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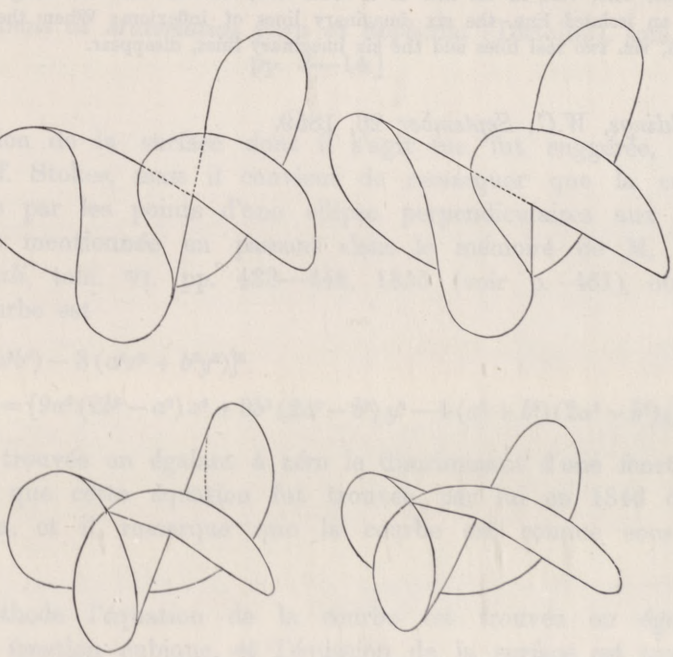
NOTE ON CONES OF THE THIRD ORDER.

[From the *Philosophical Magazine*, vol. XVIII. (1859), pp. 439—442.]

THE distinction adverted to in the preceding paper between the twin-pair sheets and single sheets of an algebraic cone is made (with respect to spherical curves, which is the same thing) by Möbius, in the interesting Memoir "Ueber die Grundformen der Linien der dritten Ordnung," *Abh. der K. Sächs. Ges. zu Leipzig*, vol. I. (1849) [and *Werke*, vol. II. pp. 86—176]. Consider the generating line POP' of a cone, vertex O, and let pOp' be any position of this line, the points P, P', and in like manner the points p, p', being on opposite sides of the vertex; then if OP originally coincides with Op (and therefore OP' with Op'), and if, in the course of the generation of the surface, OP (without having first come to coincide with Op') comes to coincide with Op, at the same time OP' (without having first come to coincide with Op) will come to coincide with Op', and we have a *twin-pair sheet*, viz. one twin-sheet generated by OP, and the other twin-sheet generated by OP'. This is the ordinary case of a cone of the second order, and requires no further explanation. It is proper to remark that, for cones of superior orders, the conical angle of each twin-sheet is not (as for a cone of the second order) necessarily less than 360°. But suppose that OP, starting from the position Op, and before it again comes to coincide therewith, comes to coincide with Op', then at the same time OP' (without having first come again to coincide with Op') will come to coincide with Op; the generation is here complete, and we have a *single sheet*, which, if the motion were continued until OP came to coincide with Op, would only be generated over again. The conical angle of a single sheet is necessarily greater than 360°; for OP in coming to coincide with Op' must describe an angle greater than 180°, and OP' describing an equal angle, the entire angle is therefore greater than 360°; in the limiting case, where the entire angle is precisely 360°, the conical surface is a plane. It is easy to cut out in paper and join together two sectors of a circle so as to form therewith a sector the angle whereof exceeds 360°; such a sector can then, by joining together the two radial edges, be converted into a cone of a single sheet; the generating lines being all finite lines equal in length, the curve formed by the circular edge is, it is clear, the spherical curve which is the intersection of the cone by a concentric sphere. It is shown by Möbius (stating his result with respect to cones instead of spherical curves) that a cone of an odd order must have at least one single sheet; a cone of the third order consists (1) of a single sheet, or else (2) of a single sheet and a twin-pair sheet. These are the two general forms of cones of the third order. But there are two special forms and one subspecial form, making in all five forms: viz., the two special forms are, (3) the cone has a nodal line; (4) the cone has an isolated line; and the subspecial form is, (5) the cone has a cuspidal line. The relation of the different forms may be explained as follows.

Starting from the form (1), as the constants of the equation change, the cone gathers itself up together so as to have a nodal line; this is the form (3). The loops of this form then detach themselves so as to form a twin-pair sheet, the remaining part of the surface reverting to a form similar to that of (1); we have thus a single sheet and twin-pair sheet, which is the form (2). The twin-pair sheet then dwindles away into an isolated line, giving the form (4); and lastly, the isolated line disappears and the cone resumes the form (1): these four forms constitute, there-



fore, a complete cycle. The constants may be such that the loop of the form (3) is evanescent, or, what is the same thing, that the forms (3) and (4) arise simultaneously; there is in this case a cuspidal line, or we have the form (5). It may be added that for the general forms (1) and (2) there are always three lines of inflexion. This is also the case with the form (4), where there is an isolated line; but in the form (3), where there is a nodal line, there is but one line of inflexion; and in the form (5), where there is a cuspidal line, there is not any line of inflexion: the equivalent theorem for the spherical curves is given by Möbius [1]. It was remarked long ago by Sir I. Newton, that all curves of the third order could be generated as the shadows of the five cubical parabolas; these are, in fact, sections in a particular manner of the above-mentioned five forms of cones of the third order: the existence of five essentially distinct forms of cones of the third order is noticed by M. Chasles in the *Aperçu Historique,* 1837. The analytical distinction between the forms (1) and (2) depends on the sign of the function $1-\frac{64S^3}{T^2}$, where S, T are the quartinvariant and sextinvariant of the cubic form. I annex stereoscopic representations of the cones of the third order of the general form (1), and of the form with a nodal line (3). The generating lines are finite lines of equal length, and the curved contours shown in the figures are consequently the spherical curves which are the intersections of the cones by concentric spheres. The figures are intended to be looked at with the glasses of a Reeves's book stereoscope.

[1] It is hardly necessary to mention that, according to the general theory of cones of the third order, there are always nine lines of inflexion,—three real and six imaginary. Six of the lines of inflexion disappear when there is a double line, viz., in the case of a nodal line, two real and four imaginary lines of inflexion; but in the case of an isolated line, the six imaginary lines of inflexion. When there is a cuspidal line, eight lines of inflexion, viz. two real lines and the six imaginary lines, disappear.

2, *Stone Buildings, W.C., September* 26, 1859.