -$$
$$+ \quad + \quad - \quad -$$

In the first case the contour $B_2$ is completely imaginary, and it is not only not possible to apply the Bring-Tschirnhausen method so as to make simultaneously $B_2 = 0$, $B_3 = 0$ by real quantities $u_1, u_2, u_3, u_4$, but it is also the case that such values of $u_1, u_2, u_3, u_4$ do not exist. This indeed is evident *à priori*, from the fact that the equation

$$y^5 + B_4 y + B_5 = 0$$

must have at least two imaginary roots and therefore the equation in $x$ would have at least two imaginary roots if the quantities $u_1, u_2, u_3, u_4$ were all real and unequal; whereas all the roots of that equation are supposed to be real.

In the second case the intersection of the contours $B_2$, $B_3$ may be real or imaginary: but even if it be real the method will not serve to determine any

it is the 5th part of the product of the differences of 1, $\rho_1$, $\rho_2$, $\rho_3$, $\rho_4$, and consequently the sum of the squares of the 5 minors is 5 times the 25th part of the squared product of the differences of the 5 roots. Here $\frac{5}{25}$ represents the general numerical multiplier $\frac{n}{n^2}$, that is, $\frac{1}{n}$.

single point in such section, because no *real* right line can be drawn to $B_2$ at any point which shall lie on the surface.

In the 3rd case at each point of $B_2$ two real right lines can be drawn each of which will intersect $B_3$ in one real point at least, and accordingly there will be a duplex-infinity of systems of real values of the $u$'s which will make $B_2 = 0$, $B_3 = 0$ capable of being found by solving only a quadratic and a cubic equation in succession, and any one of such systems will lead to an equation of the form

$$y^5 + B_4 y + B_5 = 0,$$

where $B_4$, $B_5$ (which it is hardly necessary to notice become respectively $\frac{1}{4}\Sigma y^4$, $-\frac{1}{5}\Sigma y^5$) will each be real.

The $B_2$ found by Hermite's method may be obtained from the $B_2$ above given by a real linear substitution impressed on the letters $u_1$, $u_2$, $u_3$, $u_4$, and consequently the same conclusions continue to apply, that is, the coefficient of $y$ and the constant will not in general be real unless four of the roots of the equation in $x$ are imaginary*.

I will now proceed to the principal object of this paper, namely, the elucidation and extension of the method, contained in Hamilton's report, for determining the least number of letters which must be contained in one or more equations in order that they may admit of being solved by means of equations whose degrees are subject to satisfy certain prescribed conditions.

Before proceeding to the Lemma upon which all that follows is based, it will be useful to give one or two definitions.

1. Let $S$ be a system of homogeneous equations in an indefinite number of variables $x, y, \ldots$, and let $x = a$, $y = b$, ... satisfy all the equations. I call $a, b, \ldots$ a solution of $S$.

2. If $a, b, \ldots$ is a given solution of $S$, I call the equation obtained by operating upon any of those in $S$ with $(a\partial_x + b\partial_y + \ldots)^q$ where $q$ has any integer value whatever not excluding zero, an emanant of such equation in respect to the solution $a, b, \ldots$, and the new system $S_1$ which contains all the emanants of all the equations in $S$ an emanant to $S$ in respect to the given solution.

* Hamilton remarks (*Report of British Association*, 1836, p. 307) that "the coefficients of the new or transformed equation will *often* be imaginary even when the coefficients of the original equation are real." Apparently he was not aware that the criterion for determining when this is so, depends solely on the intrinsic character of the equation to be transformed.

It should have been noticed before that when two of the roots in the given quintic are equal the quadratic surface represented by the coefficient of $y^3$ in the transformed equation becomes a cone and the reasoning employed in the text falls to the ground. But inasmuch as in this case two of the values of $y$ become equal, we know *à priori* that the equation in $y$ must be reducible to a form with real coefficients, namely,

$$y^5 - 5y + 4 = 0.$$

3. Let $a, b, \ldots, a_1, b_1, \ldots$, be any two solutions of a system of equations. I call

$$a+\lambda a_1, \quad b+\lambda b_1, \quad c+\lambda c_1, \quad \ldots$$

an alliance of the two solutions.

We may now state the following Sub-lemma. The alliance of any solution of a system $S$ with a solution of $S_1$, its emanant in respect to the first named solution, is a solution of $S_1$.

For let $a, b, \ldots$ be the solution of $S$ which gives rise to $S_1$. Then, calling

$$a\partial_x + b\partial_y + \ldots = E,$$

the general form of the equations in $S_1$ is $E^q\Phi(x, y, \ldots)$, and, supposing $\Phi$ to be of $n$ dimensions,

$$E^q\Phi(x+\lambda a, \quad y+\lambda b, \quad \ldots)$$
$$= E^q\Phi + \lambda E^{q+1}\Phi + \tfrac{1}{2}\lambda^2 E^{q+2}\Phi + \ldots + \lambda^{n-1}E^{q+n-1}\Phi.$$

Hence the effect of substituting $x+\lambda a$, $y+\lambda b, \ldots$ for $x, y, \ldots$ in $S_1$ is merely to effect upon it a linear transformation, and consequently the alliance of the solution $a, b, c, \ldots$ with any solution of $S_1$ will be a solution of that system. If now we find a solution $a_2, b_2, \ldots$ of $S_1$ and form an emanant $S_2$ of it in respect to that solution, it will follow from the sub-lemma that an alliance of the solutions $a, b, \ldots; a_1, b_1, \ldots$; with any solution $a_2, b_2, \ldots$ of $S_2$, that is, the solution $a+\lambda a_1+\mu a_2$, $b+\lambda b_1+\mu b_2, \ldots$ will be a solution of $S_2$, and so in general. This I call the Lemma.

An ordinary solution of a system of equations may be called a point solution, an alliance in which 1, 2, 3, ... parameters enter a line, a plane, a hyperplane, ... solution.

It will of course be observed that any solution of an emanant to a system is *à fortiori* a solution of the primitive which as observed forms a portion of its emanant.

If $S$ be a system of $q_i, q_{i-1}, \ldots q_1$ equations of degrees $i, i-1, \ldots 1$ respectively in the variables, it is obvious the 1st emanant will consist of

$$q_i, \quad q_i+q_{i-1}, \quad q_i+q_{i-1}+q_{i-2}, \quad \ldots \quad q_i+q_{i-1}+\ldots+q_1$$

equations of the degrees $i, i-1, i-2, \ldots 1$ respectively, and more generally in the $r$th emanant the number of equations of the degree

$i$ will be $q_i$

$i-1$ „ „ $rq_i+q_{i-1}$

$i-2$ „ „ $\tfrac{1}{2}r(r+1)q_i+(r-1)q_{i-1}+q_{i-2}$

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$1$ „ „ $[r+i-1]_i q_i + [r+i-2]_{i-1} q_{i-1} + \ldots + q_1$,

where in general $[\theta]_i$ is used to denote $\dfrac{\theta(\theta-1)\ldots(\theta-i+1)}{1 \,.\, 2 \ldots i}$.

The question now arises as to what must be the number of variables in a system $S$ in order that its $r$th emanant $S_r$ may admit of a general solution. If the total number of equations in $S_r$ be called $N$, it might at first sight be supposed that the number of variables, or letters as I prefer to call them, in $S$ must have $N+1$ as an inferior limit: but the case is not so—the least number of variables required will be $r$ greater than this, that is, $N+r+1$.

Thus, for example, suppose we consider a first emanant $S_1$; then if $a_1$, $b_1$, $c_1$, ... is a solution we know that $a_1+\lambda a$, $b_1+\lambda b$, $c_1+\lambda c$, ... is also a solution whatever $\lambda$ may be. Hence making $\lambda = -\dfrac{a_1}{a}$ and remembering that the equations are homogeneous we see that zero associated with any system of independent minors of the matrix

$$\begin{matrix} a & b & c & \dots, \\ a_1 & b_1 & c_1 & \dots \end{matrix}$$

will constitute a solution, as for instance $0$; $ab_1 - ba_1$; $ac_1 - ca_1$; ...*. Hence the number of independent quantities in $S_1$ will be 1 less than the number of letters in $S$.

* As an illustration suppose $\Phi$ is a quantic of degree $n$ in $(n+2)$ letters representing what may be termed a *contour*, the analogue in general space of a curve in 2-dimensional or a surface in 3-dimensional space. If we take all the successive emanants of $\Phi$ in respect to a point upon it $a, b, c, \dots$ the $n$ resulting functions [$\Phi$ included] being functions of the $n+1$ minors to the matrix [$(n+2)$ places in length]

$$\left|\begin{matrix} a & b & c & \dots \\ x & y & z & \dots \end{matrix}\right|$$

the contours which they represent will intersect in a faisceau of right lines—showing that on a contour of the $n$th degree in $(n+1)$-dimensional space $1 . 2 . 3 \dots n$ right lines lying in the contour will pass through every point thereof, a fact we are familiar with in the case of a quadric surface where $n=2$. We might with equal propriety and more convenience say that $\Pi n$ straight lines may be drawn upon and at every point of an $n$-fold contour of the $n$th *order*.

As I have already referred in this footnote to right lines drawn on contours I venture upon a slight digression connected with this conception. If we have a cubic twofold contour (an ordinary cubic surface) expressed as a quantic in $x$, $y$, $z$, $t$, we see that on writing $x$, $y$ as linear functions of $z$, $t$ and substituting their values in $\Phi$ in order to make the result, a cubic function of $z$, $t$ vanish, we have to satisfy 4 equations between the 4 coefficients of substitution, which at once shows that a finite number of right lines may be drawn upon such contour of which the number we see at once cannot exceed $3^4$ and which we know aliunde is $3^3$.

It would seem then that for a contour in $n$ letters of the degree $2n-5$ (unless there is some lurking fallacy in the counting of the constants) we ought in like manner to be able, by expressing $n-2$ of the letters as linear functions of the two remaining ones, to make the result vanish by solving $2n-4$ non-homogeneous equations of the degree $2n-5$ between the like number of coefficients of substitution, and as if upon such a contour we must be able to draw a definite number of straight lines of which the number, supposing that there is no latent fallacy of constant-counting, would be not greater and in all probability less than $(2n-5)^{2n-4}$, in fact $(2n-5)^{2n-5}$.

Also it may be shown that, as by Bedetti's theorem we know that every twofold contour (an ordinary surface) is cut by its linear polar (its tangent plane) in respect to a point upon it, in a curve having a double point thereat, so a contour of the 3rd order will be cut by its linear and quadratic polars in respect to any point upon it in a curve having a sextuple point thereat, and so in general an $n$-fold contour will be cut by $n-1$ consecutive polars (starting from the tangential

Similarly for the system $S_r$; $r$ zeros associated with any independent system of complete minors of the matrix

$$\begin{matrix} a & b & c & \dots \\ a_1 & b_1 & c_1 & \dots \\ a_2 & b_2 & c_2 & \dots \\ \dots & \dots & \dots & \dots \\ a_r & b_r & c_r & \dots \end{matrix}$$

may be taken as the variables, and consequently it is $N-r-1$ and not $N-1$ which has for its inferior limit the number of equations in $S_r$. We may restore to the variables their independence by associating with the equations in $S_r$ $r$ additional perfectly arbitrary linear functions and there is sometimes a convenience in substituting in place of the $r$th emanant as it stands such emanant augmented by $r$ arbitrary linear functions, which may be called the *completed* emanant.

For the purpose of greater clearness of exposition there will be an advantage in ignoring in the first instance all considerations based upon any other alliance except of the 1st order, that is, involving only one arbitrary parameter.

Suppose a system of equations $S_1$ consisting of a system $S$ and one equation more $Q$. If we are in possession of a linear solution of $S$, that is, a solution

$$x=a_1+\lambda a, \quad y=b_1+\lambda b, \quad \dots$$

by substituting these values in $Q$, $\lambda$ may be found by solving an equation whose degree is that of $Q$, and thus a point (or ordinary) solution of $S_1$ will have been found.

Let us now consider the question of a linear solution of $S$ containing $q_i, q_{i-1}, \dots q_1$ equations of degree $i, i-1, \dots 1$ respectively. This we shall call of the type $[q_i, q_{i-1}, \dots q_1]$. Let

$a, b, \dots$ be any point solution of $S$,

and $a_1, b_1, \dots$ any point solution of $ES$,

homaloid as the first of them) in respect to any point upon it in a curve having thereat a point of multiplicity $1 . 2 . 3 \dots n$.

It may be well here to notice that a uni-parametrical solution of $\Phi=0$ corresponds to drawing a straight line upon the contour represented by $\Phi$, and in like manner a bi-parametric solution corresponds to drawing a plane upon the contour, a tri-parametric solution to drawing a hyperplane upon the contour, and so in general. This is why I call such solutions linear, planar, hyperplanar, etc.

So again in this connexion it may be remarked that upon a quadratic contour in trans-hyperspace 6 planes lying on the contour pass through every point and in like manner upon a quadratic contour in $2n$ letters, $1 . 2 . 3 \dots n$ $n$-fold homaloids may be drawn upon the contour through every point thereon.

the *completed* emanant of $S$ in regard to $a, b, \ldots$. This will be of the type

$$q_i;\ q_i+q_{i-1};\ \ldots;\ q_i+q_{i-1}+\ldots+q_2;\ q_i+q_{i-1}+\ldots+q_1+1.$$

The alliance of these two point solutions will be a linear solution of $S$. Again, the number of variables required for the point solution of $ES$ need not exceed the number required for the linear solution of the system to which $S$ is reduced by the abstraction of one equation of the degree $i$. Hence if we use $[p, q, r, \ldots \eta, \theta]$ to denote the number of variables sufficient for the solution of a system of $p$ equations of degree $i$, $q$ of degree $i-1$, etc. ($i$ being the number of indices $p, q, \ldots \theta$) we obtain the formula of reduction

$$[p, q, r, \ldots \theta] = [p-1, p+q, p+q+r, \ldots p+q+r+\ldots+\theta]+1^*.$$

Continuing this process of reduction until the first index is reduced to zero a very easy calculation leads to the *formula of obliteration*

$$[p, q, r, s, \ldots \theta] = p+[q_1, r_1, s_1, \ldots \theta_1]\dagger,$$

where

$$q_1 = \frac{p(p+1)}{2}+q,$$

$$r_1 = \frac{p(p+1)(2p+1)}{6}+pq+r,$$

$$s_1 = \frac{p(p+1)(p+2)(3p+1)}{24}+\frac{p(p+1)}{2}q+pr+s,$$

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$$\theta_1 = \frac{p(p+1)\ldots(p+i-1)(ip+1)}{1.2.3\ldots(i+1)}+\frac{p(p+1)\ldots(p+i-2)}{1.2.3\ldots(i-1)}q+\ldots+\theta.$$

In applying the process of reduction in the way indicated, the system $S$ will have been replaced by two systems which we may call a diminished $S$ and a diminished emanant of $S$, that is, $S$ and $ES$ each deprived of an equation (not necessarily the same in both) of degree $i$.

In like manner each of these will give rise to two systems, namely, a diminished self-system and a diminished emanant-system; but as the object is to obtain a formula of reduction for the number of letters required to obtain a linear solution of $S$, and as this number is greater for an emanant of any system than for the system itself, it was sufficient to follow the main-stream of deduction, in which the first alone is taken account of, in order to arrive at the required formula. In doing so, $2^p$ independent equations of the degree $i$ will have been set apart each of which will have to be solved in its proper turn.

In the formula of *obliteration* the index in the first place has disappeared. Repeating the process we shall come to

$$p+q_1+[r_2, s_2, \ldots \theta_2]$$

* In this and all subsequent formulae of reduction or obliteration the sign "=" is to be understood to mean "not greater than."

† Compare Hamilton, *Report of British Association*, 1836, p. 335, formula 244, and p. 345, formula 320.

where $r_2, s_2, \dots \theta_2$ are derived from $q_1, r_1, s_1, \dots \theta_1$ in the same way as $q_1, r_1, s_1, \dots \theta_1$ from $p, q, r, s, \dots \theta$ except that $i$ will be replaced by $i-1$; and thus pursuing the same process we shall arrive at

$$[p + q_1 + r_2 + \dots + \eta_{i-1} + \theta_i]$$

or say $[\sigma]$. The number of variables required for a solution involving one arbitrary parameter of $\sigma$ homogeneous linear equations being $\sigma + 2$, this latter will be the number sufficient for $S$ to admit of a linear solution without giving occasion to solve any equation of a degree exceeding $i$, and also without having occasion to solve any simultaneous system of equations other than linear ones.

Suppose a system of equations of the respective degrees 1, 2, 3, ... $i$ and a single equation of the degree $i+1$.

The type of the former will be 1, 1, 1, ... 1 to $i$ places,
and of the latter 1, 0, 0, 0, ... 0 „ $i+1$ „ .

By the rule which has been established the number of letters required for the linear solution of the latter will be *one* more than for the former.

Hence the determination of the Tschirnhausen question of finding what the degree of an equation must be in order that $i$ consecutive terms following immediately after the first term in the transformed equation, conjoined with any more advanced term, may admit of a solution of minimum weight, contains a determination of the number of variables required to ensure the possibility of obtaining a linear solution by a system of equations of minimum weight of a single equation of degree $i+1$; for the latter number will be the former increased by a unit*. The first form of the question is the more simple in itself; but as the other is more immediately connected with the object in which the theory originated, I prefer to put it in the latter form.

We may apply the obliteration formula to the indefinite type and obtain the annexed Table.

Triangle of Obliteration.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | ...... |
| | 2 | 3 | 4 | 5 | 6 | 7 | ...... |
| | | 6 | 15 | 29 | 49 | 76 | ...... |
| | | | 36 | 210 | 804 | 2449 | ...... |
| | | | | 876 | 24570 | 401134 | ...... |
| | | | | | 408696 | 246382080 | ...... |
| | | | | | | 83762796636 | ...... |
| | | | | | | | ...... |
| | | | | | | | .... |
| | | | | | | | ... |

* For example, to take away the 2nd, 3rd, and another term the degree required is 5: and to obtain a linear solution of a cubic the number of variables required is 6.

To take away the 2nd, 3rd, 4th, and another term, employing a solution of *the lowest weight*, 11 variables are required; in order to obtain a solution, of lowest weight, of a single function of the fourth degree, 12 variables are required, and so on.

The degree of the equation sufficient to allow

$$2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots$$

consecutive terms following the first to be removed by a solution of *minimum weight* of the auxiliary equations, will be the continued sum of

$$1,\ 2,\ 6,\ 36,\ 876,\ 408\,696,\ 83\,762\,796\,636,\ \ldots$$

each increased by 2, that is,

$$3,\ 5,\ 11,\ 47,\ 923,\ 409\,619,\ 83\,763\,206\,255,\ \ldots$$

These numbers up to 923 agree with those found by Hamilton (*Report*, p. 346), the two last have been calculated here probably for the first time.

It would be too arduous a task to seek to give a much further extension to the table inasmuch as each successive term in the series 1, 2, 6, 36, ... is a fraction converging to $\frac{1}{2}$ of the square of the preceding term. This becomes obvious from inspection of the series formed by dividing each number in the above series by the square of the one before it; we thus obtain the fractions:

$$\frac{4}{1},\ \frac{6}{4},\ \frac{36}{36},\ \frac{876}{1296},\ \frac{408696}{767376},\ \frac{83762796636}{167032420416},$$

which are continually diminishing.

But if we call two successive and infinitely distant rows of the Triangle of Obliteration

$$a\quad b\quad \ldots$$

$$B\quad \ldots,$$

$$B = \frac{a^2 + a}{2} + b.$$

Hence $\frac{B}{a^2}$ converges to $\frac{1}{2} + \frac{b}{a^2}$ which is always greater than $\frac{1}{2}$. Moreover $\frac{b}{a^2}$, calculated for the successive values as far as the table extends, will be seen to be a continually decreasing fraction and assuming (what awaits exact proof) that it eventually vanishes, $\frac{B}{a^2}$ must converge to $\frac{1}{2}$.

The successive values of $\frac{b}{a^2}$ for the different rows are

$$\frac{3}{4},\ \frac{15}{36},\ \frac{210}{1296},\ \frac{24570}{767376},\ \frac{246382080}{167032420416}.$$

Inverting these fractions the values, to the nearest integer, become 1, 2, 6, 31, 678, so that there can be no doubt of the truth of the law that the asymptotic value of the square of each term divided by the square of its antecedent is $\frac{1}{2}$.

Moreover the numbers last found themselves obviously obey a parallel law to that of the original series which raises a presumption that it may be possible to obtain an exact expression for the general term in the original series or even in the Obliteration Table in its entirety. But be that as it may, as evidently the asymptotic law is equally true for the sums of the terms in the first diagonal as for the terms themselves, we arrive at the interesting fact that if $\Phi(i)$ is the minimum degree of an equation from which $i$ consecutive terms immediately following the first can be removed, $2\Phi(i+1)$ converges to a ratio of equality with $\Phi(i)^2$ when $i$ increases indefinitely.

The minimum number of letters thus found is we see a minimum, at all events in this sense that the *method employed* to obtain a solution is inapplicable if that number of letters be reduced. In the words of Jerrard as quoted by Hamilton (*Report*, pp. 326, 327) "to discover $m-1$ ratios of $m$ disposable quantities,

$$a_1, \quad a_2, \quad \ldots \quad a_m$$

which shall satisfy a given system of $h_1$ rational and integral and homogeneous equations of the first degree

$$A'=0, \quad A''=0, \quad \ldots \quad A^{(h_1)}=0,$$

$h_2$ such equations of the second degree

$$B'=0, \quad B''=0, \quad \ldots \quad B^{(h_2)}=0,$$

$h_3$ of the third degree

$$C'=0, \quad C''=0, \quad \ldots \quad C^{(h_3)}=0,$$

and so on, as far as $h_t$ equations of the $t$th degree

$$T'=0, \quad T''=0, \quad \ldots \quad T^{(h_t)}=0$$

*without being obliged, in any part of the process, to introduce any elevation of degree by elimination.*"

But this definition may be superseded by another in which only the intrinsic character of the result arrived at is in question, and not the particular method pursued to reach it.

Let us agree to consider all equations of the same degree to have the same weight and that this weight is infinitely greater than that of an equation of any lower degree. The weight of a system of equations to be regarded as the sum of the weights of the equations which it contains.

We may, extending but not altering the meaning previously attached to the word "solution," call the *ensemble* of the equations to be solved in order to obtain any solution of the given system a solution thereof. If now a system of equations is given in number and in the degree of each, and each equation is supposed to be the most general of its kind, but the number of variables in the system is left disposable, it is easy to see that the above process, when it is practicable, leads to a solution of the lowest weight, so that no increase in the number of letters will have any effect in diminishing the weight of the solution, whatever may be the process employed to obtain it. Thus the numbers given by the linear method are *minima* in regard to solutions of the *lowest weight*.

We may however suppose another and more natural condition attached to the solution to be obtained; let $n$ be the highest degree of any equation in a given general system proposed for solution; we know that it is impossible to avoid the solution of one or more equations of the $n$th degree. We may therefore propose to ourselves the problem of determining what is the least number of letters necessary in order that no equation in the solution shall be of a degree exceeding $n$. The minimum thus obtained will in general be inferior to the minimum required for obtaining a solution of the lowest weight, and to arrive at it in any particular case it becomes necessary to make use of the Lemma in its general form which introduces the notion of alliances above the first order. Hamilton has not touched upon this part of the subject except in a single case which it was impossible to overlook: namely, where he considers the problem of taking away four consecutive terms from the general equation of the tenth or any higher degree.

The process we have seen leads to the conclusion that as many letters are required as are needed for the solution of two quadratics and seven linear equations. The solution of one biquadratic equation in the application of the process being indispensable, he felt the absurdity (if I may use the word) of stickling at the introduction of one biquadratic more, the use of which has the effect of lowering the minimum from 11 to 10. See *Report of British Association*, 1836, p. 326.

The linear method however or theory of solutions of lowest weight enjoys this prerogative that the reduction formulae are of a purely algebraical kind, whereas when the other condition above referred to is introduced, questions of numerical equality and inequality have to be considered and the theory ceases to be strictly algebraical. In what follows therefore I shall confine myself to the only case of any particular interest, namely, that which arises from the original problem of removing any given number of consecutive terms (immediately following the first) from an algebraical equation.

We may accept as the general condition to be observed that the degree of no equation appearing in the solution of a system of equations shall exceed the highest degree which must perforce figure in such solution, that is, the highest degree in the system of equations to be solved. In the case then of $n$ equations of the successive degrees 1, 2, 3, ... $i$ the condition will be that no equation in the solution shall be of a higher degree than $i$.

Thus, for example, if we look back to the easy case of a quaternary succession of such terms to be removed, we find that the problem reduces itself to finding the number of letters required to obtain a line-solution of the system whose type is 1, 1, 1, and that again to finding the number of letters required to obtain a line-solution to its augmented emanant 2, 4, that is, a system of 2 quadratic and 4 linear solutions, that is, a point solution of the completed emanant to this system which will be of the type 2, 7. The condition imposed here is that no equation shall appear of a higher degree than a biquadratic. Consequently subject to this condition the number of letters required to solve a system of one linear, one quadratic, and one cubic equation, is that sufficient for the plane-solution of a system of 7 linear equations, that is, 10, which is less by 1 than the number required in order to obtain a solution of the same system which shall be of the lowest weight.

It might at first sight be supposed that in general the introduction of solutions involving 2 or more parameters would lead to a very considerable reduction of the numbers found in the obliteration table; this however is not the case, the reduction in the values obtained by this extended method bears in general a very small ratio to the number reduced. This is a consequence of the following rule:—

In passing from the point solution of a system to a solution of any kind with a reduced type, the reduction is effected by *segregating* a certain number $i$ of the given equations and obtaining a solution of the remainder which shall contain $i$ arbitrary parameters.

Now it will be found that the *literant* (by which I mean the number of letters sufficient for the solution) will never be diminished by any other kind of segregation than what may be termed an *external segregation**.

* Imagine the type of a set of equations to be represented by a broad ribbon, in which each group of equations of the same degree is represented by a band of a distinct colour occupying as many units of space as there are units in the group. The legitimate process of segregation will then consist in dividing the band into two, obeying the same conditions as the original one, and the rule of "external segregation" amounts to saying that this separation must be effected by a single straight cut so that no middle portion is to be cut out.

According to this (which is a perfectly natural) representation the rule of external segregation may in the language of logic be described as the rule of the *excluded middle*. Thus, for example, suppose we wish to find the smallest number of variables required for the solution of a system of equations of which the type is 1, 1, 1, 0 without solving an equation beyond the 8th degree. The number required may be made equal to (cf. p. 547)

$$.[1, 1, 0] \text{ or to } :[1, 0, 0].$$

But $$.[1, 1, 0]=[1, 2, 3]=:[3],$$

and $$:[1, 0, 0]=[1, 2, 5]=:[5].$$

Thus the simultaneous segregation of the equations of the 4th and 2nd degrees *contrary to the rule* not only raises the weight of the solution but also increases the number of variables required in the given system in order that the solution may be possible.

As a consequence of this rule it may easily be seen (in the problem of determining the

Let $f, g, \ldots k, l$ be the type of the system of equations segregated, this will have no effect in diminishing the literant unless $f, g, \ldots k$ are the initial numbers of the type of the given system, in such case I call the segregation *external*.

Thus in starting with a system of the type 1, 1, ... 1, 1 the first act of segregation must consist in setting apart the equation of the highest degree and finding a line-solution of the system thus reduced. Suppose, to fix the ideas, that the highest degree is 6 and that we have arrived in the course of the deduction at a system of linear, quadratic, and cubic equations denoted by the type $m, n, p$.

So far as regards observance of the limit 6 for the highest degree in any substituted system, it would be permissible to segregate one cubic and one quadratic, but according to the rule of external segregation this will not be profitable (it will in general be quite the reverse unless $m = 1$) and so in general.

Let us now proceed to obtain the literant required for the point-solution of a sequence of $i$ equations of all degrees from 1 to $i$ subject to the condition that no auxiliary system shall contain an equation of degree higher than $i$ for the values $i = 5, 6, 7, 8$ which is as far as the table of obliteration extends. The rule teaches that this is the same as the literant of a line-solution of a system of $i - 1$ equations whose degrees extend from 1 to $i - 1$.

It will be useful in what follows to obtain a general formula for the plane-literant of a system of $i$ quadratics denoted by the type $i$, 0.

Let us signify by a symbol consisting of a type preceded by $q$ points the literant to the form of solution containing $q$ parameters of the system to which the type refers.

Then calling the plane-literant for $[i, 0]$ $v_i$, we have by virtue of the Lemma

$$v_i = :[i, 0] = [i-2, 2i+2] \quad = v_{i-2} + 2i + 2,$$
$$v_1 = :[1, 0] = .\,[1, 2] \quad = .\,[4] = 6,$$
$$v_2 = :[2, 0] = .\,[2, 3] = .\,[1, 6] = [8] = 9.$$

Hence by integrating $v_i - v_{i-2} = 2i + 2$ we shall easily obtain:

$$v_{2q} \;\; = 2q^2 + 4q + 3,$$
$$v_{2q-1} = 2q^2 + 2q + 2.$$

In treating of the literant to [1, 1, 1, 1, 1, 1, 1] it will be convenient to find

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minimum degree of the equation required for taking away $i$ consecutive terms without any equation in the solution exceeding the $i$th degree) that the occasion can never arise in the act of segregation to take account of any other numerical equalities and inequalities than one or the other of the two following

$$q^i = \text{ or } < n, \quad q^i (q-1)^j = \text{ or } < n.$$

the value of $\vdots[i, 0, 0]$ the general expression of which rid of exponentials will give rise to 3 cases.

Not being desirous of encumbering this memoir with formulae, and as we shall only have occasion to consider a single case of these formulae, I adjourn the calculation until we know what the form is of $i$ in regard to 3 in the case to be calculated, and shall obtain the value of $\vdots[i, 0, 0]$ for that case alone.

I will now consider in succession the *literants* denoted by

$$.[1, 1, 1, 1] \quad .[1, 1, 1, 1, 1] \quad .[1, 1, 1, 1, 1, 1] \quad .[1, 1, 1, 1, 1, 1, 1]$$

subject to the conditions of the solution containing no equation of a degree higher than the 5th, 6th, 7th, 8th respectively

$$.[1, 1, 1, 1] = .[2, 3, 5] = .[1, 5, 11] = .[6, 18]$$
$$= :[4, \;\; 25] = 25 + 2 \,.\, 2^2 + 4 \,.\, 2 + 3 = 44.$$

This is the literant for the solution of minimum highest degree and is 3 units less than 47, the literant for the solution of lowest weight.

It will be observed that $.[6, 18]$ has been expressed in the course of the deduction by $:[4, 25]$ instead of $.[5, 25]$. In fact $.[6, 18] = [6, 25]$ and this latter according as we segregate 1 or 2 of the quadratics is expressible by $.[5, 25]$ or by $:[4, 25]$.

The expression $.[6, 18]$ might have been obtained immediately from the triangle of obliteration

$$\begin{matrix} 1 & 1 & 1 & 1 & \ldots \\ & 2 & 3 & 4 & \ldots \\ & & 6 & 15 & \ldots \\ & & & \cdot & \ldots \end{matrix}$$

by simply substituting $1 + 2 + 15$ for 18. (It is worth noticing that in the table of obliteration after the 2nd line every initial number in any line ends with 6 and after the 3rd line every second number in each line ends with 0.)

So in like manner observing that $1 + 2 + 6 + 210 = 219$, we have

$$.[1, 1, 1, 1, 1] = .[36, 219]$$

which must have been *led up to* from

$$[1, 36, 219].$$

Hence $\quad .[1, 1, 1, 1, 1] = .[1, 35, 182] = [1, 36, 219] = :[35, 219]$

$$= 219 + 2 \,.\, 18^2 + 2 \,.\, 18 + 2 = 905$$

which is 18 units less than the corresponding literant of lowest weight 923. Similarly observing that

$$1 + 2 + 6 + 36 + 24\,570 = 24\,615,$$

$$.[1, 1, 1, 1, 1, 1] = :[875, 24\,615] = 24\,615 + 2\,(438)^2 + 2\,(438) + 2 = 409\,181$$

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which is 438 less than the corresponding literant of lowest weight 409 619. In like manner calling

$$246\,382\,080 + 876 + 36 + 6 + 2 + 1 = 246\,383\,175 = s,$$

$$.\,[1, 1, 1, 1, 1, 1, 1] = s + :[408\,695, 0] = [408\,695, 0] + t = \vdots\,[408\,692, 0] + t$$

where $t = s + 2 \times 408\,695 + 2 = 247\,200\,567.$

Here $408\,695 \equiv 2$ [mod. 3].

But in general
$$\vdots\,[3q + 2, 0] = \vdots\,[3q - 1, 0] + 9q + 9$$
$$= \vdots\,[2, 0] + 9\,\{(q+1) + q + (q-1) + \ldots + 2\}$$
$$= \vdots\,[2, 0] + \frac{9\,(q^2 + 3q)}{2} = \frac{9q^2 + 27q + 24}{2}\,*$$
$$= \frac{(3q+2)^2 + 5(3q+2)}{2} + 5.$$

Therefore
$$.\,[1, 1, 1, 1, 1, 1, 1] = t + 5 + (204\,346)\,(408\,697)$$
$$= 247\,200\,572 + 83\,515\,597\,162$$
$$= 83\,762\,797\,734.$$

This number is the minimum degree of equation which admits of 8 of its terms being removed without solving any equation above the 8th degree in the same sense as 5 is the minimum degree of equation from which 3 terms can be removed without solving an equation above the 3rd degree.

The Hamiltonian numbers corresponding to the solutions of lowest weight, have been found to be

3, 5, 11, 47, 923, 409 619, 83 763 206 255

the reduced numbers due to the introduction of planar and hyperplanar solutions

3, 5, 10, 44, 905, 409 181, 83 762 797 734,

the differences are 1, 3, 18, 438, 408 521.

The ratio of these last numbers to the numbers above them constituting a rapidly decreasing series, it is obvious that the "asymptotic law" will remain good for the second as well as for the first line of numbers: so that if $\phi(i)$ expresses the minimum degree of an equation from which $i$ terms can be abstracted without solving an equation above the $i$th degree, $\frac{2\phi(i+1)}{\phi(i)^2}$ will continually decrease towards and finally (when $i$ is infinite) coincide with unity.

I have already defined the weight of a solution. According to analogy (as, for example, in the case of a given symmetric function $\Sigma a^\alpha \,.\, b^\beta \,.\, c^\gamma \ldots$) the degree of the equation of highest degree in a solution may be termed its *order*.

* For $\vdots\,[2, 0] = [2, 9] = :[9] = 12.$

Thus then the two first series of numbers which have been given express the first of them the literant of the solution of lowest *weight*, the second the literant of the solution of lowest *order*. The numbers in the first series up to 923 and in the second series up to 10 appear in Hamilton's Report, all the others are here presented (it is believed) for the first time.

A solution is of course to be understood to mean a *non-simultaneous* but *not independent* system of equations from which a solution of a given system of equations may be derived. The equations in the solution-system form an arborescence or a ramification of consecutive systems, meaning thereby that the solution of any one of them depends upon a successive process of substitution of values of variables deduced from equations which precede it in such ramification. Some of the simpler of these arborescences I propose to *delineate* graphically in a subsequent communication.

Invited to participate in the centenary number of the leading Mathematical Journal in the world, it occurred to me that compatibly with my feeble means no more suitable contribution could be made than one which at the same time celebrates the centenary of the discovery due to the long and persistently ignored author of the method which it is the object of this memoir to elucidate and extend. I offer it (an aloe-flower of 100 years' growth) as a tardy Bessarabian "satisfaction to the Manes of" Bring.