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ON THE DELINEATION OF A CUBIC SCROLL.

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IMAGINE a cubic scroll (skew surface of the third order) generated by lines each of which meets two given directrix lines. One of these is a nodal (double) line on the surface, and I call it the nodal directrix; the other is a single line on the surface, and I call it the single directrix. The section by any plane is a cubic passing through the points in which the plane meets the directrix lines; i.e. the point on the nodal directrix is a node (double point) of the curve, the point on the single directrix a single point on the curve; the two directrix lines, and the cubic curve, the section by any plane, determine the scroll. Consider the sections by a series of parallel planes. Let one of these planes be called the basic plane, and the section by this plane the basic section or basic cubic; and imagine any other section projected on the basic plane by lines parallel to the nodal directrix: such section may be spoken of simply as 'the section,' and its projection as 'the cubic.' The cubic has a node at the node of the basic cubic; that is, the two curves have at this point *four* points in common. The two curves have, moreover, in common the *three* points at infinity (or, in other words, their asymptotes are parallel); in fact the points at infinity of either curve are the points in which the line at infinity, the intersection of the basic plane and the plane of the section, meets the scroll; and these points are therefore the same for each of the two curves. The remaining *two* points of intersection of the cubic with the basic cubic are also fixed points on the basic cubic, i.e. they are the points of intersection of the basic plane by the two generating lines parallel to the nodal directrix. Hence the cubic meets the basic cubic in *nine* fixed points, viz. the node counting as four points, the three points at infinity, and the two points the feet of the generators parallel to the nodal directrix. It follows that if $U=0$ is the equation of the basic cubic, $V=0$ the equation of some other cubic meeting the basic cubic in the nine points in question, then the equation of 'the cubic' is $U+\lambda V=0$, λ being a parameter the value of which varies according to the position

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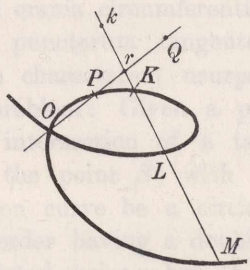
(in the series of parallel planes) of the plane of the section. Suppose that the basic cubic $U=0$ is given, and suppose for a moment that the cubic $V=0$ is also given, these two cubics having the above-mentioned relations, viz. they have a common node and parallel asymptotes: the cubic $U+\lambda V=0$ might be constructed by drawing through the node (say O) a radius vector meeting the cubics in P, P' respectively, and taking on this radius vector a point Q such that $PQ=\frac{\lambda}{1+\lambda}PP'$, or, what is the same thing, $OQ=\frac{OP+\lambda OP'}{1+\lambda}$; the locus of the point Q will then be the cubic $U+\lambda V=0$. And we may even suppose the cubic $V=0$ to break up into a line and a conic (hyperbola), and then (disregarding the line) use the hyperbola in the construction. In fact, if the hyperbola is determined by the following five conditions, viz. to pass through the node and through the feet of the two generators parallel to the nodal directrix, and to have its asymptotes parallel to two of the asymptotes of the basic cubic, and if the line be taken to be a line through the node parallel to the third asymptote of the basic cubic; then the hyperbola and line form together a cubic curve meeting the basic cubic in the nine points, and therefore satisfying the conditions assumed in regard to the cubic $V=0$. And it is to be noticed that as in general the cubic $V=0$ is the projection of some section of the scroll, so the hyperbola and line are the projection of a section of the scroll, viz. the section through one of the generating lines (there are three such lines) parallel to the basic plane. But it is better to construct 'the cubic' by a different method (using only the basic cubic $U=0$) which results more immediately from the geometrical theory. Taking the basic plane as the plane of the figure, let O be the node, or foot of the nodal directrix, K the foot of the single directrix, Kk the projection of the single directrix,



k being the projection of the point in which the single directrix meets the plane of the section. Drawing through O any radius vector meeting the basic cubic in P, and the line Kk in r, and producing it to a properly determined point Q, then $OPrQ$ will be the projection of the generating line which meets the nodal directrix, the basic cubic, the single directrix, and the section in the points the projections whereof are O, P, r, Q respectively: and the consideration of the solid figure shows easily that the condition for the determination of the point Q is

$$PQ=Kk\,.\,\frac{Pr}{rK}.$$

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Hence, starting from the basic cubic and the line Kk, we have a construction for the point Q the locus whereof is the cubic, the projection of a section of the scroll; for the projections of the parallel sections, we have only to vary the length Kk. By what precedes, the construction gives for the locus of Q a cubic having a node at O, and having its asymptotes parallel to those of the basic cubic. As P moves up to K, the distances Pr, rK become indefinitely small; but their ratio is finite, hence the cubic, the locus of Q, does *not* pass through the point K. The construction shows, however, that it does pass through the points L, M, which are the other two intersections of Kk with the basic cubic; these points L, M are in fact the feet of the generators parallel to the nodal directrix.

The general conclusion is, that a series of cubics having each of them at one and the same given point a node—having their asymptotes parallel—and besides passing through the same two given points—may be considered as the projections of a series of parallel sections of a cubic scroll; and such a series of cubics will thus afford a delineation of the scroll.

2, *Stone Buildings, W.C., April* 15, 1863.