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Water wave stability

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The first approximation to water wave motion is the linear theory of dispersive infinitesimal waves. Higher approximations for waves of small but finite amplitude describe non-linear interaction effects which are weak, but which may cause slow changes in amplitude and phase velocity. In the linear theory, any sinusoidal wave propagates over water of uniform depth with a constant velocity without change of form. When non-linear effects are included, it is only the permanent waves, which include Stokes waves and cnoidal waves, that propagate with a constant velocity without change of form. Stokes waves of almost all wavelengths have been shown by Benjamin and others to be unstable to sideband disturbances. Cnoidal waves, which are permanent waves on shallow water, are stable to most but not all disturbances. Even when a cnoidal wave is stable, the margin of stability can be made to be so small that small disturbances cause large changes to the wave properties. It is found that the large changes are periodic, indicating that in such cases cnoidal waves are stable in a non-linear sense.

Pierwszym przybliżeniem ruchu falowego w wodzie jest liniowa teoria infnitezymalnych fal dyspersyjnych. Wyższe przybliżenia dla fal o małej lecz skończonej amplitudzie opisują nieliniowe efekty oddziaływania, które choć słabe mogą powodować powolne zmiany amplitudy i prędkości fazy. W teorii liniowej dowolna fala sinusoidalna w wodzie na jednakowej głębokości rozprzestrzenia się ze stałą prędkością bez zmiany postaci. Gdy dodamy efekty nieliniowe jedynymi falami rozprzestrzeniającymi się ze stałą prędkością i bez zmiany postaci są fale o kształcie stałego nieskończonego impulsu. Do nich należą fale Stokesa i fale knoidalne. Benjamin i inni wykazali, że fale Stokesa o prawie wszystkich długościach fal są niestateczne dla zaburzeń z pasma rezonansowego (sideband). Fale knoidalne, które są falami ustalonymi na płytkiej wodzie, są stateczne dla większości lecz nie wszystkich zaburzeń. Nawet wtedy, gdy fala knoidalna jest stateczna, zakres stateczności można uczynić tak mały, że małe zaburzenia będą powodować duże zmiany własności fal. Wykazano, że duże zmiany są periodyczne, co wskazuje, że w takich przypadkach fale knoidalne są stateczne w sensie nieliniowym.

Первым приближением волнового движения в воде является линейная теория инфинитезимальных дисперсных волн. Высшие приближения для волн с малой, но конечной амплитудой описывают нелинейные эффекты взаимодействия, которые, хотя слабые, могут вызывать медленные изменения амплитуды и скорости фазы. В линейной теории произвольная синусоидальная волна в воде на одинаковой глубине распространяется с постоянной скоростью без изменения вида. Когда добавим нелинейные эффекты единственными волнами распространяющимися с постоянной скоростью и без изменения вида являются волны о форме постоянного бесконечного импульса. К ним принадлежат волны Стокса и кноидальные волны. Беньямин и другие показали, что волны Стокса, о почти всех длинах волн, неустойчивы для возмущений из резонансного диапазона. Кноидальные волны, которые являются установившимися волнами на мелкой воде, устойчивы для большинства но не для всех возмущений. Даже тогда, когда кноидальная волна устойчива, область устойчивости можно сделать так малой, что малые возмущения будут вызывать большие изменения свойств волн. Показано, что большие изменения периодически, что указывает на то, что в таких случаях кноидальные волны устойчивы в нелинейном смысле.

1. Introduction

A PERIODIC wave train on water of uniform depth may be unstable in the sense of a loss of periodicity, or as a breaking of waves. The former effect is known to occur for small but finite amplitudes, while the latter is a large amplitude phenomenon. It is the former effect which is discussed here. Much of the recent work on wave propagation and stability is described in the book by WHITHAM [6].

The linear theory for water waves of infinitesimal amplitude admits sinusoidal solutions, each propagating unchanged on the water surface. When non-linear interactions between wave components are included in the model, the class of permanent waves which propagate unchanged on the water surface is more restricted. Stokes waves are permanent waves on water of uniform depth, their main defining property being that the n-th harmonic is proportional to the n-th power of the amplitude to wavelength ratio of the wave. There is the additional restriction that for Stokes waves on shallow water, the amplitude to wavelength ratio must be very small compared with the depth to wavelength ratio, which means in effect that Stokes waves on shallow water are only of infinitesimal amplitude. The permanent periodic waves on shallow water of small but finite amplitude are called cnoidal waves.

A sinusoidal wave component on the water surface may be represented by the real part of $a(\mathbf{k})\exp i(\mathbf{k}.\mathbf{x}-\omega t)$ where $\mathbf{k}$ is the wave number on the mean horizontal water surface and $\omega = \omega(\mathbf{k})$ is the frequency given in terms of $\mathbf{k}$ by the dispersion relation. If two or more wave components interact, the sum and difference components $\mathbf{k}_1 \pm \mathbf{k}_2$ are generated with frequencies $\omega_1 \pm \omega_2$. Resonant forcing occurs if the forced frequency of one of the generated components is the same as its natural frequency, that is, for example,

$$\omega_1+\omega_2 = \omega(\mathbf{k}_1+\mathbf{k}_2). \tag{1.1}$$

Such resonant interaction is the basis of the instability of Stokes waves described by BENJAMIN [1] and others. If the harmonics of a Stokes wave are denoted by $\mathbf{k}$, $2\mathbf{k}$, ..., then a first approximation to the frequencies of the harmonics is $\omega(\mathbf{k})$, $2\omega(\mathbf{k})$, ... For a sideband modulation $\boldsymbol{\varkappa}$ of the wave, where $|\boldsymbol{\varkappa}| \ll |\mathbf{k}|$, resonant forcing occurs when

$$\omega(\mathbf{k}-\boldsymbol{\varkappa})+\omega(\mathbf{k}+\boldsymbol{\varkappa}) = 2\omega(\mathbf{k}). \tag{1.2}$$

More accurately, the dispersion relation is dependent on wave amplitude as well as wavenumber when non-linear interactions are involved, so provided the amplitude dispersion does not nullify the resonance, instability occurs in the neighbourhood of the curve in wave number space described by Eq. (1.2). BENJAMIN [1] showed that for disturbances $\boldsymbol{\varkappa}$ parallel to the wave number $\mathbf{k}$ on water of uniform depth h, such instability exists only when $kh > 1.363$. BENNEY and ROSKES [2] generalised this result by showing that instability could also occur for smaller values of kh when $\boldsymbol{\varkappa}$ is oblique to $\mathbf{k}$.

The dispersion relation for infinitesimal waves of wave number k propagating in a fixed direction on water of uniform depth h is

$$\omega = (gk\tanh kh)^{1/2}.$$

For waves on shallow water, $kh \ll 1$, and

$$\omega = kc_0\left[1-\frac{1}{6}(kh)^2+0(kh)^4\right], \tag{1.3}$$

where $c_0 = (gh)^{1/2}$ is the linear long wave velocity. It follows that

$$\omega_1+\omega_2 = \omega(k_1+k_2)\times(1+0(kh)^2), \tag{1.4}$$

so that as $kh \to 0$, the interaction between two wave components tends towards the resonant interaction of Eq. (1.1). This type of interaction is described as near-resonant. Although permanent waves are linearly stable to periodic disturbances in the same direction, the near-resonant interactions that occur between the periodic disturbance and all the harmonics of a cnoidal wave make the margin of stability to such a disturbance vanishingly small [5].

2. Theory

When the water surface consists of a number of discrete Fourier wave components, the surface displacement may be written

$$\eta(\mathbf{x}, t) = R \sum B(\mathbf{k}, t) \exp i(\mathbf{k}.\, \mathbf{x} - \omega(\mathbf{k})t), \tag{2.1}$$

where the time dependence of $B(\mathbf{k}, t)$ is due to the non-linear interactions between wave components. A permanent wave propagating with wave velocity c in the x-direction has the form

$$\eta(\mathbf{x}, t) = R \sum a(k) \exp ik(x - ct). \tag{2.2}$$

In considering the stability of such a wave, a more convenient Fourier representation than Eq. (2.1) for the disturbed wave is

$$\eta(\mathbf{x}, t) = R \sum A(\mathbf{k}, t) \exp i\mathbf{k}.\, (\mathbf{x} - \mathbf{c}t). \tag{2.3}$$

It may be shown then that the wave amplitudes $A(\mathbf{k}, t)$ satisfy

$$\frac{dA(\mathbf{k})}{dt} - i(\mathbf{k}.\, \mathbf{c} - \omega(\mathbf{k}))A(\mathbf{k}) = -i\varepsilon \sum_{\mathbf{l}+\mathbf{m}=\mathbf{k}} S(\mathbf{l}, \mathbf{m})A(\mathbf{l})A(\mathbf{m}) - i\varepsilon \sum_{\mathbf{l}-\mathbf{m}=\mathbf{k}} R(\mathbf{l}, \mathbf{m})A(\mathbf{l})A^*(\mathbf{m}) + 0(\varepsilon^2), \tag{2.4}$$

where the coefficients $S(\mathbf{l}, \mathbf{m})$, $R(\mathbf{l}, \mathbf{m})$ are functions of $\mathbf{l}$, $\mathbf{m}$, $\omega(\mathbf{l})$, $\omega(\mathbf{m})$, and kh [5], and* denotes complex conjugate.

The near resonant interaction between wave components on shallow water [Eq. (1.4)] leads to significant growth of the harmonics of those components present, a property for which experimental evidence is available [3]. Although the set of equations (2.4) may be solved analytically when a small number of wave components are present, the generation and near-resonant growth of further wave components makes it necessary, in general, to use a numerical solution for waves on shallow water [4].

The permanent waves are those solutions of Eqs. (2.4) for which $A(\mathbf{k}, t) = a(k)$ with $dA(\mathbf{k})/dt = 0$. The linear stability of the permanent waves is investigated by writing $A(\mathbf{k}, t) = a(k) + \hat{A}(\mathbf{k}, t)$ and linearising in $\hat{A}$. The detailed analysis for the case of disturbances parallel to the permanent wave (scalar k) has been described elsewhere [5].

3. Examples

Some of the stability properties of cnoidal waves are now described for two particular examples, (i) $\varepsilon = a/h = 0.05$, $h^2/l^2 = 0.2$ and (ii) $\varepsilon = a/h = h^2/l^2 = 0.05$, where a is wave amplitude and $2\pi l$ is wavelength. The first contains 6 harmonics exceeding 10^{-4} in magnitude, while the second contains 10 harmonics exceeding the same magnitude.

Equation (1.2) predicts oblique instability for the first permanent wave when, for example, $|\boldsymbol{\varkappa}| = 0.1/l$ and $\boldsymbol{\varkappa}$ makes an angle 0.127π with the wave direction. Maximum instability for $|\boldsymbol{\varkappa}| = 0.1/l$ is found numerically to occur at 0.1276π, when a disturbance of initial amplitude $\hat{a}$ grows according to

$$(3.1)\qquad \hat{A}(\boldsymbol{\varkappa}, t) = \hat{a}[(0.07\cosh 0.001\,\varepsilon t + 0.23\, i\sinh 0.001\varepsilon t)\exp(0.13\varepsilon t) + 0.93\exp(-0.23 i\varepsilon t)],$$

where t is measured in units of l/c_0. This is to be compared with the near-resonant stability of the same permanent wave when $\varkappa = 0.01/l$ parallel to the wave, when the side-band wave amplitudes are found to be

$$(3.2)\qquad \begin{aligned} \hat{A}(0.99/l, t) &= \hat{a}[-75.9\exp(0.003\, i\varepsilon t) + 65.5\exp(-0.004 i\varepsilon t) + 10.4\exp(-0.020\, i\varepsilon t)],\\ \hat{A}(1.01/l, t) &= \hat{a}[75.6\exp(-0.003\, i\varepsilon t) - 64.2\exp(0.004\, i\varepsilon t) - 11.4\exp(0.020\, i\varepsilon t))]. \end{aligned}$$

The error in Eqs. (2.4) is $0(\varepsilon^2)$, which means that the solutions in Eqs. (3.1), (3.2) are of doubtful validity for times t exceeding $\varepsilon^2 t = 0(1)$. Within this time interval, the amplification in the near-resonant case is over 100 times greater than in the resonant case. A small but finite parallel disturbance is amplified to such an extent that non-linear modification of the permanent wave occurs readily, dominating any resonant interaction due to the presence of an oblique disturbance.

Resonant instability is found not to occur for the more non-linear second example described above. Although Eq. (1.2) still predicts the possibility of instability, the modification of the linear dispersion relation by the increased number of harmonics is such that resonance does not occur. The near-resonant interaction between a parallel disturbance and the permanent wave is stronger than in the first example because the permanent wave in the second example has more harmonics closer to resonance. A typical interaction is illustrated in the figure. A disturbance of 0.1 times the amplitude and 4 times the wavelength is applied in phase with the permanent wave. The first crest of the disturbed wave train is increased in height and therefore speed, the second and fourth crests retain their undisturbed height and speed, and the third crest is reduced in height and speed. Although the disturbance is small enough that a linear perturbation analysis describes the initial behaviour of the wave train, consecutive wave crests become sufficiently close that a non-linear interaction occurs between them. The non-linear interaction takes the form of wave crests passing through one another, the property being the same as that found recently for solitary waves [6].

4. Conclusion

The sideband instability of Stokes waves has been advanced by Benjamin as an explanation of the observed disintegration with increasing distance of a Stokes wave train generated by a wavemaker. Although this instability is still present for some permanent waves on shallow water, it is overtaken in practical importance by near-resonant interaction with parallel periodic disturbances. For this reason, difficulty can be expected in generating a strictly periodic permanent wave train on shallow water, because any modulation of the wavemaker interacts significantly with the wave generated. It leads to non--linear modifications of the wave train and the appearance of secondary wave trains, as is illustrated in the figure.

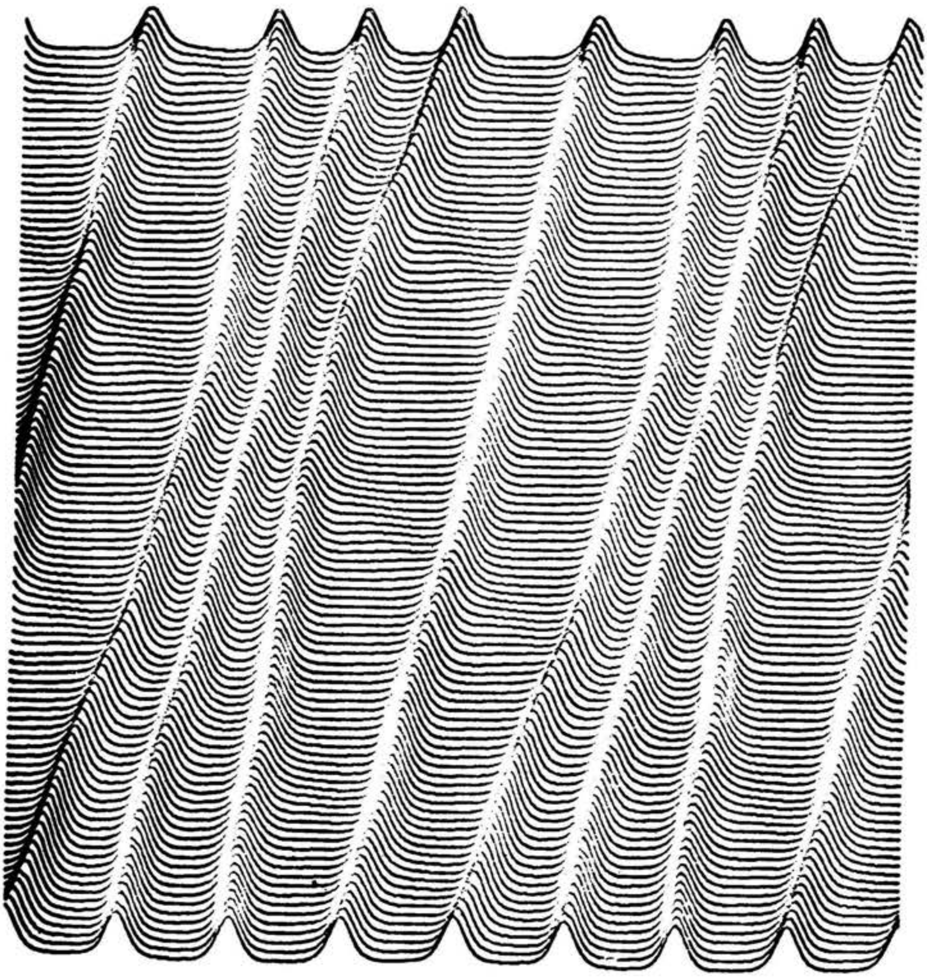

FIG. 1. The perturbation of a permanent wave by a disturbance of 4 times the wavelength and 0.1 times the amplitude of the wave. The x-axis extends over 8 wavelengths across the page and the t-axis extends over 50 periods up the page away from the viewer. The development of the wave is seen relative to an observer moving with the long wave velocity.

References

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