315.

ON THE CUBIC CENTRES OF A LINE WITH RESPECT TO THREE LINES AND A LINE.

[From the *Philosophical Magazine*, vol. XXII. (1861), pp. 433—436.]

On referring to my Note on this subject (*Phil. Mag.* vol. XX. pp. 418—423, 1860 [257]), it will be seen that the cubic centres of the line

$$\lambda x + \mu y + \nu z = 0$$

in relation to the lines $x=0$, $y=0$, $z=0$, and the line $x+y+z=0$, are determined by the equations

$$x : y : z = \frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu},$$

where θ is a root of the cubic equation

$$\frac{1}{\theta+\lambda} + \frac{1}{\theta+\mu} + \frac{1}{\theta+\nu} - \frac{2}{\theta} = 0;$$

or as it may also be written,

$$\theta^3 - \theta(\mu\nu + \nu\lambda + \lambda\mu) - 2\lambda\mu\nu = 0.$$

Two of the centres will coincide if the equation for θ has equal roots; and this will be the case if

$$\lambda^{-\frac{1}{3}} + \mu^{-\frac{1}{3}} + \nu^{-\frac{1}{3}} = 0,$$

or, what is the same thing, if λ, μ, $\nu = a^{-3}$, b^{-3}, c^{-3}, where $a+b+c=0$. In fact, if $a+b+c=0$, then $a^3+b^3+c^3=3abc$, and the equation in θ becomes

$$\theta^3 - \frac{3\theta}{a^2b^2c^2} - \frac{2}{a^3b^3c^3} = 0;$$

that is

$$(abc\theta)^3 - 3(abc\theta) - 2 = 0,$$

which is

$$(abc\theta + 1)^2 (abc\theta - 2) = 0;$$

so that the values of θ are $\frac{-1}{abc}, \frac{2}{abc}$.

First, if $\theta = -\frac{1}{abc}$, then x, y, z will be the coordinates of the double centre. And we have

$$\theta + \lambda = \frac{1}{a^3} - \frac{1}{abc} = \frac{1}{2a^3bc}(2bc - 2a^2) = \frac{1}{2a^3bc}(-a^2 - b^2 - c^2);$$

or putting for shortness $\square = a^2 + b^2 + c^2$,

$$\theta + \lambda = -\frac{1}{2a^3bc}\square, \quad = -\frac{3}{abc}\cdot\frac{\square}{6a^2},$$

with similar values for $\theta + \mu$, $\theta + \nu$. But $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are proportional to $\theta + \lambda$, $\theta + \mu$, $\theta + \nu$; and we may therefore write

$$\frac{P}{x} = \frac{\square}{6a^2}, \quad \frac{P}{y} = \frac{\square}{6b^2}, \quad \frac{P}{z} = \frac{\square}{6c^2};$$

whence, in virtue of the equation $a + b + c = 0$, we have for the locus of the double centre,

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = 0;$$

or this locus is a conic touching the lines $x = 0$, $y = 0$, $z = 0$ harmonically in respect to the line $x + y + z = 0$, a result which was obtained somewhat differently in the paper above referred to.

Next, if $\theta = \frac{2}{abc}$, x, y, z will be the coordinates of the single centre. And we now have

$$\theta + \lambda = \frac{1}{a^3} + \frac{2}{abc} = \frac{1}{2a^3bc}(2bc - 2a^2 + 6a^2) = \frac{1}{2a^3bc}(-\square + 6a^2) = -\frac{3}{abc}\,\frac{\square - 6a^2}{6a^2},$$

with similar values for $\theta + \mu$, $\theta + \nu$. But $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are proportional to $\theta + \lambda$, $\theta + \mu$, $\theta + \nu$, and we may therefore write

$$\frac{P}{x} = \frac{\square - 6a^2}{6a^2}, \quad \frac{P}{y} = \frac{\square - 6b^2}{6b^2}, \quad \frac{P}{z} = \frac{\square - 6c^2}{6c^2},$$

from which equations, and the equation $a + b + c = 0$, the quantities P, a, b, c have to be eliminated. I at first effected the elimination as follows: viz., writing the equations under the form

$$\frac{x}{x + P} = \frac{6a^2}{\square}, \quad \frac{y}{y + P} = \frac{6b^2}{\square}, \quad \frac{z}{z + P} = \frac{6c^2}{\square},$$

we obtain

$$\frac{x}{x+P}+\frac{y}{y+P}+\frac{z}{z+P}=6,$$

$$\sqrt{\frac{x}{x+P}}+\sqrt{\frac{y}{y+P}}+\sqrt{\frac{z}{z+P}}=0,$$

which are easily transformed into

$$\frac{x}{x+P} \quad + \quad \frac{y}{y+P} \quad + \quad \frac{z}{z+P} \quad =6,$$

$$\frac{yz}{(y+P)(z+P)}+\frac{zx}{(z+P)(x+P)}+\frac{xy}{(x+P)(y+P)}=9\,;$$

or, what is the same thing,

$$6(P+x)(P+y)(P+z)-x(P+y)(P+z)-y(P+z)(P+x)-z(P+x)(P+y)=0,$$

$$9(P+x)(P+y)(P+z)-yz(P+x) \quad -zx(P+y) \quad -xy(P+z)=0,$$

which give

$$6P^3+5P^2(x+y+z)+4P(yz+zx+xy)+3xyz=0,$$

$$9P^3+9P^2(x+y+z)+8P(yz+zx+xy)+6xyz=0\,;$$

or, multiplying the first equation by 2, and subtracting the second,

$$3P + \quad (x+y+z)=0\,;$$

and we thus obtain for the locus of the single centre the equation

$$\frac{x}{-2x+y+z}+\frac{y}{-2y+z+x}+\frac{z}{-2z+x+y}=2,$$

or, what is the same thing,

$$x^3+y^3+z^3-(yz^2+zx^2+xy^2+y^2z+z^2x+x^2y)+3xyz=0,$$

which may also be written,

$$-(-x+y+z)(x-y+z)(x+y-z)+xyz=0.$$

The same result may also be obtained as follows: viz., observing that

$$\square-6a^2=b^2+c^2-5a^2=-4a^2-2bc,$$

we have

$$\frac{x}{P}=\frac{-3a^2}{2a^2+bc},\quad \frac{y}{P}=\frac{-3b^2}{2b^2+ca},\quad \frac{z}{P}=\frac{-3c^2}{2c^2+ab},$$

and then by means of the equation

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}-1=0,$$

which is identically true in virtue of $a+b+c=0$ (in fact, multiplying out, this gives

$$12a^2b^2c^2+4(b^3c^3+c^3a^3+a^3b^3)+abc(a^3+b^3+c^3)$$
$$-8a^2b^2c^2-4(b^3c^3+c^3a^3+a^3b^3)-2abc(a^3+b^3+c^3)-a^2b^2c^2=0;$$

that is

$$3a^2b^2c^2-abc(a^3+b^3+c^3)=0,\ \text{or}\ abc(a^3+b^3+c^3-3abc)=0,$$

where the second factor divides by $a+b+c$), we find the above-mentioned equation,

$$x+y+z+3P=0.$$

We then have

$$\frac{-x+y+z}{P}=\frac{x+y+z}{P}-\frac{2x}{P}=-3+\frac{6a^2}{2a^2+bc}=-\frac{3bc}{2a^2+bc};$$

that is

$$\frac{-x+y+z}{P}=\frac{-3bc}{2a^2+bc},\quad \frac{x-y+z}{P}=\frac{-3ca}{2b^2+c},\quad \frac{x+y-z}{P}=\frac{-3ab}{2c^2+ab};$$

and forming the product of these functions, and that of the foregoing values of $\frac{x}{P}$, $\frac{y}{P}$, $\frac{z}{P}$, we find as before,

$$-(-x+y+z)(x-y+z)(x+y-z)+xyz=0$$

for the equation of the locus of the single centre. The equation shows that the locus is a cubic curve which touches the lines $x=0$, $y=0$, $z=0$ at the points where these lines are intersected by the lines $y-z=0$, $z-x=0$, $x-y=0$ (that is, it touches the lines $x=0$, $y=0$, $z=0$ harmonically in respect to the line $x+y+z=0$), and besides meets the same lines $x=0$, $y=0$, $z=0$ at the points in which they are respectively met by the line $x+y+z=0$.

2, *Stone Buildings, W.C., September* 25, 1861.