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ON A NEW ANALYTICAL REPRESENTATION OF CURVES IN SPACE.

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THE employment of a new kind of coordinates for the analytical representation of curves in space is suggested in my former paper under the same title *Journal*, t. III., pp. 225—236 (1859). The idea was as follows: viz. if (x, y, z, w) are current coordinates of a point in space (ordinary point coordinates), and $(\alpha, \beta, \gamma, \delta)$ the coordinates of a particular point, then taking (p, q, r, s, t, u) to represent the minor determinants formed out of the matrix

$$\begin{pmatrix} x, & y, & r, & w \\ \alpha, & \beta, & \gamma, & \delta \end{pmatrix},$$

viz.

$$p = \gamma y - \beta z, \quad s = \delta x - \alpha w,$$
$$q = \alpha z - \gamma x, \quad t = \delta y - \beta w,$$
$$r = \beta x - \alpha y, \quad u = \delta z - \gamma w,$$

values which satisfy identically

$$ps + qt + ru = 0,$$

then the equation of a cone passing through a given curve and having for its vertex the arbitrary point $(\alpha, \beta, \gamma, \delta)$, is of the form

$$V = 0,$$

V being a homogeneous function of the six new coordinates (p, q, r, s, t, u). And it was proposed to consider $V = 0$ as the equation of the curve.

But as remarked in the paper, it is not every function V of the coordinates (p, q, r, s, t, u) which equated to zero, does in fact represent a curve. In order that the equation $V=0$ may represent a curve, it is necessary, that when any infinitesimal variations whatever are given to the constants $(\alpha, \beta, \gamma, \delta)$, thus converting the equation into $V+\delta V=0$, the two equations $V=0$, $\delta V=0$ (considered as equations in ordinary point coordinates) shall represent one and the same curve, whatever the system of infinitesimal variations attributed to $\alpha, \beta, \gamma, \delta$ may be. Let P, Q, R, S, T, U denote the differential coefficients of V in regard to p, q, r, s, t, u respectively, then the equation $\delta V=0$, breaks up into the equations

$$\begin{array}{rrrr} & -Ry & +Qz & -Sw=0, \\ Rx & . & -Pz & -Tw=0, \\ -Qx & +Py & . & -Uw=0, \\ Sx & +Ty & +Uz & . \quad =0, \end{array}$$

and the system composed of these four equations and the equation $V=0$ (considered as equations in ordinary point coördinates) must belong to one and the same curve.

The four equations gave

$$PS+QT+RU=0,$$

a relation between the differential coefficients of V which must be satisfied either identically or in virtue of the equation $V=0$. And this relation existing, any two of the four equations lead to the other two. Attending exclusively to the coordinates (p, q, r, s, t, u) and considering (x, y, z, w) as mere arbitrary multipliers, the above equation

$$PS+QT+RU=0$$

is the only relation between the differential coefficients of V which is deducible from the four equations.

But it was noticed that the equation $V=0$, even when V is a function such that we have (identically or in virtue of the equation $V=0$) the equation $PS+QT+RU=0$, does not of necessity represent a curve. Some further relation or relations between the differential coefficients of V must therefore exist, either identically or in virtue of the equation $V=0$; and such relations can be found by resorting to the second differential $\delta^2 V$ of the function V. In fact not only the equation $\delta V=0$ but the entire series of relations $\delta^2 V=0$, $\delta^3 V=0, \ldots$ should be satisfied by the coordinates of any point of the curve. I find by means of the equation $\delta^2 V=0$ a plexus of equations, which are consequently necessary, and I am inclined to believe sufficient, in order that the equation $V=0$ may in fact represent a curve; the equations of the plexus are, it will be seen, very numerous, and certainly only a small number of them are independent, but this is a question which I have not as yet investigated.

Attending to the expressions for p, q, r, s, t, u, we have

$$\begin{array}{rl} d_\alpha = & \quad . \quad -yd_r + zd_q - wd_s = (1), \\ d_\beta = & xd_r \quad . \quad -zd_p - wd_t = (2), \\ d_\gamma = & -xd_q + yd_p \quad . \quad -wd_u = (3), \\ d_\delta = & xd_s + yd_t + zd_u \quad . \quad = (4), \end{array}$$

and writing for convenience a, b, c, d instead of d_α, d_β, d_γ, d_δ, we have

$$d = (1)\,\text{a} + (2)\,\text{b} + (3)\,\text{c} + (4)\,\text{d}.$$

It was in effect by operating on V with this symbol and equating to zero the coefficients of a, b, c, d, that the before-mentioned equations

$$\begin{aligned} \cdot \quad - Ry + Qz - Sw &= 0, \\ Rx \quad \cdot \quad - Pz - Tw &= 0, \\ -Qx + Py \quad \cdot \quad - Uw &= 0, \\ Sx + Ty + Uz \quad \cdot \quad &= 0, \end{aligned}$$

were found.

If to these equations we join the equation

$$Ax + By + Cz + Dw = 0,$$

where A, B, C, D are arbitrary multipliers, we can express x, y, z, w in terms of A, B, C, D in such manner as to satisfy the four equations, viz. we have

$$\begin{aligned} x &= \quad \cdot \quad BU - CT + DP, \\ y &= -AU \quad \cdot \quad + CS + DQ, \\ z &= AT - BS \quad \cdot \quad + DR, \\ w &= -AP - BQ - CR \quad \cdot \quad , \end{aligned}$$

and if in the expressions for (1), (2), (3), (4) we substitute for x, y, z, w these values, and form therewith the value of d, which value I will for distinction call $\mathfrak{D}$, we have

$$\mathfrak{D} = \left(\begin{array}{cccc} Ud_r + Td_q + Pd_s, & Qd_s - Sd_q, & Rd_s - Sd_r, & Rd_q - Qd_r \\ Pd_t - Td_p, & Ud_r + Qd_t + Sd_p, & Rd_t - Td_r, & Pd_r - Rd_p \\ Pd_u - Ud_p, & Qd_u - Ud_q, & Rd_u + Sd_p + Td_q, & Qd_p - Pd_q \\ Td_u - Ud_t, & Ud_s - Sd_u, & Sd_t - Td_s, & Pd_s + Qd_t + Rd_u \end{array}\right)(A, B, C, D)(\text{a}, \text{b}, \text{c}, \text{d}),$$

viz. $\mathfrak{D}$ is a lineo-linear function of the two sets of indeterminate quantities (A, B, C, D), (a, b, c, d), the coefficients thereof being the operators

$$Ud_r + Td_q + Pd_s,\ Qd_s - Sd_q,\ \&\text{c}.$$

It may be remarked that we have identically

$$\mathfrak{D}V = (PS + QT + RU)(A\text{a} + B\text{b} + C\text{c} + D\text{d}),$$

since obviously each term such as $(Qd_s - Sd_q)V$, which is equal to $QS - SQ$, vanishes identically. The equation $\mathfrak{D}V = 0$ gives therefore only the before-mentioned equation $PS + QT + RU = 0$, which is as it should be.

The equation $\mathfrak{D}^2 V = 0$, is then to be satisfied independently of the values of (A, B, C, D) and (a, b, c, d), and as $\mathfrak{D}$ contains 16 distinct terms, $\mathfrak{D}^2$ will contain in all $\frac{1}{2}16.17$ or 136 distinct terms. The equation $\mathfrak{D}^2 V = 0$ gives therefore a plexus of 136 equations, and the equations in each succeeding plexus, involved in $\mathfrak{D}^3 V = 0$, $\mathfrak{D}^4 V = 0$, &c. will, of course, be still be more numerous.

If $V = 0$ be the plane conic which is the intersection of the surfaces

$$x^2 + y^2 + z^2 + w^2 = 0,$$

$$ax + by + cz + dw = 0,$$

then we have

$$V = \left(\begin{array}{cccccc} b^2 + c^2, & -ab, & -ac, & \cdot, & cd, & -bd \\ -ba, & c^2 + a^2, & -bc, & -cd, & \cdot, & ad \\ -ca, & -cb, & a^2 + b^2, & bd, & -ad, & \cdot \\ \cdot, & -cd, & bd, & a^2 + d^2, & ab, & ac \\ cd, & \cdot, & -ad, & ba, & b^2 + d^2, & bc \\ -bd, & ad, & \cdot, & ca, & cb, & c^2 + d^2 \end{array} \right) (p, q, r, s, t, u)^2.$$

The values of P, Q, R, S, T, U (omitting a common factor 2) are

$$P = (b^2 + c^2, -ab, -ac, \cdot, +cd, -bd)(p, q, r, s, t, u),$$
&c.,

and if we proceed to form a term in $\mathfrak{D}^2 V$, say the coefficient of $A^2\text{a}^2$, this is $(Ud_r + Td_q + Pd_s)^2\ V$, or

$$(a^2 + b^2, c^2 + a^2, a^2 + d^2, -cd, +bd, -cb)(U, T, P)^2.$$

The coefficient therein of p^2 is

$$(a^2 + b^2, c^2 + a^2, a^2 + d^2, -cd, +bd, -cb)(-bd, cd, b^2 + c^2)^2,$$

that is, it is

$$\begin{aligned} &(a^2 + b^2)\, b^2 d^2 \quad - 2cd\, .\ cd\,(b^2 + c^2) \\ &+ (c^2 + a^2)\, c^2 d^2 \quad + 2bd\, .\, -bd\,(b^2 + c^2) \\ &+ (a^2 + d^2)(b^2 + c^2)^2 - 2cb\, .\, -bd\, .\, cd, \end{aligned}$$

where the terms in which $(b^2 + c^2)$ does not appear as a factor are together equal to

$$a^2 d^2 (b^2 + c^2) + d^2 (b^2 + c^2)^2,$$

the entire expression thus divides by $b^2 + c^2$, the quotient being

$$(a^2 + d^2)(b^2 + c^2) - 2c^2 d^2 - 2b^2 d^2 + a^2 d^2 + d^2 (b^2 + c^2),$$

which is equal to $a^2 (b^2 + c^2 + d^2)$, or restoring the factor $b^2 + c^2$, we see that in $\mathfrak{D}^2 V$ the coefficient of $A^2\text{a}^2$ is

$$a^2 (b^2 + c^2 + d^2)(b^2 + c^2)\, p^2 + \&\text{c}.$$

The complete value must, it is clear, be of the form

$$a^2(b^2+c^2+d^2)\,V+k\,(ps+qt+ru),$$

vanishing in virtue of the equations $V=0$, $ps+qt+ru=0$, and this being so, observing that V contains no term in ps, we have $k=$ coefficient ps in

$$(a^2+b^2,\ c^2+a^2,\ a^2+d^2,\ -cd,\ +bd,\ -cb)\,(U,\ T,\ P)^2,$$

that is

$$k=2\,(a^2+b^2,\ c^2+a^2,\ a^2+d^2,\ -cd,\ +bd,\ -cb)\,(-bd,\ cd,\ b^2+c^2)\,(ca,\ ba,\ 0),$$

or

$$\begin{aligned}\tfrac{1}{2}k = \ &(a^2+b^2)\,.-bd\,.\,ca \ - \ cd\,\{cd\,.\,0+(b^2+c^2)\,ba\}\\ &+(c^2+a^2)\,.\quad cd\,.\,ba \ + \ bd\,\{(b^2+c^2)\,ca-bd\,.\,0\}\\ &+(a^2+d^2)\,.\qquad 0 \ - \ cb\,\{-bd\,.\,ba+cd\,.\,ca\},\end{aligned}$$

which is

$$=abcd\left\{\begin{matrix}-(a^2+b^2) & -(b^2+c^2)\\ +(c^2+a^2) & +(b^2+c^2)\\ & +(b^2-c^2)\end{matrix}\right\},\ =0.$$

The coefficient k consequently vanishes, and therefore in $\mathfrak{D}^2V$ the coefficient of $A^2\mathfrak{a}^2$ is $a^2(b^2+c^2+d^2)\,V$, but I have not worked out the coefficients of the other terms.

2, *Stone Buildings, W.C.*, 30*th October*, 1860.