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ON THE DEMONSTRATION OF A THEOREM RELATING TO THE MOMENTS OF INERTIA OF A SOLID BODY.

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CONSIDERING in the first instance the analogous question *in plano*, let $a=\int x^2dm$, $b=\int y^2dm$, $h=\int xydm$, where the integration extends over any closed figure whatever, then it is to be shown that the equations

$$(a,\ h,\ b)(p,\ q)^2=1, \text{ and } (a+b)(p^2+q^2)-(a,\ h,\ b)(p,\ q)^2=1,$$

represent respectively ellipses.

If in the first case, by a transformation of coordinates,

$$(a,\ h,\ b)(p,\ q)^2=a_1p_1^2+b_1q_1^2,$$

then a_1, b_1 are the roots of the quadratic equations,

$$\begin{vmatrix} a-\rho, & h \\ h\ , & b-\rho \end{vmatrix}=0,$$

and if in the second case,

$$(a+b)(p^2+q^2)-(a,\ h,\ b)(p,\ q)^2=a_1p_1^2+b_1q_1^2,$$

then a_1, b_1 are the roots of

$$\begin{vmatrix} b-\rho, & -h \\ -h, & a-\rho \end{vmatrix}=0,$$

the two equations being in fact the same equation,

$$\rho^2-(a+b)\rho+ab-h^2=0,$$

and the conditions that the curve may be an ellipse, are

$$a+b=+,$$
$$ab-h^2=+,$$

the former of which requires no demonstration; to prove the latter, changing merely the variables under the integral sign, I write

$$a'=\int x'^2 dm',\quad b'=\int y'^2 dm',\quad h'=\int x'y' dm',$$

these quantities being of course respectively equal to *a*, *b*, *h*, we have then

$$ab'+a'b-2hh'=\iint (xy'-x'y)^2\, dm dm' = 2\,(ab-h^2),$$

or since the quantity under the integral sign is a square, $ab-h^2$ is positive.

For the analogous problem *in solido*, we have

$$a=\int x^2 dm,\ b=\int y^2 dm,\ c=\int z^2 dm,\ f=\int yz dm,\ g=\int zx dm,\ h=\int xy dm,$$

and it is to be shown that the equations

$$(a,\ b,\ c,\ f,\ g,\ h)(p,\ q,\ r)^2=1,$$
$$(a+b+c)(p^2+q^2+r^2)-(a,\ b,\ c,\ f,\ g,\ h)(p,\ q,\ r)^2=1,$$

represent respectively ellipsoids.

The conditions in the first problem are

$$a+b+c=+,$$
$$bc+ca+ab-f^2-g^2-h^2=+,$$
$$abc-af^2-bg^2-ch^2+2fgh=+,$$

the first of which is obviously true: as regards the second, the theorem *in plano* shows that each of the quantities $bc-f^2$, $ca-g^2$, $ab-h^2$ is positive, or merely reproducing the investigation, we find

$$2\,(bc+ca+ab-f^2-g^2-h^2)=\iint [(yz'-y'z)^2+(zx'-z'x)^2+(xy'-x'y)^2]\, dm dm',$$

which proves the theorem, and where it is to be observed that the integral may also be written

$$\iint [(x^2+y^2+z^2)(x'^2+y'^2+z'^2)-(xx'+yy'+zz')^2]\, dm dm';$$

and for the third, we find in a precisely similar manner,

$$6\,(abc-af^2-bg^2-ch^2+2fgh)=\iint \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix}^2 dm dm',$$

which proves the theorem. The integral may also be written

$$\iint \begin{vmatrix} x^2 + y^2 + z^2, & xx' + yy' + zz', & xx'' + yy'' + zz'' \\ x'x + y'y + z'z, & x'^2 + y'^2 + z'^2, & x'x'' + y'y'' + z'z'' \\ x''x + y''y + z''z, & x''x' + y''y' + z''z', & x''^2 + y''^2 + z''^2 \end{vmatrix} dm\,dm'.$$

The conditions in the second problem are

$$(b+c)+(c+a)+(a+b)=+,$$
$$(c+a)(a+b)+(a+b)(b+c)+(b+c)(c+a)-f^2-g^2-h^2=+,$$
$$(a+b)(b+c)(c+a)-(b+c)f^2-(c+a)g^2-(a+b)h^2-2fgh=+,$$

the first and second of which are respectively equivalent to

$$a+b+c=+,$$
$$(a+b+c)^2+bc+ca+ab-f^2-g^2-h^2=+,$$

which are already proved. The last may be written

$$(a+b+c)(bc+ca+ab-f^2-g^2-h^2)-(abc-af^2-bg^2-ch^2+2fgh)=+,$$

which, putting for shortness,

$$A = x^2 + y^2 + z^2, \quad B = x'^2 + y'^2 + z'^2, \quad C = x''^2 + y''^2 + z''^2,$$
$$F = x'x'' + y'y'' + z'z'', \quad G = x''x + y''y + z''z, \quad H = xx' + yy' + zz',$$

is by what precedes expressible in the form

$$\tfrac{1}{6}\iint \{A(BC-F^2)+B(CA-G^2)+C(AB-H^2)-(ABC-AF^2-BG^2-CH^2+2FGH)\}\,dm\,dm'$$
$$=\tfrac{1}{3}\iint (ABC-FGH)\,dm\,dm',$$

or, since $\sqrt{BC}>F$, $\sqrt{CA}>G$, $\sqrt{AB}>H$, we have $ABC>FGH$, or $ABC-FGH=+$, and therefore the value of the integral is also positive.

2, *Stone Buildings, W.C.*, 6*th March*, 1860.