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A VERIFICATION IN REGARD TO THE LINEAR TRANSFORMATION OF THE THETA-FUNCTIONS.

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THE notation is that of Smith's* "Memoir on the Theta and Omega Functions," differing from Hermite's only in a factor, $\vartheta_{m,n}$ (Smith) $= i^{mn}\,\vartheta_{m,n}$ (Hermite); and it is in consequence of this that the factor $i^{\mu\nu-mn}$ occurs in the expression presently given for δ. Hermite's Memoir "Sur quelques formules relatives à la transformation des fonctions elliptiques" appeared in *Liouville*, t. III. (1858), pp. 27—36.

Writing

$$\omega = \frac{c + d\Omega}{a + b\Omega},$$

where $ad - bc = 1$, the formula of transformation is

$$\vartheta_{\mu,\nu}\left\{(a + b\Omega)\frac{\pi x}{h},\ \Omega\right\} = C\ \exp\left\{-i\pi b\,(a + b\Omega)\frac{x^2}{h^2}\right\}\vartheta_{m,n}\left(\frac{\pi x}{h},\ \omega\right),$$

where

$$m = a\mu + b\nu + ab,$$
$$n = c\mu + d\nu + cd.$$

We have, according as b is positive or negative,

$$C = \frac{\delta H}{\sqrt{\{-i\,(a + b\Omega)\}}},\ \text{ or }\ C = \frac{\delta H}{\sqrt{\{i\,(a + b\Omega)\}}},$$

where for each case the square root is to be taken in such wise that the real part is positive; δ, H are eighth roots of unity; viz. in each case

$$\delta = \exp\{-\tfrac{1}{4}\pi i\,(ac\mu^2 + 2bc\mu\nu + bd\nu^2 + 2abc\mu + 2abd\nu + ab^2c)\}\ i^{\mu\nu-mn},$$

[* H. J. S. Smith, *Collected Mathematical Papers*, vol. II., pp. 415—621.]

and, according as b is positive or negative,

$$H=\left(\frac{b}{a}\right) i^{-\frac{1}{2}a}, \quad a \text{ odd}, \qquad\qquad H=\left(\frac{-b}{-a}\right) i^{\frac{1}{2}a}, \quad a \text{ odd},$$

$$H=\left(\frac{a}{b}\right) i^{-\frac{1}{2}a-\frac{1}{2}(a-1)(b-1)}, \quad b \text{ odd}, \qquad H=\left(\frac{-a}{-b}\right) i^{\frac{1}{2}a-\frac{1}{2}(a+1)(b+1)}, \quad b \text{ odd},$$

where $\left(\frac{a}{b}\right)$, &c., are the Legendrian symbols as generalised by Jacobi; if a and b are each of them odd, the formulæ for the cases a odd and b odd respectively are equivalent to each other, and either may be used indifferently.

To fix the ideas, I assume throughout that b is positive and odd; the proper formulæ thus are

$$C=\frac{\delta H}{\sqrt{\{-i(a+b\Omega)\}}},$$

δ as above, and

$$H=\left(\frac{a}{b}\right) i^{-\frac{1}{2}a-\frac{1}{2}(a-1)(b-1)}.$$

I will also, for greater convenience, assume that c is odd; ad is consequently even, viz. the numbers a and d are each of them even, or else one is odd and the other even.

The equation $\omega=\frac{c+d\Omega}{a+b\Omega}$ gives $\Omega=\frac{c-a\omega}{-d+b\omega}$, and thence

$$a+b\Omega=\frac{-1}{-d+b\omega}.$$

Comparing the expressions for ω, Ω, it appears that we may interchange ω and Ω, writing also $-d, b, c, -a$ for a, b, c, d; and changing the other letters, we may for

$$a,\ b,\ c,\ \ d,\ \omega,\ \Omega,\ \mu,\ \nu,\ m,\ n,$$

write

$$-d,\ b,\ c,\ -a,\ \Omega,\ \omega,\ m,\ n,\ m',\ n',$$

where

$$m'=-dm+bn-bd,$$

$$n'=\ \ cm-an-ac.$$

The formula thus becomes

$$\vartheta_{m,n}\left\{(-d+b\omega)\frac{\pi x}{h},\ \omega\right\}=C'\exp\left\{-i\pi b(-d+b\omega)\frac{x^2}{h^2}\right\}\vartheta_{m',n'}\left(\frac{\pi x}{h},\ \Omega\right),$$

or, for x writing $(a+b\Omega)x$, this is

$$\vartheta_{m,n}\left\{-\frac{\pi x}{h},\ \omega\right\}=C'\exp\left\{i\pi(a+b\Omega)\frac{x^2}{h^2}\right\}\vartheta_{m',n'}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\},$$

or, observing that the left-hand side is $=(-)^{mn}\vartheta\left(\frac{\pi x}{h},\ \omega\right)$, we have

$$\vartheta_{m',\,n'}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\}=\frac{(-)^{mn}}{C'}\exp\left\{-i\pi(a+b\Omega)\frac{x^2}{h^2}\right\}\vartheta_{m,\,n}\left(\frac{\pi x}{h},\ \omega\right),$$

an equation which is of the same form as the original equation of transformation, and is to be identified with it.

We have

$$\begin{aligned} m' &= -dm+bn-bd = -d\,(a\mu+b\nu+ab) & &= -\mu+bd\,(-a+c-1),\\ & \qquad\qquad\qquad\quad +b\,(c\mu+d\nu+cd)-bd, & & \\ n' &= \quad cm-an-ac = \quad c\,(a\mu+b\nu+ab) & &= -\nu+ac\,(\ \ b-d-1);\\ & \qquad\qquad\qquad\quad -a\,(c\mu+d\nu+cd)-ac, & & \end{aligned}$$

values which may be written

$$\begin{aligned} m' &= 2P-\mu, \text{ where } P=\tfrac{1}{2}bd\,(-a+c-1),\\ n' &= 2Q-\nu, \quad \text{,,} \qquad Q=\tfrac{1}{2}ac\,(\ \ b-d-1), \end{aligned}$$

P and Q being integers. In fact, P is an integer if b or d is even, and if they are each of them odd, then from the equation $ad-bc=1$, a and c will be one of them odd, the other even, and $-a+c-1$ will be even; and similarly for Q.

The left-hand side is

$$\vartheta_{2P-\mu,\,2Q-\nu}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\},$$

which is

$$=(-)^{\mu\,(Q-\nu)}\,\vartheta_{\mu,\,\nu}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\},$$

and the equation finally is

$$(-)^{\mu\,(Q-\nu)}\,\vartheta_{\mu,\,\nu}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\}=\frac{(-)^{mn}}{C'}\exp\left\{-i\pi(a+b\Omega)\frac{x^2}{h^2}\right\}\vartheta_{m,\,n}\left(\frac{\pi x}{h},\ \omega\right).$$

Comparing this with the original equation

$$\vartheta_{\mu,\nu}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\}=C\exp\left\{-i\pi(a+b\Omega)\frac{x^2}{h^2}\right\}\vartheta_{m,\,n}\left(\frac{\pi x}{h},\ \omega\right),$$

the two equations will be identical if only

$$CC'=(-)^{-\mu Q+\mu\nu-mn}.$$

We have

$$C=\frac{\delta H}{\sqrt{\{-i\,(a+b\Omega)\}}},\quad C'=\frac{\delta' H'}{\sqrt{\{-i\,(-d+b\omega)\}}},$$

where the square roots are taken in such wise that the real part is positive; hence

$$CC'=\frac{\delta\delta'\,HH'}{\sqrt{(+1)}},$$

where $\surd(+1)$ means

$$\surd\{-i(a+b\Omega)\} \,.\, \surd\{-i(-d+b\omega)\},$$

the last-mentioned two square roots being as just explained; and we have moreover

$$\delta = \exp\{-\tfrac{1}{4}i\pi(\quad ac\mu^2 + 2bc\mu\nu + bd\nu^2 + 2abc\mu + 2abd\nu + ab^2c)\}\, i^{\mu\nu - mn},$$
$$\delta' = \exp\{-\tfrac{1}{4}i\pi(-dcm^2 + 2bcmn - abn^2 - 2dbcm + 2abdn - db^2c)\}\, i^{mn - m'n'},$$

viz. the value of δ' is obtained from that of δ by the change $a, b, c, d, \mu, \nu, m, n$ into $-d, b, c, -a, m, n, m', n'$.

Representing these by

$$\delta = \exp\{-\tfrac{1}{4}(i\pi)\Delta\}\, i^{\mu\nu - mn}, \quad \delta' = \exp\{-\tfrac{1}{4}(i\pi)\Delta'\}\, i^{mn - m'n'},$$

we have

$$\delta\delta' = \exp\{-\tfrac{1}{4}i\pi(\Delta + \Delta')\}\, i^{\mu\nu - m'n'}.$$

But

$$m'n' = (2P - \mu)(2Q - \nu), \ = 4PQ - 2P\nu - 2Q\mu + \mu\nu,$$

that is,

$$\mu\nu - m'n' = -4PQ + 2P\nu + 2Q\mu;$$

or, omitting the term divisible by 4,

$$i^{\mu\nu - m'n'} = i^{2P\nu + 2Q\mu}, \ = (-)^{P\nu + Q\mu}.$$

To calculate $\Delta + \Delta'$, we have

$$dm - bn = \mu + bd(\quad a - c),$$
$$-cm + an = \nu + ac(-b + d),$$

and thence

$$-cdm^2 + (ad + bc)mn - abn^2 = \quad \mu\nu + \mu ac(-b + d) + \nu bd(a - c) + abcd(d - b)(a - c)$$
$$(-ad + bc)mn \qquad = -ac\mu^2 - bd\nu^2 - (ad + bc)\mu\nu - \mu ac(b + d) - \nu bd(a + c) - abcd,$$

consequently

$$-cdm^2 + 2bcmn - abn^2 = -ac\mu^2 - bd\nu^2 - 2bc\mu\nu - 2abc\mu - 2bcd\nu + abcd(2bc - ab - cd);$$

also

$$-2bcdm \quad = \quad -2abcd\mu - 2b^2cd\nu - 2ab^2cd,$$
$$2abdn \quad = \quad 2abcd\mu + 2abd^2\nu + 2abcd^2,$$
$$-db^2c \quad = \quad -db^2c,$$

whence, adding, we obtain

$$\Delta' = -ac\mu^2 - 2bc\mu\nu - bd\nu^2 - 2abc\mu + (-2bcd - 2b^2cd + 2abd^2)\nu$$
$$+ abcd(2bc - ab - cd - 2b + 2d) - db^2c,$$

and, adding to this the expression of Δ, we find

$$\Delta + \Delta' = (2abd - 2bcd - 2b^2cd + 2abd^2)\nu + ab^2c - db^2c + abcd(2bc - ab - cd - 2b + 2d).$$

The coefficient of ν is $=2bd\,(a-c-bc+ad)=2bd\,(a-c+1)$, which is $=-4P\nu$. Hence writing

$$\Theta=\tfrac{1}{4}abcd\,(2bc-ab-cd-2b+2d)+\tfrac{1}{4}b^2c\,(a-d),$$

where observe that 4Θ, but not in every case Θ itself, is an integer, we have $\Delta+\Delta'=-4P\nu+4\Theta$, and consequently

$$\delta\delta'=(-)^{\Theta+P\nu}\,(-)^{P\nu+Q\mu},\ =(-)^{\Theta+2P\nu+Q\mu},$$

or, omitting the even number $2P\nu$,

$$\delta\delta'=(-)^{\Theta+Q\mu}.$$

Observe that $(-)^{\Theta}$ denotes, and it might properly have been written, $\exp i\pi\Theta$. The foregoing equation

$$\frac{\delta\delta' HH'}{\sqrt{(+1)}}=(-)^{-Q\mu+\mu\nu-mn}$$

becomes thus

$$(-)^{\Theta+Q\mu}\,\frac{HH'}{\sqrt{(+1)}}=(-)^{-Q\mu+\mu\nu-mn},$$

that is,

$$\frac{HH'}{\sqrt{(+1)}}=(-)^{-\Theta-2Q\mu+\mu\nu-mn},$$

where the even term $-2Q\mu$ may be omitted. We have moreover

$$mn=ac\mu^2+(ad+bc)\,\mu\nu+bd\nu^2+ac\,(b+d)\,\mu+bd\,(a+c)\,\nu+abcd,$$
$$-\mu\nu=\qquad(-ad+bc)\,\mu\nu,$$

and thence

$$mn-\mu\nu=ac\,(\mu^2-\mu)+2bc\mu\nu+bd\,(\nu^2-\nu)+ac\,(b+d+1)\,\mu+bd\,(a+c+1)\,\nu+abcd,$$

where each term is even; hence $mn-\mu\nu$ is even, and we have simply

$$\frac{HH'}{\sqrt{(+1)}}=(-)^{-\Theta},$$

where

$$\Theta=\tfrac{1}{4}abcd\,(ab+cd-2bc+2b-2d)-\tfrac{1}{4}b^2c\,(a-d).$$

I write $M=\tfrac{1}{4}b\,(a-d)$, then

$$\Theta-M=\tfrac{1}{4}abcd\,(ab+cd-2bc+2b-2d)-\tfrac{1}{4}\,(bc+1)\,b\,(a-d),$$

where the second term is $=-\tfrac{1}{4}abd\,(a-d)$, and we have therefore

$$\Theta-M=\tfrac{1}{4}abd\,(abc+c^2d-2bc^2+2bc-2cd-a+d).$$

I assume, as above, that b and c are each of them odd; therefore ad is even. I suppose, first, that ad divides by 4, then $\tfrac{1}{4}abd$ is an integer, and in the expression of $\Theta-M$, omitting even numbers, we have

$$\Theta-M=\tfrac{1}{4}abd\,(abc+c^2d-a+d),$$

which, putting therein $bc = ad - 1$, becomes

$$= \tfrac{1}{4}abd\,(a^2d + c^2d - 2a + d),$$
$$= \tfrac{1}{4}abd\,\{a^2d + (c^2 - c)\,d + (c+1)\,d - 2a\},$$

where inside the { } each term is even; hence $\Theta - M$ is even.

Next, if ad is even but not divisible by 4, then $bc = ad - 1$, which is $\equiv 1$ (mod. 4), thus b and c are

$$= 4\sigma + 1 \text{ and } 4\tau + 1,$$

or else

$$= 4\sigma - 1 \text{ and } 4\tau - 1,$$

and, moreover, $bc = 4\theta + 1$, and $c^2 = 4\phi + 1$; hence

$$\Theta - M = \tfrac{1}{2}b\,.\,\tfrac{1}{2}ad\,\{a\,(4\theta+1) + d\,(4\phi+1) - 2b\,(4\phi+1) + 2\,(4\theta+1) - 2cd - a + d\},$$

or, omitting even numbers, that is, inside the { } numbers which contain the factor 4, this is

$$= \tfrac{1}{4}abd\,(a + d - 2b + 2 - a + d),$$
$$= \tfrac{1}{2}abd\,(d - b + 1 - cd),$$
$$= \tfrac{1}{2}abd\,\{d\,(1-c) + 1 - b\},$$

or, since each term within the { } is even, we have in this case also $\Theta - M$ even. And this being so, the foregoing equation for HH' becomes

$$\frac{HH'}{\sqrt{(+1)}} = (-)^{-M}, \; = (-)^{\frac{1}{4}b\,(d-a)}, \text{ or say } = i^{\frac{1}{2}b\,(d-a)}.$$

The values of H, H', refer each of them to a positive odd value of b, and they thus are

$$H = \left(\frac{a}{b}\right) \quad i^{-\frac{1}{2}a - \frac{1}{2}(a-1)\,(b-1)},$$

$$H' = \left(-\frac{d}{b}\right) i^{\frac{1}{2}d - \frac{1}{2}\,(-d-1)\,(b-1)};$$

hence

$$HH' = \left(\frac{a}{b}\right)\left(\frac{-d}{b}\right) i^{-\frac{1}{2}\,(a-d) - \frac{1}{2}\,(a-d-2)\,(b-1)} = \left(\frac{a}{b}\right)\left(\frac{-d}{b}\right) i^{b-1+\frac{1}{2}b\,(d-a)},$$

or, since

$$\left(\frac{-d}{b}\right) = (-)^{\frac{1}{2}(b-1)}\left(\frac{d}{b}\right) = i^{b-1}\left(\frac{d}{b}\right),$$

and $2\,(b-1)$ divides by 4, this is

$$HH' = \left(\frac{a}{b}\right)\left(\frac{d}{b}\right) i^{\frac{1}{2}b\,(d-a)}.$$

Also $\left(\frac{a}{b}\right)\left(\frac{d}{b}\right) = \left(\frac{ad}{b}\right)$, but from the equation $ad - bc = 1$, or $\frac{ad}{b} = c + \frac{1}{b}$, we have $\left(\frac{ad}{b}\right) = \left(\frac{1}{b}\right) = 1$; whence

$$HH' = i^{\frac{1}{2}b\,(d-a)}.$$

We have $\omega = x + iy$, where y is positive; hence

$$-d + b\omega = -d + bx + iby = \alpha + 2by, \text{ if } \alpha = -d + bx;$$

hence

$$a + b\Omega = \frac{-1}{\alpha + iby} = \frac{-\alpha + iby}{\alpha^2 + \beta^2 y^2},$$

$$-i(-d - b\Omega) = by - i\alpha = R(\cos\theta + i\sin\theta),$$

$$-i(\quad a + b\Omega) = \frac{by + i\alpha}{\alpha^2 + \beta^2 y^2} = \frac{R(\cos\theta - i\sin\theta)}{\alpha^2 + \beta^2 y^2},$$

where R is positive; and $\cos\theta$ is positive since y is positive, and thus θ lies between $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$. Hence

$$\sqrt{\{-i(-d + b\Omega)\}} = \sqrt{R}(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta),$$

$\cos\frac{1}{2}\theta$ being positive,

$$\sqrt{\{-i(\quad a + b\Omega)\}} = \frac{\sqrt{R}(\cos\frac{1}{2}\theta - i\sin\frac{1}{2}\theta)}{\sqrt{(\alpha^2 + \beta^2 y^2)}},$$

and thus

$$\sqrt{\{-i(a + b\Omega)\}}\sqrt{\{-i(-d + b\Omega)\}} = \frac{R}{\sqrt{(\alpha^2 + \beta^2 y^2)}} = +1,$$

that is, $\sqrt{(+1)} = +1$, and we thus have, as we should do,

$$\frac{HH'}{\sqrt{(+1)}} = i^{\frac{1}{2}b(d-a)},$$

the equation which was to be verified.