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BRIEF NOTES

Integration of boundary layer equations at low speeds

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THE METHOD of integral relations [1] is applied to the boundary layer flow where viscosity and conductivity are assumed to depend on temperature. Viscous heating is not neglected but the flow speed is taken to be low so that the fluid can be assumed incompressible.

1. Governing equations

IN LOW speed flow [2] where the difference between the temperature of the stream and that of the plate is not too great (so that density is sensibly constant), the set of differential equations of laminar boundary layer is of the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\varrho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right), \tag{1.2}$$

$$\begin{aligned}\frac{u\partial}{\partial x}\left(C_pT+\frac{1}{2}u^2\right)+v\frac{\partial}{\partial y}\left(C_pT+\frac{1}{2}u^2\right)\\ =\frac{\partial}{\partial y}\left(\nu\frac{\partial}{\partial y}\left(C_pT+\frac{1}{2}u^2\right)\right)+\left(\frac{1}{\sigma}-1\right)\frac{\partial}{\partial y}\left(\nu\frac{\partial}{\partial y}(C_pT)\right),\end{aligned} \tag{1.3}$$

where

$$\sigma=\frac{C_p\mu}{R},\qquad \nu=\frac{\mu}{\varrho},$$

u and v are tangential and normal velocity components while P, ϱ, T are pressure, density and temperature, respectively. C_p, μ, x are the specific heat at the constant pressure coefficient of viscosity and the coefficient of thermal conductivity, respectively.

Now transform to new variables given by [4]

$$\xi=\int_0^x U dx,\qquad \eta=\frac{Uy}{\sqrt{\nu_d}}.$$

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The subscript d denotes the stagnation value. Equations (1.1) to (1.3) are then put in the divergence form

$$\frac{\partial q}{\partial \xi}+\frac{\partial \omega}{\partial \eta}=0, \tag{1.4}$$

$$q\frac{\partial q}{\partial \xi}+\frac{\omega\partial q}{\partial \eta}=\frac{1}{U}\frac{dU}{d\xi}(1-q^2)+\frac{\partial}{\partial \eta}\left(b\frac{\partial q}{\partial \eta}\right), \tag{1.5}$$

$$q\frac{\partial h}{\partial \xi}+\omega\frac{\partial h}{\partial \eta}=\frac{1}{\sigma}\frac{\partial}{\partial \eta}\left(b\frac{\partial h}{\partial \eta}\right)-n^2\left(\frac{1}{\sigma}-1\right)\frac{\partial}{\partial \eta}\left(b\frac{\partial q^2}{\partial \eta}\right), \tag{1.6}$$

where $U(x)$ is the upstream velocity and

$$\begin{gathered} q=\frac{u}{U}, \quad \omega=\frac{v}{U\sqrt{\nu_d}}+\frac{u}{U^2}\frac{dU}{d\xi}\eta, \\ h=\frac{T}{T_d}+\frac{1}{2}\frac{U^2}{C_pT_d}, \quad q^2=\frac{T}{T_d}+q^2n^2, \\ n^2=\frac{1}{2}\frac{U^2}{C_pT_d}, \quad b=\frac{\nu}{\nu_d}. \end{gathered} \tag{1.7}$$

The boundary conditions are

$$\begin{aligned} q=\omega=0 \quad &\text{when} \quad \eta=0, \\ q\to 1, \quad h\to 1 \quad &\text{as} \quad \eta\to\infty. \end{aligned} \tag{1.8}$$

2. Integral relations

Multiplying Eq. (1.4) by $f'(q)$ and Eq. (1.5) by $f(q)$ and adding the results, we have

$$\frac{\partial}{\partial \xi}(fq)+\frac{\partial}{\partial \eta}(f\omega)=\frac{1}{U}\frac{dU}{d\xi}(1-q^2)f'(q)+f'(q)\frac{\partial}{\partial \eta}\left(\frac{b\partial q}{\partial \eta}\right), \tag{2.1}$$

where $f(q)$ is an arbitrary function of q to be determined later. Integrating Eq. (2.1) across the boundary layer with respect to η and then changing to the variable $q(0\leqslant q\leqslant 1)$, we have

$$\frac{d}{d\xi}\int_0^1 pqf\,dq+[f\omega]_0^1=\frac{1}{U}\frac{dU}{d\xi}\int_0^1 p(1-q^2)f'(q)dq+\left[f'(q)\frac{b}{p}\right]_0^1-\int_0^1\frac{bf''(q)}{p}dq, \tag{2.2}$$

where $p=1\Big/\dfrac{\partial q}{\partial \eta}$.

This is the first integral relation. To obtain the second we multiply Eq. (1.4) by $(1-p)f(q)$, Eq. (1.5) by $(1-p)f'(q)$ and Eq. (1.6) by $f(q)$ and add all the three results to obtain

$$\begin{aligned} \frac{\partial}{\partial \xi}(fqh)+\frac{\partial}{\partial \eta}(f\omega h)=hf'(q)\frac{1}{U}\frac{dU}{d\xi}(1-q^2) \\ +hf'(q)\frac{\partial}{\partial \eta}\left(b\frac{\partial q}{\partial \eta}\right)+\frac{1}{\sigma}f\frac{\partial}{\partial \eta}\left(b\frac{\partial h}{\partial \eta}\right)+2^2_n\left(1-\frac{1}{\sigma}\right)f\frac{\partial}{\partial \eta}\left(bq\frac{\partial q}{\partial \eta}\right). \end{aligned} \tag{2.3}$$

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Integrating this with respect to η and again passing to the variable q, we have

$$(2.4)\quad \frac{d}{d\xi}\int_0^1 phqf\,dq+[f\omega h]_0^1 = \frac{1}{U}\frac{dU}{d\xi}\int_0^1 phf'(q)(1-q^2)dq$$
$$+\int_0^1 hf'(q)\frac{\partial}{\partial q}\left(\frac{b}{p}\right)dq+\int_0^1 \frac{1}{\sigma}f\frac{\partial}{\partial q}\left(\frac{b}{p}\frac{\partial h}{\partial q}\right)dq+2\int_0^1 n^2\left(1-\frac{1}{\sigma}\right)f\frac{\partial}{\partial q}\left(\frac{b}{p}q\right)dq.$$

This is the second integral relation.

For the first approximation we have to choose $f(q)$ to ensure the convergence of the integrals in Eqs. (2.2) and (2.4) and also such that $f(q)$ shall tend to zero sufficiently quickly as $q\to 1$. We therefore take $f(q) = 1-q$. Equation (2.2) becomes

$$(2.5)\quad \frac{d}{d\xi}\int_0^1 pq(1-q)\,dq = -\frac{1}{U}\frac{dU}{d\xi}\int_0^1 p(1-q^2)dq-\left[\frac{b}{p}\right]_0^1$$

and Eq. (2.4) becomes

$$(2.6)\quad \frac{d}{d\xi}\int_0^1 phq(1-q)dq = -\frac{1}{U}\frac{dU}{d\xi}\int_0^1 ph(1-q^2)dq-\int h\frac{\partial}{\partial q}\left(\frac{b}{p}\right)dq$$
$$+\int_0^1 \frac{1}{\sigma}(1-q)\frac{\partial}{\partial q}\left(\frac{b}{p}\frac{\partial h}{\partial q}\right)dq+2\int_0^1 n^2\left(1-\frac{1}{\sigma}\right)(1-q^2)\frac{\partial}{\partial q}\left(\frac{b}{p}q\right)dq.$$

3. Solution

Now let us represent the integrands by

$$(3.1)\quad p = \frac{1}{1-q}a_0,\quad h = \delta_0+\delta_1 q,\quad b = \left(\frac{T}{T_d}\right)^i.$$

Considering the case $U(x) = U_0x^m$ whence

$$\xi = \frac{U_0x^{m+1}}{m+1},\quad \eta = \frac{U_0x^m y}{\sqrt{\nu_d}},$$

Eq. (2.5) becomes

$$(3.2)\quad \frac{da_0^2}{d\xi}+\frac{6m}{m+1}\frac{a_0^2}{\xi} = 4,$$

which has a solution

$$(3.3)\quad a_0^2 = \frac{4(m+1)}{7m+1}\xi,$$
$$\text{i.e.}\quad a_0 = \left\{\frac{4U_0x^{m+1}}{7m+1}\right\}^{1/2}$$

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and p is then given by

(3.4) $$p = \frac{1}{1-q}\left(\frac{4(m+1)\xi}{7m+1}\right)^{1/2} = \frac{\partial\eta}{\partial q},$$

which, on integration, gives

(3.5) $$q = 1-e^{-\left(\frac{\eta}{2\alpha^{1/2}}\xi^{1/2}\right)},$$

where $\alpha = \dfrac{m+1}{7m+1}$.

In terms of x and y Eq. (3.5) becomes

(3.6) $$q = 1-e^{-\left(\beta y x^{\frac{m-1}{2}}\right)},$$

where

$$\beta = \frac{U_0^{1/2}}{2\sqrt{\nu_d}\,(7m+1)^{1/2}}.$$

If $m = 0$, which corresponds to the flat plate, $\alpha = 1$ and

(3.6′) $$q = 1-e^{-\eta/2\xi^{1/2}}.$$

4. $\sigma = 1$

When the Prandtl number (σ) is equal to unity [3] and there is no pressure gradient (U = constant), Eqs. (1.5) and (1.6) are identical if h is replaced by q in Eq. (1.6). In this case

(4.1) $$h = Bq,$$

where B is a constant.

Also Eq. (1.6) always has a solution

(4.2) $$h = \text{constant} = A.$$

Hence the general solution is

(4.3) $$h = A+Bq.$$

This means that wherever $\sigma = 1$, regardless of the viscosity-temperature relationship, Eq. (4.3) always satisfies Eq. (1.6) for the zero pressure gradient and Eq. (1.5) is satisfied by Eq. (3.6′).

If T_w is the wall temperature which is uniform, then Eq. (4.3) becomes

(4.4) $$h = \frac{T_w}{T_d}+\frac{(T_d-T_w)}{T_d}q,$$

where q is given by Eq. (3.6′).

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