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NOTE ON A THEOREM RELATING TO SURFACES.

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THE following apparently self-evident geometrical theorem requires, I think, a proof; viz. the theorem is—"If every plane section of a surface of the order $m+n$ break up into two curves of the orders m and n respectively, then the surface breaks up into two surfaces of the orders m, n respectively."

To fix the ideas, suppose $n=2$. Imagine any line meeting the surface in $m+2$ points, the section includes a conic which meets the line in two of the $m+2$ points, say the points A, A'(1). Suppose that the plane revolves round the line AA', the section will always include a conic which passes through *these same two points* A, A'; and it is to be shown that the sheet, the locus of this conic, is a surface of the second order. In fact the conic in question, say APA', by its intersection with an arbitrary plane traces out a branch of the intersection of the given surface with the arbitrary plane. And if $ABA'B'$ be the conic in any particular plane through A, A', and if the arbitrary plane meet this conic in the points B, B', then the branch passes through these points B, B'. Imagine the plane $ABA'B'$ revolving round BB' until it coincides with the arbitrary plane; the section includes a conic passing through the points B, B', and the before-mentioned branch is this conic; that is, the conic APA' by its intersection with an arbitrary plane traces out a conic; or, what is the same thing, the sheet, the locus of the conic APA', is met by an arbitrary plane in a conic, that is, the sheet is a surface of the second order; and the given surface thus includes a surface of the second order, and is therefore made up of two surfaces of the orders m and 2 respectively. The demonstration seems to me to add at least

1 The figure referred to will be at once understood by considering A, A' as the poles of an ellipsoid, or say of a sphere, $ABA'B'$ the meridian of projection, APA' any other meridian, BPB' the equator or any other great circle.

something to the evidence of the theorem asserted, but I should be glad if a more simple one could be found. Analytically, the theorem is—"If

$$(x, y, z, \alpha x + \beta y + \gamma z)^{m+n},$$

where (α, β, γ) are arbitrary, break up into factors $(x, y, z)^m$, $(x, y, z)^n$, rational in regard to (x, y, z), then $(x, y, z, w)^{m+n}$ breaks up into factors $(x, y, z, w)^m$, $(x, y, z, w)^n$, rational in regard to (x, y, z, w)." It would at first sight appear that (α, β, γ) being arbitrary, these quantities can only enter into the factors of $(x, y, z, \alpha x + \beta y + \gamma z)^{m+n}$ through the quantity $\alpha x + \beta y + \gamma z$; that is, that the factors in question, considered as functions of $(x, y, z, \alpha, \beta, \gamma)$, are of the form

$$(x, y, z, \alpha x + \beta y + \gamma z)^m, \quad (x, y, z, \alpha x + \beta y + \gamma z)^n;$$

and then replacing the arbitrary quantity $\alpha x + \beta y + \gamma z$ by w, the factors of $(x, y, z, w)^{m+n}$ will be $(x, y, z, w)^m$, $(x, y, z, w)^n$. But the objection proves too much; for in a similar way it would follow that if $(x, y, \alpha x + \beta y)^{m+n}$, where α, β are arbitrary, breaks up into the factors $(x, y)^m$, $(x, y)^n$, rational in regard to (x, y) (and quà homogeneous function of two variables *it always does so break up*), then $(x, y, z)^{m+n}$ would in like manner break up into the factors $(x, y, z)^m$, $(x, y, z)^n$, rational in regard to (x, y, z): and a simple example will show that it is not true that the factors of $(x, y, \alpha x + \beta y)^{m+n}$ only contain (α, β) through $\alpha x + \beta y$; in fact, if the function be $= x^2 + y^2 + (\alpha x + \beta y)^2$, then the factor is

$$\frac{1}{\sqrt{\alpha^2+1}} \{(\alpha^2+1)\, x + (\alpha\beta + i\sqrt{\alpha^2+\beta^2+1})\, y\},$$

which cannot be exhibited as a function of α, β, $\alpha x + \beta y$.

I am not acquainted with any analytical demonstration; the geometrical one cannot easily be exhibited in an analytical form.

2, *Stone Buildings, W.C., November* 26, 1862.