# 317.

## NOTE ON THE TRANSFORMATION OF A CERTAIN DIFFERENTIAL EQUATION.

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THE differential equation

$$(1+\theta^2)\frac{d^2y}{d\theta^2}+\theta\frac{dy}{d\theta}-m^2y=0,$$

if we put therein $i\theta=2x^2+1$ ($i=\sqrt{-1}$ as usual), becomes

$$(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4m^2y=0.$$

In fact an integral of the second equation is $(\sqrt{1+x^2}+x)^{2m}$; this is

$$=\left(\sqrt{(2x^2+1)^2-1}+2x^2+1\right)^m;$$

or putting $i\theta=2x^2+1$, it is

$$=(\sqrt{-\theta^2-1}+i\theta)^m,$$

which is

$$=\{i(\sqrt{\theta^2+1}+\theta)\}^m;$$

so that an integral of the transformed equation in $\theta$ is

$$=\quad(\sqrt{\theta^2+1}+\theta)^m;$$

and writing in the second equation $\theta$ for $x$, and $\frac{1}{2}m$ for $m$, we see that the last-mentioned function, viz. $(\sqrt{\theta^2+1}+\theta)^m$, is an integral of

$$(1+\theta^2)\frac{d^2y}{d\theta^2}+\theta\frac{dy}{d\theta}-m^2y=0;$$

whence the transformed equation in $\theta$ must be this very equation, that is, it must be the first equation. I have for shortness used the particular integral $(\sqrt{1+x^2}+x)^{2m}$; but the reasoning should have been applied, and it is in fact applicable, without alteration, to the general integral

$$C(\sqrt{1+x^2}+x)^m + C'(\sqrt{1+x^2}-x)^m.$$

There is of course no difficulty in a direct verification. Thus, starting from the first equation, or equation in $\theta$, the relation $i\theta = 2x^2+1$ gives

$$\frac{dy}{d\theta} = \frac{i}{4x}\frac{dy}{dx},\quad \frac{d^2y}{d\theta^2} = \frac{i}{4x}\frac{d}{dx}\left(\frac{i}{4x}\frac{dy}{dx}\right) = -\frac{1}{16x^2}\left(\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx}\right),$$

$$1+\theta^2 = -4x^2(1+x^2);$$

so that the equation becomes

$$\tfrac{1}{4}(1+x^2)\left(\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx}\right) + \frac{1+2x^2}{4x}\frac{dy}{dx} - m^2y = 0,$$

or multiplying by 4,

$$(1+x^2)\frac{d^2y}{dx^2} + \left(-\frac{1+x^2}{x} + \frac{1+2x^2}{x}\right)\frac{dy}{dx} - 4m^2y = 0;$$

that is

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4m^2y = 0,$$

the second equation. But the first method shows the reason why the two forms are thus connected together.

2, *Stone Buildings, W.C., February* 19, 1862.