Archives of Mechanics • Archiwum Mechaniki Stosowanej • 28, 5-6, pp. 947-968, Warszawa 1976

Magnetohydrodynamic flow in a rectangular duct under a uniform transverse magnetic field at high Hartmann number

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IN THIS paper we consider fully developed, laminar, unidirectional flow of uniformly conducting, incompressible fluid through a rectangular duct of uniform cross-section. An externally applied magnetic field acts parallel to one pair of opposite walls and induced velocity and magnetic fields are generated in a direction parallel to the axis of the duct. The governing equations and boundary conditions for the latter fields are introduced and study is then concentrated on the special case of a duct having all walls non-conducting. For values of the Hartmann number $M \gg 1$, classical asymptotic analysis reveals the leading terms in the expansions of the induced fields in all key regions, with the exception of certain boundary layers near the corners of the duct. The order of magnitude of the affect of the latter layers on the flow-rate is discussed and closed-form solutions are obtained for the induced fields near the corners of the duct. Attempts were made to formulate a concise Principle of Minimum Singularity to enable the correct choice of eigen functions for the various field components in the boundary layers on the walls parallel to the applied field. It was found, however, that these components are best found by taking the outer expansion of the closed-form solution in those boundary-layers near the corners of the duct where classical asymptotic analysis is not applicable.

W niniejszej pracy rozważany jest w pełni rozwinięty jednokierunkowy przepływ laminarny nieściśliwej i nieprzewodzącej cieczy przez kanał prostokątny o równomiernym przekroju poprzecznym. Przyłożone z zewnątrz pole magnetyczne działa równolegle do jednej z par przeciwległych ścian, a wzbudzona prędkość i pole magnetyczne są wygenerowane w kierunku równoległym do osi kanału. Po wyprowadzeniu równań konstytutywnych i warunków brzegowych dla wygenerowanych pól magnetycznych i prędkości uwagę skoncentrowano na szczególnym przypadku kanału mającego wszystkie ścianki nieprzewodzące. Dla wartości liczb Hartmanna $M \gg 1$ klasyczna analiza asymptotyczna daje pierwsze wyrazy rozwinięcia pól indukowanych we wszystkich kluczowych obszarach z wyjątkiem pewnych warstw przyściennych w pobliżu naroży kanału. Przedyskutowano rząd wielkości oddziaływania warstw przyściennych na prędkość przepływu oraz otrzymano rozwiązania w postaci zamkniętej dla pól indukowanych w pobliżu naroży kanału. Poczyniono próbę sformułowania zwięzłej Zasady Minimum Osobliwości pozwalającej na prawidłowy wybór funkcji własnych dla poszczególnych składowych pola w warstwie przyściennej na ściankach równoległych do przyłożonego pola. Wykazano, że składowe te można najlepiej wyznaczyć biorąc zewnętrzne rozwinięcie rozwiązania ścisłego w tych warstwach przyściennych w pobliżu naroży, gdzie analiza asymptotyczna nie może być stosowana.

В настоящей работе рассмотрено вполне развернутое однонаправленное ламинарное течение несжимаемой и непроводящей жидкости через прямоугольный канал с равномерным поперечным сечением. Приложенное внешнее магнитное поле действует параллельно одной из пар противолежащих стенок, а возбужденные скорость и магнитное поле генерированы в параллельном направлении к оси канала. После введения определяющих уравнений и граничных условий для генерированных магнитных полей и скоростей внимание сосредоточено на частном случае канала имеющего все непроводящие стенки. Для значений чисел Гартмана $M \gg 1$ классический асимптотический анализ дает первые члены разложения полей индуцированных во всех главных областях за исключением некоторых пристеночных слоев вблизи углов канала. Обсужден порядок величины взаимодействия пристеночных слоев на скорость течения и получены решения в замкнутом виде для индуцированных полей вблизи углов канала. Предпринята попытка формулировки краткого Принципа Минимума Особенностей, позволяющего правильно подобрать собственные функции для отдельных составляющих поля в пристеночном слое на стенках параллельных приложенному полю. Показано, что эти составляющие можно наилучшим образом определить, принимая внешнее разложение точного решения в этих пристеночных слоях вблизи углов, где асимптотический анализ не может применяться.

1. Introduction

THE FULLY developed, laminar, unidirectional flow of uniformly conducting, incompressible fluid through rectangular ducts of uniform cross-section, subject to the application of a uniform transverse magnetic field, has attracted much attention from theoreticians and experimentalists over the past twenty years. Much of this work has been summarised by REGIRER *et al.* [1], HUNT and STEWARTSON [2], and HUNT and SHERCLIFF [3]. The channel walls are usually assumed thin with respect to the internal dimensions of the duct, as is normally the case in experimental work.

Similar problems to that considered herein arise in heat convection (see BOUSSINESQ [4]), in contained rotating fluids (see GREENSPAN [5]), and in the static response of a membrane strip (see OLUNLOYO *et al.* [6]), This is however, only a partial list of other areas of application.

2. The general MHD problem

The governing equations (see, for example, TEMPERLEY and TODD [7]) are

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+M\frac{\partial b}{\partial y} = -1, \tag{2.1}$$

$$\frac{\partial^2 b}{\partial x^2}+\frac{\partial^2 b}{\partial y^2}+M\frac{\partial v}{\partial y} = 0, \tag{2.2}$$

where v and b are the normalised induced velocity and magnetic fields in the direction of flow (z direction) and $M \gg 1$ is a parameter, the Hartmann number, defined by $M = B_0 a'(\sigma_f)^{\frac{1}{2}}/(\varrho\nu)^{\frac{1}{2}}$. Here B_0 and a' represent the magnitude of the applied magnetic field and the semi-width of the fluid cross-section (both prior to normalisation), and σ_f, ϱ, ν represent the electrical conductivity, density and shear diffusivity of the fluid ($\nu = \bar{\eta}\varrho^{-1}$, $\bar{\eta}$ being the viscosity of the fluid).

Figure 1 shows the typical duct cross-section, internal dimensions $2l$ units by 2 units (l taken as $0(1)$ for simplicity), the origin for coordinates being taken at the centre of the cross-section. The boundary conditions on v, b are

$$v = 0 \quad \text{at} \quad x = \pm l \quad \text{and at} \quad y = \pm 1, \tag{2.3}$$

$$\frac{\partial b}{\partial x} = \mp D_A b \quad \text{at} \quad x = \pm l, \qquad \frac{\partial b}{\partial y} = \mp D_B b \quad \text{at} \quad y = \pm 1, \tag{2.4}$$

where

$$D_{\text{wall}} = \left[\frac{\sigma_f}{\sigma_{\text{wall}}}\right]\cdot(\text{dimensionless wall-thickness})^{-1}.$$

If $\sigma_{\text{wall}} = 0$, then $D_{\text{wall}} = \infty$ and the relevant boundary condition on b at such a fluid-wall interface becomes $b = 0$. For ducts where $\sigma_A = 0$, exact solutions to the problem exist for all M (see SHERCLIFF [8] for $\sigma_A = 0 = \sigma_B$, and SLOAN and SMITH [9] for $\sigma_A = 0$

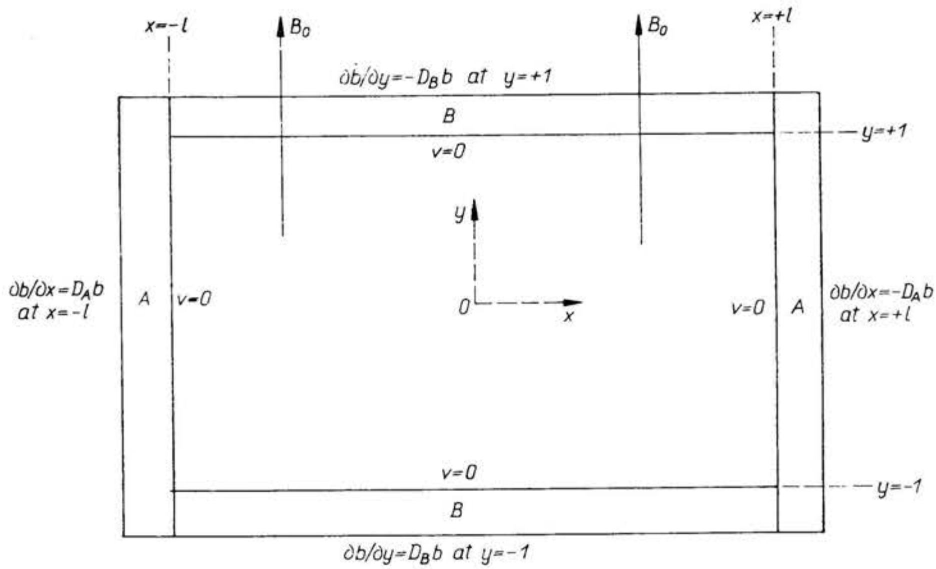

FIG. 1. Cross-section taken through the rectangular duct with externally applied magnetic field B_0 in the y-direction.

$\sigma_B > 0$ and any wall-thickness at the top and bottom boundaries of the cross-section(*)). HUNT [10] has shown the uniqueness of the solution to this class of problems.

3. The sub-class of problems for which $D_A = \infty$, $D_B = \infty$, i.e., $b = 0$ at $x = \pm l$ and at $y = \pm 1$

We wish to obtain the asymptotic solution, as a power series in M^{-1}, of the coupled partial differential Eqs. (2.1) and (2.2) subject to the boundary conditions $v = 0 = b$ at $x = \pm l$ and at $y = \pm 1$, utilising the features that v is even in x and y and b is even in x and odd in y (see [2], p. 567). Introducing

$$u = v+b+M^{-1}(1+y), \quad w = v-b-M^{-1}(1+y), \tag{3.1}$$

$\big($so that $2v = u+w$, $2b = u-w-2M^{-1}(1+y)\big)$ it follows that

$$\nabla^2 u + M\frac{\partial u}{\partial y} = 0, \quad \text{and} \quad \nabla^2 w - M\frac{\partial w}{\partial y} = 0. \tag{3.2}$$

We will obtain an expansion for $u(x, y)$ in $-l \leqslant x \leqslant 0$; other results for u and w then follow using symmetry considerations. Figure 2 shows the boundary conditions satisfied by u and in Fig. 3 the boundary layers for u when $M \gg 1$ are illustrated.

(*) Note, however, that the problem of obtaining an exact solution to the configuration where $\sigma_A = 0$, $\sigma_B > 0$ and the regions where the walls overlap have conductivity σ_B, rather than σ_A, seems intractable (see TEMPERLEY [11]).

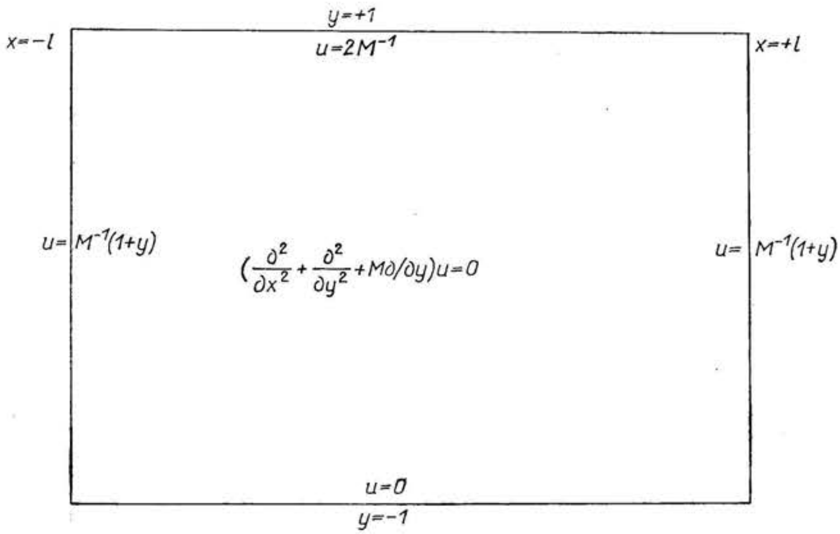

FIG. 2. Governing equation and boundary conditions satisfied by u.

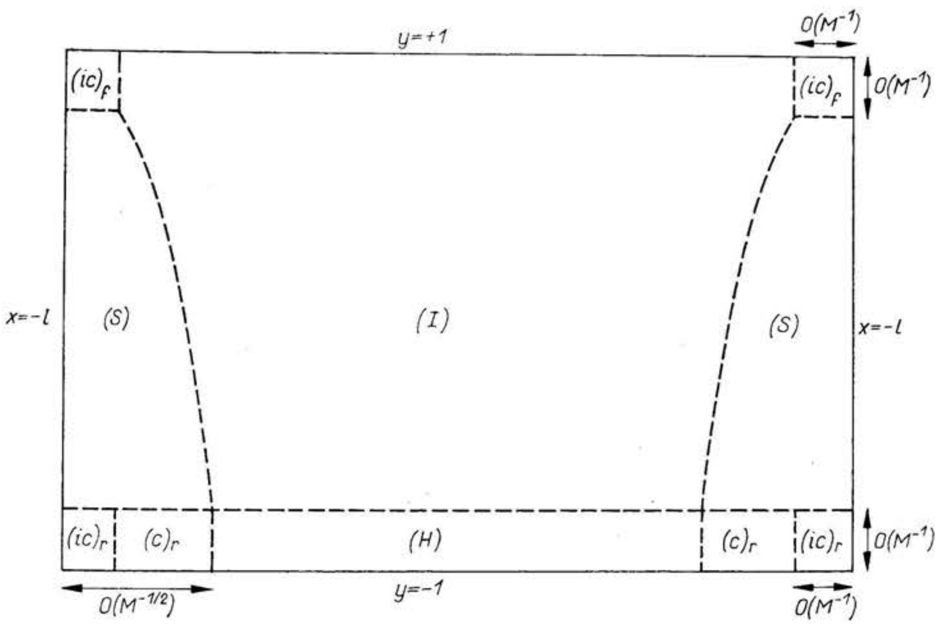

FIG. 3. Boundary layers for u when $M \gg 1$ (not to scale).

The lettering (I), (H), (s), $(ic)_f$, $(c)_r$ and $(ic)_r$ denotes respectively the interior (or core) region, the Hartmann (or primary), side (or secondary), front (or forward) inner-corner, rear corner and rear inner-corner boundary layers for u. The front inner-corners are situated at the leading edges of the parabolic boundary-layers on the side walls and the rear corner and rear inner-corner layers lie at the trailing edges.

Boundary-layer coordinates relevant to an analysis for the region $-l \leqslant x \leqslant 0$, $-1 \leqslant y \leqslant +1$ are

$$(3.3) \qquad X = M^{\frac{1}{2}}(l+x), \quad \chi = M(l+x), \quad Y^{\circ} = M(1+y), \quad Y = M(1-y).$$

This configuration (often referred to as the Shercliff duct) is the only one for which both the governing equations and the boundary conditions decouple. One could, of course, analyse the original pair of coupled Eqs. (2.1) and (2.2) directly; however, the analysis presented herein will most clearly reveal the nature of the difficulties which arise when an asymptotic expansion of the solution in powers of M^{-1} is sought. Similar difficulties occur for other rectangular duct problems.

4. Classical asymptotic analysis for $M \gg 1$

In the (I) region, the solution for u may be taken as $u \sim u_I = 2M^{-1}$; in the (H) layer a correction term $u_H(x, Y^{\circ})$ must be added to u_I in order to satisfy $u(x, -1) = 0$. Similarly, a correction $u_s(X, y)$ must be added to u_I in the (s) layer in order to satisfy $u(-l, y) = M^{-1}(1+y)$, and further corrections $(u_c)_r$ and $(u_{ic})_r$ must be added in the $(c)_r$ and $(ic)_r$ layers, for similar reasons. In the $(ic)_f$ region one adds to u_I a correction term $\{u_{ic}(\chi, Y)\}_f$. However, $(u_{ic})_f$ is not a correction to (u_I+u_s); instead $((u_{ic})_f+u_I)$ matches with (u_s+u_I) as we move from the $(ic)_f$ layer into the (s) layer (see Sec. 6).

We shall give closed-form solutions for u_I and u_H, the errors in which are transcendentally small in M, and shall find the leading terms in the expansions of u_s, $(u_c)_r$ and $(u_{ic})_r$ in powers of M^{-1}. Though $(u_{ic})_f$ will not be obtained in detail, sufficient information about $(u_{ic})_f$ will be obtained in order to estimate the flow-rate to $0(M^{-4})$. This is what constitutes the classical approach. Based on these results we can estimate the leading terms in the volumetric flow-rate in powers of M^{-1} (this will be done in a later paper).

4.1. The (I) and (H) Regions

Away from the boundary-layers on the walls,

$$(4.1) \qquad u \sim u_I = u(x, 1) = 2M^{-1},$$

while in the (H) layer the condition $u(x, -1) = 0$ is satisfied by adding to u_I a correction term

$$(4.2) \qquad u_H = -2M^{-1}e^{-M(1+y)} = -2M^{-1}e^{-Y^{\circ}}.$$

That is, the solution outside the boundary-layers on the side walls, neglecting asymptotically exponentially small terms, may be taken as

$$(4.3) \qquad u \sim u_I + u_H = 2M^{-1}(1-e^{-Y^{\circ}}).$$

4.2. The side layer

In the (s) layer on $x = -l$, a correction term $u_s(X, y)$ is added to u_I to satisfy $u(-l,y) = M^{-1}(1+y)$. Assuming that u_s may be expanded asymptotically for $M \gg 1$ as

$$u_s \sim \sum_{n=1}^{\infty} u_s^{(n)}(X, y) M^{-n}, \tag{4.4}$$

substituting into the first of Eqs. (3.2) and the boundary conditions of Fig. 2 yields

$$\begin{aligned} &\left(\frac{\partial^2}{\partial X^2} + \frac{\partial}{\partial y}\right) u_s^{(1)} = 0, \\ &\left(\frac{\partial^2}{\partial X^2} + \frac{\partial}{\partial y}\right) u_s^{(n)} = -\frac{\partial^2 u_s^{(n-1)}}{\partial y^2}, \quad \text{for} \quad n \geqslant 2, \end{aligned} \tag{4.5}$$

with

$$\begin{aligned} u_s^{(1)}(0, y) &= y-1, \\ u_s^{(n)}(0, y) &= 0, \quad \text{for} \quad n \geqslant 2, \\ u_s^{(n)}(X, 1) &= 0, \quad \text{for} \quad n \geqslant 1, \quad X > 0, \\ u_s^{(n)} &\to 0 \quad \text{as} \quad X \to \infty, \quad \text{for} \quad n \geqslant 1. \end{aligned} \tag{4.6}$$

This set of equations may be solved using Laplace transforms with respect to $(1-y)$, a method seemingly adopted for a similar problem by COOK, LUDFORD and WALKER [12]. Due to the passivity of the $(c)_r$ and $(ic)_r$ layers on $y = -1$ with respect to the (s) layer, we can treat the (s) layer problem as extending to $y = -\infty$; appropriate $(c)_r$ and $(ic)_r$ correction terms may then be added in order to satisfy the boundary condition at $y = -1$.

Introducing

$$\tilde{u}_n(X, s) = \int_{1-y=0}^{1-y=\infty} u_s^{(n)} \cdot e^{-s(1-y)} d(1-y), \tag{4.7}$$

multiplication of Eq. $(4.5)_1$ by $e^{-s(1-y)}$ and integration from $(1-y) = 0$ to $(1-y) = \infty$ yields, using condition $(4.6)_3$,

$$\left(\frac{d^2}{dX^2} - s\right) \tilde{u}_1 = 0. \tag{4.8}$$

From condition $(4.6)_4$ and on applying the "automatic" boundary condition

$$\tilde{u}_1(0, s) = \mathscr{L}\{u_s^{(1)}(0, y)\} = \mathscr{L}\{y-1\} = -s^{-2}, \tag{4.9}$$

the solution of Eq. (4.8) is

$$\tilde{u}_1 = -s^{-2} e^{-Xs^{\frac{1}{2}}}. \tag{4.10}$$

Hence (see ERDÉLYI et al. ([13], p. 245))

$$u_s^{(1)}(X, y) = X(1-y)^{\frac{1}{2}} \pi^{-\frac{1}{2}} \exp\{-X^2/4(1-y)\} - \left(1 - y + \frac{1}{2}X^2\right) \operatorname{erfc}\{X/2(1-y)^{\frac{1}{2}}\}. \tag{4.11}$$

The latter result may also be written in the forms

$$(4.11')\qquad u_s^{(1)} = -\int\limits_0^{1-y} \operatorname{erfc}\{X/2(1-y-\beta)^{\frac{1}{2}}\}d\beta = -\int\limits_0^{1-y} \operatorname{erfc}(X/2\theta^{\frac{1}{2}})d\theta$$

$$= -(1-y)+\int\limits_0^{1-y} \operatorname{erf}(X/2\theta^{\frac{1}{2}})d\theta.$$

Expression (4.11′) is that first obtained by CHANG and LUNDGREN [14] and both (4.11) and (4.11′) are alternative forms to that given by SHERCLIFF ([8], p. 140).

Our results for $u_s^{(1)}$ is bounded and continuous at all points inside and on the boundary of the rectangle. If the "automatic" condition (4.9) were to be relaxed, the addition of a term

$$A(s)e^{-Xs^{\frac{1}{2}}} = e^{-Xs^{\frac{1}{2}}} \sum_{i=0}^{\infty} A_i s^i,$$

the $A_i's$ being constants, to expression (4.10) would introduce eigen functions into the solution for $u_s^{(1)}$; these correspond to multiples of the various $(1-y)$ derivatives of $\operatorname{erfc}\{X/2(1-y)^{\frac{1}{2}}\}$ (see [13], p. 245) and are rejected, as they are singular at the corner $X = 0 = 1-y$.

Setting $n = 2$ in Eq. $(4.5)_2$ and proceeding as for $\tilde{u}_1, u_s^{(1)}$, we have (using condition $(4.6)_3$)

$$(4.12)\qquad \left(\frac{d^2}{dX^2}-s\right)\tilde{u}_2 = -\mathscr{L}\{\partial^2 u_s^{(1)}/\partial y^2\} = -s^2\mathscr{L}\{u_s^{(1)}\} = e^{-Xs^{\frac{1}{2}}}.$$

From condition $(4.6)_4$ and use of the "automatic" condition

$$(4.13)\qquad \tilde{u}_2(0,s) = \mathscr{L}\{u_s^{(2)}(0,y)\} = 0,$$

we conclude that

$$(4.14)\qquad \tilde{u}_2 = -Xe^{-Xs^{\frac{1}{2}}}/2s^{\frac{1}{2}},$$

and hence (see [13], p. 246)

$$(4.15)\qquad u_s^{(2)} = -X\exp\{-X^2/4(1-y)\}/2\pi^{\frac{1}{2}}(1-y)^{\frac{1}{2}} = (1-y)\frac{\partial}{\partial(1-y)}(\operatorname{erf}\{X/2(1-y)^{\frac{1}{2}}\});$$

$u_s^{(2)}$ is bounded everywhere inside and on the boundary of the rectangle, despite the fact that the right-hand side of the governing equation for $u_s^{(2)}$ equals

$$(4.16)\qquad \frac{-\partial^2 u_s^{(1)}}{\partial y^2} = X\exp\{-X^2/4(1-y)\}/2\pi^{\frac{1}{2}}(1-y)^{\frac{3}{2}},$$

which may be shown to behave like a multiple of $\delta'(X)$ as $(1-y)\to 0$. Use of the "automatic" condition (4.13) may thus be suspect and liable to cause difficulties, but all turns out well in the final result. Relaxation of condition (4.13), and proceeding as suggested in the analysis for $\tilde{u}_1, u_s^{(1)}$, leads to eigen-functions for $u_s^{(2)}$ which must be rejected for being singular at the corner. Note that $u_s^{(2)}$, though bounded, is not uniquely defined at the corner; approaching $X = 0 = 1-y$ along any parabola $X^2 = 4\mu^2(1-y)$, $\mu > 0$ and constant, gives a limiting value $-\mu e^{-\mu^2}\pi^{-\frac{1}{2}}$ at the corner.

An interesting, and more direct, alternative method of solving the $u_s^{(2)}$ problem was devised by Professor L. TODD (Laurentian University, Sudbury, Ontario, Canada) during collaborative research between himself and the author in 1974. Since the governing equation for $u_s^{(2)}$ takes the form (see (4.11′))

$$(4.17)\qquad \left(\frac{\partial^2}{\partial X^2}+\frac{\partial}{\partial y}\right)u_s^{(2)} = -\frac{\partial}{\partial y}\left(\frac{\partial u_s^{(1)}}{\partial y}\right) = -\frac{\partial}{\partial(1-y)}\left(\operatorname{erf}\{X/2(1-y)^{\frac{1}{2}}\}\right),$$

there thus exists a particular integral

$$(4.18)\qquad (u_s^{(2)})_p = (1-y)\frac{\partial}{\partial(1-y)}\left(\operatorname{erf}\{X/2(1-y)^{\frac{1}{2}}\}\right).$$

The complementary function, $(u_s^{(2)})_c$, is readily seen to be more singular at $X = 0 = 1-y$ than is expression (4.18) and so is excluded using a Minimum Singularity argument. Hence, $u_s^{(2)} \equiv (u_s^{(2)})_p$, which matches with result (4.15).

The governing equation for $u_s^{(3)}$ (see result (4.18)) is

$$(4.19)\qquad \left(\frac{\partial^2}{\partial X^2}+\frac{\partial}{\partial y}\right)u_s^{(3)} = -\frac{\partial^2 u_s^{(2)}}{\partial y^2} = -\frac{\partial^2}{\partial(1-y)^2}\left\{(1-y)\frac{\partial}{\partial(1-y)}\left(\operatorname{erf}\{X/2(1-y)^{\frac{1}{2}}\}\right)\right\}$$
$$= -\left\{2\frac{\partial^2}{\partial(1-y)^2}+(1-y)\frac{\partial^3}{\partial(1-y)^3}\right\}\left(\operatorname{erf}\{X/2(1-y)^{\frac{1}{2}}\}\right),$$

which, using the "improved" Todd method, has a solution

$$(4.20)\qquad u_s^{(3)} = (u_s^{(3)})_p = \left\{2(1-y)\frac{\partial^2}{\partial(1-y)^2}+\frac{1}{2}(1-y)^2\frac{\partial^3}{\partial(1-y)^3}\right\}\left(\operatorname{erf}\{X/2(1-y)^{\frac{1}{2}}\}\right),$$

in which the individual terms are singular at the corner $X = 0 = 1-y$.

The Laplace transform method, on the other hand, leads to the equation

$$(4.21)\qquad \left(\frac{d^2}{dX^2}-s\right)\tilde{u}_3 = \frac{1}{2}Xs^{\frac{3}{2}}e^{-Xs^{\frac{1}{2}}},$$

having a general solution (see condition $(4.6)_4$)

$$(4.22)\qquad \tilde{u}_3 = \{A(s)-Xs^{\frac{1}{2}}(1+Xs^{\frac{1}{2}})/8\}e^{-Xs^{\frac{1}{2}}}.$$

If the "automatic" condition $\tilde{u}_3(0, s) = 0$ is applied, then $A(s) \equiv 0$ and hence (see [13], pp. 245/6)

$$(4.23)\qquad u_s^{(3)} = U_s^{(3)} = X\left\{4+\frac{4X^2}{(1-y)}-\frac{X^4}{(1-y)^2}\right\}\frac{\exp\{-X^2/4(1-y)\}}{64\pi^{\frac{1}{2}}(1-y)^{\frac{3}{2}}}.$$

On multiplying the latter expression by any "good" trial function $F(x)$ an integrating the product over $-\infty < X < \infty$, it may be shown (see LIGHTHILL [15], p. 15) that $U_s^{(3)}$ behaves like $2\delta'(X)$ as $(1-y) \to 0$; a similar procedure, using any "good" trial function $G(1-y)$ and integrating over $0 \leqslant 1-y < \infty$, shows that $U_s^{(3)}$ displays no $\delta^{(p)}(1-y)$, $p \geqslant 0$, behaviour as $X \to 0$.

The $\delta'(X)$ behaviour may be curbed by adding to expression (4.23) an appropriate eigen function. Relaxing the "automatic" condition for $\tilde{u}_3(0, s)$ and expanding $A(s)$ in

(4.22) as $\sum_{i=0}^{\infty} A_i s^i$, the A_i' being constants, those terms corresponding to $i \geqslant 1$ are found to give rise to eigen-functions for $u_s^{(3)}$ that are more singular than $U_s^{(3)}$ as $(1-y) \to 0$, and so are rejected. The eigen function corresponding to $A(s) \equiv A_0$, however, is

$$(4.24) \qquad u_{sef}^{(3)} = A_0 X \exp\{-X^2/4(1-y)\}/2\pi^{\frac{1}{2}}(1-y)^{\frac{3}{2}},$$

which behaves like $-2A_0\delta'(X)$ as $(1-y) \to 0$. Hence all $\delta'(X)$ behaviour can be annihilated by choosing $A_0 = 1$. However, the addition of $u_{sef}^{(3)}$ introduces into the solution for $u_s^{(3)}$ a lower-order singularity $A_0\,\delta(1-y) = \delta(1-y)$ as $X \to 0$. One is thus faced with the highly complex problem of formulating a suitable, "appropriate" Principle of Minimum Singularity, applicable to choosing the correct $u_s^{(n)}$ for all $n \geqslant 3$. We have seen how the choice of $A_0 = 1$ in $u_{sef}^{(3)}$ deletes the $\delta'(X)$-type singularity whilst introducing one of type $\delta(1-y)$. Attempts were made to justify the choice $A_0 = 1$ by two further methods; these, however, proved only partially successful. A brief description of the methods adopted is given in the Appendix.

Our final solution for $u_s^{(3)}$ is

$$(4.25) \qquad u_s^{(3)} = U_s^{(3)} + u_{sef}^{(3)} = X\{36 + 4X^2/(1-y) - X^4/(1-y)^2\}\frac{\exp\{-X^2/4(1-y)\}}{64\pi^{\frac{1}{2}}(1-y)^{\frac{3}{2}}}.$$

This result checks with (4.20), obtained by the "improved" Todd method, which automatically produces the least singular solution at the corner. Our solution for $u_s^{(3)}$ is, of course, singular at the corner, as can be seen by approaching the latter via any straight line path $X/(1-y) = \text{constant}$ or any parabola $X^2/(1-y) = \text{constant}$, for example.

The governing equation for $u_s^{(4)}$ is (see result (4.20))

$$(4.26) \qquad \left(\frac{\partial^2}{\partial X^2} + \frac{\partial}{\partial y}\right) u_s^{(4)} = \frac{-\partial^2 u_s^{(3)}}{\partial y^2}$$
$$= -\left\{5\frac{\partial^3}{\partial(1-y)^3} + 4(1-y)\frac{\partial^4}{\partial(1-y)^4} + \frac{1}{2}(1-y)^2\frac{\partial^5}{\partial(1-y)^5}\right\}\left(\operatorname{erf}\{X/2(1-y)^{\frac{1}{2}}\}\right),$$

and the Todd method leads immediately to a solution

$$(4.27) \qquad u_s^{(4)} \equiv (u_s^{(4)})_p$$
$$= \left\{5(1-y)\frac{\partial^3}{\partial(1-y)^3} + 2(1-y)^2\frac{\partial^4}{\partial(1-y)^4} + \frac{1}{6}(1-y)^3\frac{\partial^3}{\partial(1-y)^5}\right\}\left(\operatorname{erf}\{X/2(1-y)^{\frac{1}{2}}\}\right),$$

while the Laplace transform method gives

$$(4.28) \qquad \left(\frac{d^2}{dX^2} - s\right)\tilde{u}_4 = -\mathscr{L}\left\{\frac{\partial^2 u_s^{(3)}}{\partial y^2}\right\} = -s^2\tilde{u}_3 = \frac{s^2}{8}(X^2 s + X s^{\frac{1}{2}} - 8)e^{-Xs^{\frac{1}{2}}},$$

and hence (see condition $(4.6)_4$

$$(4.29) \qquad u_s^{(4)} = \mathscr{L}^{-1}\left\{\frac{1}{8}\left(48B(s) + Xs^{\frac{3}{2}}(21 - 3Xs^{\frac{1}{2}} - X^2 s)\right)e^{-Xs^{\frac{1}{2}}}\right\}.$$

Applying the "automatic" condition $\tilde{u}_4(0, s) = 0$ yields $B(s) \equiv 0$ and hence (see [13], pp. 245/6)

(4.30) $u_s^{(4)} = U_s^{(4)}$

$$= \frac{X}{3\pi^{\frac{1}{2}} \cdot 2^{10}(1-y)^{\frac{5}{2}}} \left\{1008 - \frac{1248X^2}{1-y} + \frac{24X^4}{(1-y)^2} + \frac{24X^6}{(1-y)^3} - \frac{X^8}{(1-y)^4}\right\} \exp\{-X^2/4(1-y)\}.$$

$U_s^{(4)}$ behaves like $\{2\delta'''(X) + 0 \cdot \delta'(V)\}$ as $(1-y) \to 0$, but contains no singularity of the type $\delta^{(p)}(1-y)$, $p \geqslant 0$, as $X \to 0$. To annihilate the $\delta'''(X)$ singularity, the "automatic" condition is relaxed. The choice $B(s) \equiv B_0 + B_1 s$, B_0 and B_1 constants, adds to $u_s^{(4)}$ an eigen function

$$(4.31) \qquad u_{sef_i}^{(4)} = \left\{\frac{3B_0 X}{(1-y)^{3/2}} + \frac{B_1 X\big(-6 + X^2/(1-y)\big)}{8(1-y)^{5/2}}\right\} \pi^{-\frac{1}{2}} \exp\{-X^2/4(1-y)\},$$

which behaves like $\{-12B_0\delta'(X) - B_1\delta'''(X)\}$ as $(1-y) \to 0$. Thus all $\delta'''(X)$, $\delta'(X)$ behaviour can be annihilated by choosing $B_0 = 0$ and $B_1 = 2$.

Any terms in $B(s)$ of $O(s_s^k)$, $k \geqslant 2$, introduce $\delta^{(2k+1)}(X)$-type singularities as $(1-y) \to 0$, and so are excluded.

The addition of expression (4.31) to $U_s^{(4)}$ introduces into $u_s^{(4)}$ behaviour of the type $6B_0\,\delta(1-y) + \dfrac{1}{4} B_1\,\delta'(1-y)$ as $X \to 0$, and hence our final (least singular) solution,

$$(4.32) \qquad u_s^{(4)} = \frac{X}{3\pi^{\frac{1}{2}} \cdot 2^{10}(1-y)^{\frac{5}{2}}} \left\{-3600 - \frac{480X^2}{1-y} + \frac{24X^4}{(1-y)^2} + \frac{24X^6}{(1-y)^3} - \frac{X^8}{(1-y)^4}\right\} \times$$
$$\times \exp\{-X^2/4(1-y)\},$$

behaves like $\dfrac{1}{2}\delta'(1-y)$ as $X \to 0$. It may readily be shown that expressions (4.32) and (4.27) are equivalent.

For the general term $u_s^{(n)}$, $n \geqslant 2$, the "improved" Todd method always provides the least singular solution at the corner $X = 0 = 1-y$, without any reference to the solution in the $(ic)_f$ layer, contrary to the beliefs expressed in [12], where it was held that such a procedure could not be carried out. The general Todd result for $n \geqslant 2$ is

(4.33) $u_s^{(n)} \equiv (u_s^{(n)})_p$

$$= \left\{C_q(1-y)\frac{\partial^{n-1}}{\partial(1-y)^{n-1}} + C_{q+1}(1-y)^2\frac{\partial^n}{\partial(1-y)^n} + \ldots + C_{q+n-2}(1-y)^{n-1}\frac{\partial^{2n-3}}{\partial(1-y)^{2n-3}}\right\}$$
$$\times \operatorname{erf}\{X/2/(1-y)^{\frac{1}{2}}\},$$

where $q = (n^2 - 3n + 4)/2$, the C_i' being appropriately chosen constants, which may be expressed as functions of n by setting up a system of recurrence relations.

The Laplace transform method, on the other hand, leads to an initial expression $U_s^{(n)}$, if an "automatic" condition for $\tilde{u}_n(0, s)$ is applied, containing a $\delta^{(2n-5)}(X)$-type singularity, $n \geqslant 3$, as $(1-y) \to 0$. This singularity is then annihilated by introducing into $\tilde{u}_n$ a complementary function of the form $E_{n-3}\, s^{n-3}\, e^{-Xs^{\frac{1}{2}}}$; for an appropriate choice of E_{n-3}, the corresponding eigen solution $u_{sef}^{(n)}$ combines with $U_s^{(n)}$ to form a final solution $u_s^{(n)}$ which exhibits no $\delta^{(p)}(X)$, $p \geqslant 0$, behaviour as $(1-y) \to 0$ but which behaves like a multiple of $\delta^{(n-3)}(1-y)$ as $X \to 0$. That is, at each level beyond $n = 2$, a "strong" singularity of the $\delta^{(2n-5)}(X)$-type is replaced by a "weaker" one in $\delta^{(n-3)}(1-y)$.

A further alternative method of obtaining u_s to all orders is the use of Fourier sine transforms with respect to X. Introducing

$$\bar{u}_n(\lambda, y) = \int_0^\infty u_s^{(n)} \sin(\lambda X)\, dX, \tag{4.34}$$

and multiplying the governing equation for each $u_s^{(n)}$ by sin (λX), then integrating from $X = 0$ to $X = \infty$, etc., as outlined, for example, in SNEDDON[16], the solution for each $\bar{u}_n$ is found using conditions $(4.6)_1 \to (4.6)_4$ and each $u_s^{(n)}$ then follows using the inversion formula

$$u_s^{(n)} = \frac{2}{\pi} \int_0^\infty \bar{u}_n(\lambda, y) \sin(\lambda X)\, d\lambda. \tag{4.35}$$

This method yields a unique solution, the least singular solution, at all levels. That is, a minimum singularity principle is "built-in" to the Fourier sine transform method which is lacking in the Laplace transform method, and the former is thus to be preferred when dealing with the (s) layer in this particular configuration.

4.3. The rear corner layer

In the $(c)_r$ layer on $y = -1$, one adds to $(u_I + u_H + u_s)$ a correction term $\{u_c(X, Y_0)\}_r$ in order to satisfy the condition $u(x, -1) = 0$. Since $(u_I + u_H)$ satisfies this condition identically, we simply substitute $(u_c)_r$ into the first of Eqs. (3.2) and use the boundary conditions from Fig. 2 to deduce that

$$\left(M^{-1} \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^{\circ 2}} + \frac{\partial}{\partial Y^\circ}\right)(u_c)_r = 0, \tag{4.36}$$

with

$$\begin{aligned} &\big(u_c(X, 0)\big)_r = -u_s(X, -1), \\ &(u_c)_r \to 0 \quad \text{as} \quad X \quad \text{and/or} \quad Y^\circ \to \infty. \end{aligned} \tag{4.37}$$

Expanding $(u_c)_r$ asymptotically as

$$(u_c)_r = \sum_{n=1}^\infty u_c^{(n)}(X, Y^\circ) M^{-n}, \tag{4.38}$$

and substituting into Eq. (4.36), we have

$$\begin{aligned} &\left(\frac{\partial^2}{\partial Y^{\circ 2}} + \frac{\partial}{\partial Y^\circ}\right) u_c^{(1)} = 0, \\ &\left(\frac{\partial^2}{\partial Y^{\circ 2}} + \frac{\partial}{\partial Y^\circ}\right) u_c^{(n)} = -\frac{\partial^2 u_c^{(n-1)}}{\partial X^2}, \quad n \geqslant 2. \end{aligned} \tag{4.39}$$

From Eq. $(4.39)_1$ and conditions (4.37) plus results (4.11'),

$$u_c^{(1)} = -u_s^{(1)}(X, -1)e^{-Y^\circ} = e^{-Y^\circ}\int_0^2 \operatorname{erfc}(X/2\theta^{\frac{1}{2}})\,d\theta. \tag{4.40}$$

$u_c^{(2)}$ satisfies

$$\left(\frac{\partial^2}{\partial Y^{\circ 2}}+\frac{\partial}{\partial Y^\circ}\right)u_c^{(2)} = -\frac{\partial^2 u_c^{(1)}}{\partial X^2} = e^{-Y^\circ}\frac{d^2}{dX^2}\{u_s^{(1)}(X, -1)\} = -e^{-Y^\circ}\operatorname{erfc}(X/2^{\frac{3}{2}}), \tag{4.41}$$

with

$$u_c^{(2)}(X, 0) = -u_s^{(2)}(X, -1) = \frac{Xe^{-X^2/8}}{(8\pi)^{\frac{1}{2}}}, \tag{4.42}$$

$$u_c^{(2)} \to 0 \quad \text{as} \quad X \quad \text{and/or} \quad Y^\circ \to \infty.$$

Hence

$$u_c^{(2)} = e^{-Y^\circ}\left\{\frac{X}{(8\pi)^{\frac{1}{2}}}e^{-X^2/8} - Y^\circ\frac{d^2}{dX^2}u_c^{(1)}(X, -1)\right\} = e^{-Y^\circ}\left\{\frac{X}{(8\pi)^{\frac{1}{2}}}e^{-X^2/8} + Y^\circ\operatorname{erfc}(X/2^{\frac{3}{2}})\right\}. \tag{4.43}$$

$u_c^{(3)}$ satisfies

$$\left(\frac{\partial^2}{\partial Y^{\circ 2}}+\frac{\partial}{\partial Y^\circ}\right)u_c^{(3)} = -\frac{\partial^2 u_c^{(2)}}{\partial X^2} = \frac{Xe^{-Y^\circ}}{(2^{11}\pi)^{\frac{1}{2}}}\left\{(12-X^2)e^{-X^2/8} + 2Y^\circ\int_0^1 (X^2s^2-12)s^2e^{-X^2s^2/8}ds\right\}, \tag{4.44}$$

with (see result (4.25))

$$u_c^{(3)}(X, 0) = -u_s^{(3)}(X, -1) = \frac{X(X^4-8X^2-144)e^{-X^2/8}}{(2^{19}\pi)^{\frac{1}{2}}}, \tag{4.45}$$

$$u_c^{(3)} \to 0 \quad \text{as} \quad X \quad \text{and/or} \quad Y^\circ \to \infty.$$

Hence

$$u_c^{(3)} = \frac{Xe^{-Y^\circ}}{(2^{19}\pi)^{\frac{1}{2}}}\left\{\left(X^4-8X^2-144+16Y^\circ(X^2-12)\right)e^{-X^2/8} - 16Y^\circ(Y^\circ+2)\int_0^1 (X^2s^2-12)s^2e^{-X^2s^2/8}ds\right\}. \tag{4.46}$$

We can continue indefinitely in this manner, expanding $(u_c)_r$ to the same order as u_s.

4.4. The rear inner-corner layer

In the $(ic)_r$ layer near $x = -l$, $y = -1$, a further correction term $\{u_{ic}(\chi, Y^\circ)\}_r$ must be added to those contributions already introduced, in order to satisfy $u(-l, y) = M^{-1}(1+y)$; $(u_{ic})_r$ satisfies

$$\left(\frac{\partial^2}{\partial\chi^2}+\frac{\partial^2}{\partial Y^{\circ 2}}+\frac{\partial}{\partial Y^\circ}\right)(u_{ic})_r = 0, \tag{4.47}$$

with

$$\begin{aligned} \{u_{ic}(0, Y^\circ)\}_r &= -u_H(-l, Y^\circ) - \{u_c(0, Y^\circ)\}_r; \\ \{u_{ic}(\chi, 0)\}_r &= 0, \\ (u_{ic})_r \to 0 \quad &\text{as} \quad \chi \quad \text{and/or} \quad Y^\circ \to \infty. \end{aligned} \tag{4.48}$$

Expanding $(u_{ic})_r$ asymptotically as

$$(u_{ic})_r = \sum_{n=1}^{\infty} u_{ic}^{(n)}(\chi, Y^\circ) M^{-n}, \tag{4.49}$$

it follows that since $u_{ic}^{(1)}$ satisfies Eq. (4.47) and conditions (4.48), and (see results (4.2), (4.40))

$$u_{ic}^{(1)}(0, Y^\circ) = -u_H(-l, 0) - u_c^{(1)}(0, Y^\circ) = 0, \tag{4.50}$$

so

$$u_{ic}^{(1)} \equiv 0. \tag{4.51}$$

From result (4.43),

$$u_{ic}^{(2)}(0, Y^\circ) = -u_c^{(2)}(0, Y^\circ) = -Y^\circ e^{-Y^\circ}, \tag{4.52}$$

and, on introducing

$$U_2(k, Y^\circ) = \int_0^{\infty} u_{ic}^{(2)} \sin(k\chi) d\chi, \tag{4.53}$$

multiplying the governing equation for $u_{ic}^{(2)}$ by $\sin(k\chi)$ and integrating over $0 \leqslant \chi < \infty$, using conditions $(4.48)_3$, (4.52) it follows that

$$\left(\frac{d^2}{dY^{\circ 2}} + \frac{d}{dY^\circ} - k^2\right) U_2 = kY^\circ e^{-Y^\circ}. \tag{4.54}$$

Since (see conditions (4.48)) $U_2 \to 0$ as $Y^\circ \to \infty$ and $U_2(k, 0) = 0$, so

$$U_2 = -k^{-1} Y^\circ e^{-Y^\circ} + k^{-3}(e^{-Y^\circ} - e^{-\beta Y^\circ}), \tag{4.55}$$

where $\beta = \dfrac{1}{2} + \left(\dfrac{1}{4} + k^2\right)^{\frac{1}{2}}$, and hence

$$u_{ic}^{(2)} = \frac{2}{\pi} \int_0^{\infty} k^{-3}\{(1 - k^2 Y^\circ) e^{-Y^\circ} - e^{-\beta Y^\circ}\} \sin(k\chi) dk. \tag{4.56}$$

Further terms may be obtained by similar procedures and $(u_{ic})_r$ obtained to the same order as u_s and $(u_c)_r$.

5. The volumetric flow-rate

Before turning finally to the $(u_{ic})_f$ layer near $x = -l$, $y = +1$, mention must be made of the volumetric flow-rate, expressed (using symmetry) in the form

$$F = 2 \int_{y=-1}^{1} \int_{x=-l}^{0} v\, dx dy = 2 \int_{y=-1}^{1} \int_{x=-l}^{0} \{u - M^{-1}(1+y)\} dx dy, \tag{5.1}$$

and which involves summing the integrals of the various contributions to u over the left-hand half of the fluid cross-section. It is important to note that the results for u_s are not defined so as to be valid in the $(ic)_f$ layer; integration of u_s over this layer would seem therefore to introduce an error into the solution for F. However, as will be shown in Sec. 6, u_s is, in fact, the outer expansion of $(u_{ic})_f$ as one moves out from the $(ic)_f$ layer into the (s) layer. Thus, by integrating $\{(u_{ic})_f - u_s\}$ over the cross-section the suggested error referred to above is cancelled, and contributions from the $(ic)_f$ layers are simply added to those from the (I), (H), $(c)_r$ and $(ic)_r$ regions.

We here stress that $(u_{ic})_f$ cannot be obtained as an asymptotic series using the classical approach; however, we can estimate the order of magnitude of the contribution of $\{(u_{ic})_f - u_s\} = \hat{u}$ to the flow-rate. Since $u_s^{(1)}$ is continuous and non-singular at all points, whilst $u_s^{(2)}$, though bounded, is discontinuous at the corner $X = 0 = (1-y)$, it follows that $\hat{u}$ is $0(M^{-2})$. The (ic) layers being of dimensions $0(M^{-1})$ we conclude that $\hat{u}$ first contributes to F at the $0(M^{-4})$ level. Integration of $(u_{ic})_f$ over the entire cross-section would, of course, yield the full flow-rate contribution from the $(ic)_f$ and (s) layers.

All errors due to integrating the correction terms u_H, u_s, $(u_c)_r$ and $(u_{ic})_r$ outside their "regions of influence" (excepting that referred to above) may readily be shown to be exponentially small in M. Those terms in F up to and including those of $0(M^{-7/2})$ may be evaluated by integration of the leading terms in u_I, u_H, u_s, $(u_c)_r$ and $(u_{ic})_r$, though we will not perform the calculations here. The $0(M^{-4})$ term in F cannot be found without knowing the full solution in the $(ic)_f$ layer (see earlier comments).

6. Closed form solutions

We will now derive a closed form solution for u in the $(ic)_f$ layer, from which the (s) layer solution can be derived to all orders; the procedure is due to TODD (see [17]). The correction u_r that must be added to $(u_I + u_H + u_s)$ in order to satisfy the boundary conditions on $u(x, -1)$ and $u(-l, y)$ on the boundaries of the $(c)_r$ and $(ic)_r$ layers will also be obtained in closed form.

A. In the $(ic)_f$ layer, we may take

$$u \sim u_I + \{u_{ic}(\chi, Y)\}_f = 2M^{-1} + (u_{ic})_f. \tag{6.1}$$

Substituting into the first of Eqs. (3.2) and the boundary conditions of Fig. 2 we obtain

$$\left(\frac{\partial^2}{\partial \chi^2} + \frac{\partial^2}{\partial Y^2} - \frac{\partial}{\partial Y}\right)(u_{ic})_f = 0, \tag{6.2}$$

with

$$\{u_{ic}(\chi, 0)\}_f = 0, \quad \{u_{ic}(0, Y)\}_f = -YM^{-2}, \tag{6.3}$$

and

$$(u_{ic})_f \to 0 \quad \text{as} \quad \chi \to \infty. \tag{6.4}$$

Introducing $U^*(k, Y) = \int\limits_0^\infty (u_{ic})_f \sin(k\chi) d\chi$, and proceeding as in SNEDDON [16], we obtain

$$U^* = -M^{-2}k^{-1}Y + M^{-2}k^{-3}(1 - e^{-\alpha Y}), \tag{6.5}$$

where

(6.5′) $$\alpha = \alpha(k) = -\frac{1}{2}+\left(\frac{1}{4}+k^2\right)^{\frac{1}{2}}.$$

Hence

(6.6) $$\{u_{ic}(\chi, Y)\}_f = -YM^{-2}+\frac{2M^{-2}}{\pi}\int_0^\infty k^{-3}(1-e^{-\alpha Y})\sin(k\chi)\,dk.$$

Setting $\chi = M^{\frac{1}{2}}X$, $Y = M(1-y)$, $k = \tilde{k}M^{-\frac{1}{2}}$ yields the following closed form for the outer expansion of $(u_{ic})_f$:

(6.7) $$\{u_{ic}(X, y)\}_f = M^{-1}(y-1)+\frac{2M^{-1}}{\pi}\int_0^\infty \tilde{k}^3(1-e^{-\alpha M(1-y)})\sin(\tilde{k}X)\,d\tilde{k},$$

with

(6.8) $$\alpha = \alpha(\tilde{k}) = -\frac{1}{2}+\frac{1}{2}(1+4\tilde{k}^2M^{-1})^{\frac{1}{2}}$$
$$\sim \tilde{k}^2M^{-1}-\tilde{k}^4M^{-2}+2\tilde{k}^6M^{-3}+\ldots, \quad \text{for} \quad M \gg 1.$$

Thus, for $M \gg 1$,

(6.9) $$\{u_{ic}(X, y)\}_f = M^{-1}(y-1)+\frac{2M^{-1}}{\pi}\int_0^\infty \tilde{k}^{-3}(1-e^{-\tilde{k}^2(1-y)})\sin(\tilde{k}X)\,d\tilde{k}$$
$$-\frac{2M^{-2}}{\pi}(1-y)\int_0^\infty \tilde{k}e^{-\tilde{k}^2(1-y)}\sin(\tilde{k}X)\,d\tilde{k}+\frac{4M^{-3}}{\pi}(1-y)\int_0^\infty \tilde{k}^3e^{-\tilde{k}^2(1-y)}\sin(\tilde{k}X)\,d\tilde{k}$$
$$-\frac{M^{-3}}{\pi}(1-y)^2\int_0^\infty \tilde{k}^5e^{-\tilde{k}^2(1-y)}\sin(\tilde{k}X)\,d\tilde{k}+0(M^{-4}).$$

The first integral term vanishes when $(1-y) = 0$ and its first partial derivative with respect to $(1-y)$ is

$$+\frac{2M^{-1}}{\pi}\int_0^\infty \tilde{k}^{-1}e^{-\tilde{k}^2(1-y)}\sin(\tilde{k}X)\,d\tilde{k} = M^{-1}\operatorname{erf}\{X/2(1-y)^{\frac{1}{2}}\};$$

it therefore equals

(6.10) $$M^{-1}(y-1)+M^{-1}\int_0^{1-y}\operatorname{erf}(X/2\theta^{\frac{1}{2}})\,d\theta,$$

the $M^{-1}u_s^{(1)}$ obtained in result (4.11′).

The next integral terms equals

(6.11) $$M^{-2}(1-y)\frac{\partial}{\partial(1-y)}\left\{\frac{2}{\pi}\int_0^\infty \tilde{k}^{-1}e^{-\tilde{k}^2(1-y)}\sin(\tilde{k}X)\,d\tilde{k}\right\}$$
$$= M^{-2}(1-y)\frac{\partial}{\partial(1-y)}\left(\operatorname{erf}\{X/2(1-y)^{\frac{1}{2}}\}\right),$$

which checks with the $M^{-2}u_s^{(2)}$ of result $(4.15)_2$.

The $0(M^{-3})$ integral term may likewise be shown to check with result (4.20). Thus the series expansion for u, can be obtained to all orders as the outer expansion of the closed form (6.6).

B. We now seek a closed form solution for $(u_c+u_{ic})_r$ near $x=-l$, $y=-1$. A correction must be added to the closed form solution for the (s) and $(ic)_f$ regions in order to satisfy $u(x,-1)=0$ and $u(-l,y)=M^{-1}(1+y)$. We denote this correction by u_r, the (r) region being defined as the union of the $(c)_r$ and $(ic)_r$ layers. In the $(ic)_r$ layer, $u_r(\chi, Y^\circ)$ satisfies Eq. (4.47) and the conditions (see results (4.2), (6.7))

$$u_r(0,Y^\circ) = -u_H(-l,Y^\circ) = 2M^{-1}e^{-Y^\circ}, \tag{6.12}$$

$$u_r(\chi,0) = -u_s(X,-1) = -\frac{2M^{-1}}{\pi}\int_0^\infty \tilde{k}^{-3}(1-2\tilde{k}^2-e^{-2M\alpha(\tilde{k})})\sin(\tilde{k}X)\,d\tilde{k} \tag{6.13}$$
$$= \frac{-2M^{-2}}{\pi}\int_0^\infty k^{-3}(1-2Mk^2-e^{-2M\alpha(k)})\sin(k\chi)\,dk,$$

after setting $X=\chi M^{-\frac{1}{2}}$, $k=\tilde{k}M^{-\frac{1}{2}}$ and with $\alpha(k)$ defined in line (6.5′), and

$$u_r \to 0 \quad \text{as} \quad \chi \quad \text{and/or} \quad Y^\circ \to \infty. \tag{6.14}$$

Introducing

$$U_r = \int_0^\infty u_r \sin(k\chi)\,d\chi \tag{6.15}$$

and proceeding in the usual manner yields

$$\left(\frac{d^2}{dY^{\circ 2}}+\frac{d}{dY^\circ}-k^2\right)U_r = -2kM^{-1}e^{-Y^\circ}, \tag{6.16}$$

with

$$U_r(k,0) = -M^{-2}k^{-3}(1-2Mk^2-e^{-2M\alpha(k)}), \tag{6.17}$$

and

$$U_r \to 0 \quad \text{as} \quad Y^\circ \to \infty. \tag{6.18}$$

Thus

$$U_r = 2(kM)^{-1}e^{-Y^\circ} - M^{-2}k^{-3}(1-e^{-2M\alpha(k)})e^{-Y^\circ\beta(k)}, \tag{6.19}$$

where

$$\beta(k) = 1+\alpha(k) = \frac{1}{2}+\left(\frac{1}{4}+k^2\right)^{\frac{1}{2}}, \tag{6.20}$$

and hence

$$u_r(\chi,Y^\circ) = \frac{2M^{-2}}{\pi}\int_0^\infty k^{-3}\{2k^2Me^{-Y^\circ}-(1-e^{-2\alpha M})e^{-\beta Y^\circ}\}\sin(k\chi)\,dk. \tag{6.21}$$

Adding results (6.21) and (6.6) gives the correction to $(u_I + u_H)$ "near" the wall $x = -l$ in the form

$$(6.22) \quad \{u_{ic}(\chi, Y)\}_f + u_r(\chi, Y^0)$$
$$= \frac{2M^{-2}}{\pi} \int_0^\infty k^{-3}\{1 - k^2 Y - e^{-\alpha Y} + 2k^2 M e^{-Y^0} - (1 - e^{-2\alpha M}) e^{-\beta Y^0}\} \sin(k\chi)\, dk,$$

which matches with results obtained in [17] as $D_B \to \infty$.

For $M \gg 1$, the major contribution to the integral in result (6.21) comes at small k; from (6.20) for $M \gg 1$ and k small,

$$(6.23) \qquad \alpha(k) = \beta(k) - 1 \sim k^2 - k^4 + 2k^6 + \ldots$$

and

$$(6.24) \qquad (1 - e^{-2\alpha M}) e^{-\beta Y^0} \sim 2k^2 M (1 - k^2)(1 - k^2 Y^0) e^{-Y^0}.$$

Substituting the latter approximations into (6.21) it is readily seen that the $0(M^{-1})$ correction term from the $(ic)_r$ layer is identically zero; this checks with the earlier result (4.51′).

Setting $\chi = M^{\frac{1}{2}} X$, $k = M^{-\frac{1}{2}} \tilde{k}$ in (6.21), (6.23) and (6.24),

$$(6.25) \qquad u_r(X, Y^0) = \frac{2M^{-1}}{\pi} \int_0^\infty \tilde{k}^{-3}\{2\tilde{k}^2 e^{-Y^0} - (1 - e^{-2\alpha(\tilde{k})M}) e^{-\beta(\tilde{k})Y^0}\} \sin(\tilde{k}X)\, d\tilde{k},$$

where

$$(6.26) \qquad \alpha(\tilde{k}) = \beta(\tilde{k}) - 1 \sim \tilde{k}^2 M^{-1} - \tilde{k}^4 M^{-2} + 2\tilde{k}^6 M^{-3} + \ldots$$

and

$$(6.27) \qquad e^{-2\alpha(\tilde{k})M} \sim e^{-2\tilde{k}^2}(1 + 2\tilde{k}^4 M^{-1}), \quad e^{-\beta(\tilde{k})Y^0} \sim e^{-Y^0}(1 - \tilde{k}^2 M^{-1} Y^0).$$

The leading contribution is

$$2M^{-1} e^{-Y^0}\left\{1 - \pi^{-1} \int_0^\infty (1 - e^{-2\tilde{k}^2}) \tilde{k}^{-3} \sin(\tilde{k}X)\, d\tilde{k},\right.$$

which reduces, after successive integrations by parts, to the form

$$(6.28) \quad M^{-1} e^{-Y^0}\Big\{2 + \pi^{-1} \int_0^\infty \tilde{k}^{-1}(\{X^2(1 - e^{-2\tilde{k}^2}) - 4(1 - 4\tilde{k}^2 e^{-2\tilde{k}^2})\} \sin(\tilde{k}X)$$
$$- 8\tilde{k} X e^{-2\tilde{k}^2} \cos(\tilde{k}X))\, d\tilde{k}\Big\}$$
$$= M^{-1} e^{-Y^0}\left\{2 + \frac{1}{2} X^2 (1 - \operatorname{erf}\{X/2^{\frac{3}{2}}\}) - 2\operatorname{erf}(X/2^{\frac{3}{2}}) - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} X e^{-X^2/8}\right\}$$

(see [13], pp. 15, 73).

This matches with the $M^{-1}u_c^{(1)}$ of result (4.40), for

$$u_c^{(1)} = e^{-Y^0}\left\{2-[\theta\,\mathrm{erf}(X/2\theta^{\frac{1}{2}})]_0^2-\int_0^2 \frac{Xe^{-X^2/4\theta}}{2(\pi\theta)^{\frac{1}{2}}}d\theta\right\}$$

$$= e^{-Y^0}\left\{2-2\,\mathrm{erf}(X/2^{\frac{3}{2}})-X\left(\frac{2}{\pi}\right)^{\frac{1}{2}}e^{-X^2/8}+\frac{X^2}{2}(1-\mathrm{erf}\,\{X/2^{\frac{3}{2}}\})\right\}.$$

The next contribution is

$$(6.29)\qquad \frac{2M^{-2}e^{-Y^0}}{\pi}\int_0^\infty \tilde{k}^{-1}\{2\tilde{k}^2e^{-2\tilde{k}^2}+Y^0(1-e^{-2\tilde{k}^2})\}\sin(\tilde{k}X)\,d\tilde{k}$$

$$= M^{-2}e^{-Y^0}\left\{Y^0(1-\mathrm{erf}\,\{X/2^{\frac{3}{2}}\})+X(8\pi)^{-\frac{1}{2}}e^{-X^2/8}\right\},$$

which matches the $M^{-2}u_c^{(2)}$ from result (4.43). Thus

$$(6.30)\qquad u_r = M^{-1}u_c^{(1)}+M^{-2}u_c^{(2)}$$

$$+\frac{2M^{-2}}{\pi}\int_0^\infty k^{-3}\left\{e^{-\beta Y^0}(1-e^{-2\alpha M})+e^{-Y^0}\big((1-k^2Y^0)(1-e^{-2Mk^2})-2Mk^4e^{-2Mk^2}\big)\right\}\sin(k\chi)\,dk,$$

$\alpha(k)$, $\beta(k)$ being defined in line (6.20).

The leading $(ic)_r$ correction term is thus

$$(6.31)\qquad \frac{2M^{-2}}{\pi}\int_0^\infty k^{-3}\{e^{-Y^0}(1-k^2Y^0)-e^{-\beta Y^0}\}\sin(k\chi)\,dk,$$

which matches the $M^{-2}u_{ic}^{(2)}$ of result (4.56).

One can proceed indefinitely in this manner, working in terms of X and $\tilde{k}$ when seeking $(c)_r$ layer correction terms and in terms of χ, k in deriving $(ic)_r$ corrections.

7. Conclusions

The actual calculation of the flow-rate, F, and its matching with the result of WILLIAMS [18] will be dealt with in a future paper (see also TODD [19]).

In the present paper we have outlined an expansion scheme yielding the solution in all regions except the $(ic)_f$ layer to all orders, and have derived a closed form solution for u in the latter layer, from which the (s) layer solution can be obtained to all orders as an outer expansion. A closed form solution in the (r) region was also found, from which $(c)_r$ and $(ic)_r$ layer correction terms can be obtained to any order. The superiority of the "improved" Todd method over the Laplace transform method in deriving (s) layer correction terms has been firmly established. Both the former and the Fourier sine transform methods are clearly to be preferred when dealing with this and similar problems. A principle of minimum singularity is "built-in" to the Fourier sine transform method, at least as applied to the configuration presently under consideration, which is lacking in the Laplace transform method. We are not clear as to why this is so, but Professor W. D. LAKIN (Toronto, Canada) has given some thought to this matter and is believed to have found an explanation for this difference between the results obtained by the two transform methods.

In a further paper, we will consider a duct having walls BB of arbitrary, non-zero conductivity and will endeavour to derive the solution for v, b to all orders in each of the key regions.

Appendix

Further attempts to justify the choice $A_0 = 1$ in expression (4.24) for $u_{sef}^{(3)}$

(*i*) On adding the eigen function $u_{sef}^{(3)}$ (result (4.24)) to the "provisional" solution $U_s^{(3)}$ (result (4.23)), we will integrate the expression $\{u_{sef}^{(3)} + U_s^{(3)}\}F(X) = u_s^{(3)}F(X)$, where $F(X)$ is any "good" trial function (as defined in LIGHTHILL [15]), along a path consisting of the line $y = 1$ indented below to avoid the corner $X = 0 = 1-y$ and symmetric about $x = -l$, in an attempt to pick up the singularities in $u_s^{(3)}$ at the corner when the indentation shrinks to zero. We hope to show that $A_0 = 1$ is the only choice of A_0 which annihilates the $\delta'(X)$ behaviour in $u_s^{(3)}$ as $(1-y) \to 0$, hence minimising the overall singularity at the corner.

The initial indentation cuts across the wake of the parabolic side-layer near $x = -l$, $y = 1$ (see Fig. 4); it consists of that portion of the line $1-y = \beta - K|X|$ for which $|X| \leqslant \beta K^{-1}$, K and β being small and positive.

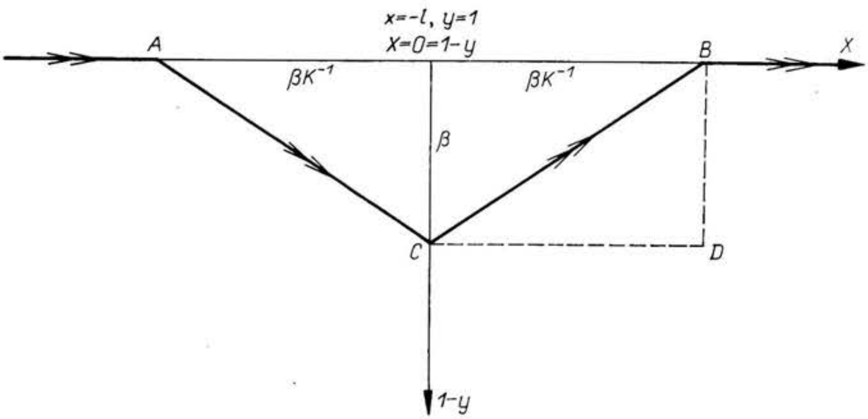

FIG. 4. Simple path of integration, indented to avoid the corner $X = 0 = 1-y$.

As $\beta \to 0$, the indentation vanishes; the lines AC, CB reduce to the corner point if $L_1 = \lim_{\beta \to 0} (\beta K^{-1}) = 0$, to the finite interval $|X| \leqslant \mu$ of the line $y = 1$ if $L_1 = \mu$, a positive constant, or to the entire line $y = 1$ if $\beta K^{-1} \to \infty$ when $\beta \to 0$ (i.e., $K = K(\beta) \to 0$ more rapidly than does β itself). In what follows, we will take results (4.23), (4.24) as being valid for $X < 0$, i.e., $U_s^{(3)}$, $u_{sef}^{(3)}$ will be treated as odd in X.

Now, for those sections of $y = 1$ on which $|X| > \beta K^{-1}$, $u_s^{(3)}$ is identically zero (see condition $(4.6)_3$), while on AC and CB,

$$1-y = \beta - K|X| \quad \text{and} \quad |d(1-y)/dX| = K.$$

Consider $I = \int_{-\infty}^{\infty} F(X) u_s^{(3)}\, ds$ taken over the indented contour. Since $u_s^{(3)}$ is odd in X and $ds = (1+K^2)^{\frac{1}{2}} dX$, so

$$I = 2(1+K^2)^{\frac{1}{2}} \int_0^{\beta K^{-1}} F_0(X) u_s^{(3)} dX,$$

where $F_0(X) = \frac{1}{2}\{F(X) - F(-X)\}$, the odd part of $F(X)$.

$$\text{(A.1)} \quad \text{i.e.} \quad I = \frac{(1+K^2)}{32\pi^{\frac{1}{2}}} \int_0^{\beta K^{-1}} \frac{XF_0(X)}{(\beta - KX)^{\frac{3}{2}}} \left\{4 + 32A_0 + \frac{4X^2}{\beta - KX} - \frac{X^4}{(\beta - KX)^2}\right\} \times \exp\{-X^2/4(\beta - KX)\}\, dX.$$

Introducing $t = X/2(\beta - KX)^{\frac{1}{2}}$, so that $X = -2Kt^2 + 2t(\beta + K^2t^2)^{\frac{1}{2}}$ and $(X + 2Kt^2)\,dX = 4(\beta - KX)\,t\,dt = 2(\beta + K^2t^2)^{\frac{1}{2}}\, t\,dX$, (A.1.) reduces to

$$\text{(A.2)} \quad I = \frac{(1+K^2)^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}} \int_0^{\infty} \frac{F_0\left(-2Kt^2 + 2t(\beta + K^2t^2)^{\frac{1}{2}}\right)}{(\beta + K^2t^2)^{\frac{1}{2}}} e^{-t^2}\{(8A_0 + 1)t + 4t^3 - 4t^5\}\, dt.$$

As $\beta \to 0$, the limiting value of I depends on the value of $L_2 = \lim_{\beta \to 0} \beta K^{-2}$, though the argument of F_0 clearly tends to zero as $\beta \to 0$, for all K. For "small" values of β, F_0 in (A.2) can be expanded as a Taylor series about $\beta = 0$ and the integration then performed. Alternatively, one may integrate by parts in (A.2) and then substitute a Taylor series expansion for F_0. It is readily shown that both procedures yield an identical limiting value for I *as* $\beta \to 0$. Using the former method and taking

$$\text{(A.3)} \qquad F_0 \sim 2t\{-Kt + (\beta + K^2t^2)^{\frac{1}{2}}\} F_0'(0) + \ldots,$$

it may be shown that all terms after the first in (A.3) contribute nothing to I in the limit $\beta \to 0$, for all $K \geqslant 0$. Substituting (A.3) into (A.2), and noting that $F_0'(0) = F'(0)$,

$$\text{(A.4)} \quad I = \frac{(1+K^2)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} F'(0) \int_0^{\infty} \{1 - Kt(\beta + K^2t^2)^{-\frac{1}{2}}\}\{(8A_0 + 1)t^2 + 4t^4 - 4t^6\} e^{-t^2} dt + (\text{contributions which} \to 0 \quad \text{as} \quad \beta \to 0).$$

Neglecting the latter terms and integrating the leading one by parts yields, for "small" β,

$$\text{(A.5)} \qquad I \sim (1+K^2)^{\frac{1}{2}} F'(0) \left\{2(A_0 - 1) - \pi^{-\frac{1}{2}} \int_0^{\infty} \frac{t^3(8A_0 + 1 + 4t^2 - 4t^4) e^{-t^2} dt}{(t^2 + \beta K^{-2})^{\frac{1}{2}}}\right\}.$$

If $L_2 = \lim_{\beta \to 0} \beta K^{-2} = 0$, then $I \to 0$ as $\beta \to 0$ for all A_0; this represents a failure to pick up the $\delta'(X)$ behaviour as $(1-y) \to 0$. If L_2 does not exist (i.e. $K^2 \to 0$ more rapidly than does β, e.g., $K = 0(\beta)$), then $I \to 0$ as $\beta \to 0$ only if $A_0 = 1$; this means that the $\delta'(X)$ behaviour is removed only by this chosen value of A_0.

If $L_2 = \lambda$, a positive constant, then as $\beta \to 0$

$$I \to F'(0)\left\{2(A_0-1)-\pi^{-\frac{1}{2}}\int\limits_0^\infty \frac{t^3(8A_0+1+4t^2-4t^4)e^{-t^2}dt}{(t^2+\lambda)^{\frac{1}{2}}}\right\}, \tag{A.6}$$

$$= \lambda F'(0)\pi^{-\frac{1}{2}}\int\limits_0^\infty \frac{t^3(2t^2+3)e^{-t^2}}{(t^2+\lambda)^{\frac{3}{2}}}dt \neq 0, \quad \text{if} \quad A_0 = 1; \tag{A.6'}$$

thus, in this third case, the $\delta'(X)$ singularity can only be removed by an appropriate choice of $A_0 = A_0(\lambda)$ (see (A6)).

Summarising the results obtained above we see that the choice $A_0 = 1$ cannot be fully justified using the indented contour of Fig. 4. For a suitable subset of the class C_∞ of curves, the choice may perhaps be validated using the above procedure; an initial contour $1-y = \beta > 0$, with $\beta \to 0$ in the limit, may readily be shown to justify the choice $A_0 = 1$, for example, and integration along a path $X = \sigma > 0$, with $\sigma \to 0$ in the limit, picks up the $\delta(1-y)$ behaviour as $X \to 0$, for all $A_0 \neq 0$, $u_s^{(3)}$ being defined as zero for $(1-y) < 0$.

A physical interpretation of the three limiting values of L_2 discussed above is apparent from Fig. 4. Since $(CD)^2/BD = \beta K^{-2}$, so $L_2 = 0$ corresponds to the point D lying well within the wake, $L_2 \to \infty$ corresponds to D lying well outside the wake and $L_2 = \lambda > 0$ corresponds to D lying on one of the parabolic streamlines $X^2/4(1-y) = $ constant.

(*ii*) A better way of justifying the choice $A_0 = 1$ in $u_{sef}^{(3)}$ is by consideration of an analogous physical situation, the heat diffusion problem for an infinitely long, thin rod. In Eq. (4.19), suppose that $u_s^{(3)}$ represents the temperature in such a rod, having unit diffusivity; X, $(1-y)$ represent the length and time coordinates. The right-hand side of the equation represents $-A^*(X, 1-y)/K^*$, A^* being the rate at which heat is supplied to the rod per unit length per unit time and K^* being the rod's conductivity. The initial temperature at all points of the rod is zero and the temperature at station $X = 0$ is held zero for all time $(1-y) > 0$. Expression (4.23) would represent a temperature distribution that is singular as $(1-y) \to 0$, while the eigen function (4.24) represents the change in distribution due to the insertion of a doublet. Beginning with a source plus sink each of strength Q at stations $X = X'+dX'$, X' respectively, suppose $dX' \to 0$ and define $2A_0 = \lim\limits_{dX'\to 0}(Q\cdot dX')$, and then finally let $X' \to 0$; the result is an instantaneous doublet at $X = 0$. The choice $A_0 = 1$ corresponds then to minimising the singularity in the temperature distribution along the rod as the time $(1-y) \to 0$ (because $\int\limits_{-\infty}^{\infty} F(X)\cdot u_s^{(3)}dX = 0$ when $A_0 = 1$).

We have not been able to specify a truly satisfactory Principle of Minimum Singularity in this paper. The choice $A_0 = 1$ in $u_{sef}^{(3)}$ and selection of the appropriate eigen functions for $u_s^{(n)}$, $n \geqslant 4$, can only be completely justified using the outer expansion of the closed-form solution for $(u_{ic})_f$, derived in Sec. 6.

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Received November 18, 1975.