

## Generalized coupled thermoplasticity

### Part. I. Fundamental equations and identities

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IN THIS PAPER the field equations of the generalized coupled thermoplasticity theory are derived using the postulates of classical thermodynamics of irreversible processes. The most general form for the thermodynamic potentials (e.g. for the free energy) is assumed instead of the usually used additive form. Due to this assumption, it is possible to describe all the effects of thermo-mechanical couples and also the elastic-plastic coupling effects. The plastic flow constitutive equations have the character of non-associated flow laws even when the Gyarmati postulate is assumed.

W pracy wyprowadzono równania pola sprzężonej uogólnionej termoplastyczności wykorzystując postulaty klasycznej termodynamiki procesów nieodwracalnych. Przyjęto tutaj najogólniejszą postać dla potencjałów termodynamicznych (np. dla energii swobodnej), nie zaś w postaci addytywnej. Dzięki przyjęciu takiego założenia możliwe stało się nie tylko wyspecyfikowanie i opis wszystkich efektów sprzężeń termo-mechanicznych, lecz również efektu sprzężenia sprężysto-plastycznego, a równania konstytutywne plastycznego płynięcia mają charakter niestowarzyszonych praw nawet w przypadku przyjęcia postulatu Gyarmatego.

В работе выведены уравнения поля сопряженной обобщенной термопластичности, используя постулаты классической термодинамики необратимых процессов. Принят здесь самый общий вид для термодинамических потенциалов (например для свободной энергии), но не в аддитивном виде. Благодаря принятию такого предположения стало возможным не только выспецифировать и описать все эффекты термо-механических сопряжений, но также эффект упруго-пластического сопряжения, а определяющие уравнения пластического течения имеют характер неассоциированных законов, даже в случае принятия постулата Гармати.

#### Symbols and abbreviations

$$\mathbf{A} \cdot \mathbf{B} \Rightarrow A_i B_i \quad \text{or} \quad A_{ij} B_{ij} \quad (i, j, k, l, m, n, \dots = 1, \dots, 3),$$

$$\mathbf{A}^{(K)} \cdot \mathbf{B}^{(K)} = \sum_{K=1}^{K=M} \mathbf{A}^{(K)} \cdot \mathbf{B}^{(K)},$$

$$A^{(I)} B^{(I)} = \sum_{I=1}^{I=M} A^{(I)} B^{(I)},$$

$$\text{tr} \mathbf{A} = A_{ii}, \quad \text{sym} \mathbf{A} \Rightarrow \frac{1}{2} (A_{ij} + A_{ji}),$$

$$\mathbf{A} \mathbf{B} \Rightarrow A_{ij} B_j \quad \text{or} \quad A_{mnl} B_{lj},$$

$$\text{dev} \mathbf{A} = \mathbf{A} - \frac{1}{3} (\text{tr} \mathbf{A}) \mathbf{1},$$

$$\mathbf{A} \otimes \mathbf{B} \Rightarrow A_i B_j \quad \text{or} \quad A_{ij} B_{mn},$$

$$\mathbf{1} \Rightarrow \delta_{ij} \quad (\text{Kronecker's delta}),$$

$$A_{i,j} = \frac{\partial A_i}{\partial x_j} \quad (x_j - \text{coordinates of a particle})$$

$$\dot{\mathbf{A}} = \frac{\partial \mathbf{A}}{\partial t} \quad (t - \text{time}),$$

$$\text{grad} \mathbf{A} \Rightarrow A_{i,j},$$

$$\text{div} \mathbf{A} \Rightarrow A_{i,i} \quad \text{or} \quad A_{i,j,j}$$

$$\frac{\partial \mathbf{A}}{\partial \mathbf{B}} d\mathbf{B} \Rightarrow \frac{\partial A_{ij}}{\partial B_{mn}} dB_{mn},$$

If  $\Pi$  and  $K$  denote "vectors of pairs" of tensors of the second and zero order, the operation  $\Pi \cdot K$  gives a scalar

$$\Pi \cdot K = \sum_{N=1}^{N=n} \Pi_{ij}^{(N)} K_{ij}^{(N)} + \sum_{M=1}^{M=m} \Pi^{(M)} K^{(M)}.$$

If  $\mathbf{Z}$  is a "vector of pairs" of tensors of the fourth and second order

$$\mathbf{Z} \Leftrightarrow \{Z_{mnlk}^{(N)}; Z_{mn}^{(M)}\}_{\substack{N=1 \dots n \\ M=1 \dots m}}$$

and  $\mathbf{M}$  is a tensor of fourth order, then  $\mathbf{MZ}$  is the following "vector of pairs" of the sum of tensors of the fourth and second order

$$\mathbf{MZ} \Leftrightarrow \left\{ \sum_{N=1}^{N=n} M_{ijkl} Z_{mnlk}^{(N)}; \sum_{M=1}^{M=m} M_{ijkl} Z_{mn}^{(M)} \right\}.$$

If  $\alpha$  is a tensor of the second order, the operation  $\alpha \odot (\mathbf{MZ})$  gives a "vector of pairs" of a sum of tensors of the second and zero orders

$$\alpha \odot (\mathbf{MZ}) \Leftrightarrow \left\{ \sum_{N=1}^{N=n} \alpha_{ij} M_{ijkl} Z_{mnlk}^{(N)}; \sum_{M=1}^{M=m} \alpha_{ij} M_{ijkl} Z_{mn}^{(M)} \right\}.$$

If  $\mathbf{Z}$  is, as before, a "vector of pairs" of tensors of the fourth and second orders and  $K$  is a "vector of pairs" of tensors of the second and zero orders, the expression  $\mathbf{Z} \square K$  denotes the vector of the sum of tensors of the second order

$$\mathbf{Z} \square K = \sum_{N=1}^{N=n} Z_{ijkl}^{(N)} K_{mn}^{(N)} + \sum_{M=1}^{M=m} Z_{ij}^{(M)} K^{(M)}.$$

If  $\alpha$  is, as before, a tensor of the second order, the expression  $\alpha \cdot (\mathbf{Z} \square K)$  is the following scalar

$$\alpha \cdot (\mathbf{Z} \square K) = \sum_{N=1}^{N=n} \alpha_{ij} Z_{ijkl}^{(N)} K_{mn}^{(N)} + \sum_{M=1}^{M=m} \alpha_{ij} Z_{ij}^{(M)} K^{(M)}.$$

If  $\Pi$  is a "vector of pairs" of tensors of the second and zero orders and  $\epsilon$  is a tensor of the second order, the differential of the "vector of pairs"  $\Pi$  with respect to  $\epsilon$  is the following "vector of pairs" of a sum of tensors of second and zero orders

$$\frac{\partial \Pi}{\partial \epsilon} \odot d\epsilon \Leftrightarrow \left\{ \sum_{N=1}^{N=n} \frac{\partial \Pi_{ij}^{(N)}}{\partial \epsilon_{kl}} d\epsilon_{kl}; \sum_{M=1}^{M=m} \frac{\partial \Pi^{(M)}}{\partial \epsilon_{kl}} d\epsilon_{kl} \right\}.$$

The differential of the "vector of pairs"  $\Pi$  of tensors of the second and zero orders with respect to the "vector of pairs"  $K$  of tensors of the second and zero orders is a "vector of pairs" of a sum of tensors of the second and zero orders as follows

$$\frac{\partial \Pi}{\partial K} \ast dK \Leftrightarrow \left\{ \sum_{N=1}^{N=n} \left( \sum_{r=1}^{r=n} \frac{\partial \Pi_{ij}^{(N)}}{\partial K_{kl}^{(r)}} dK_{kl}^{(r)} + \sum_{s=1}^{s=m} \frac{\partial \Pi_{ij}^{(N)}}{\partial K^{(s)}} dK^{(s)} \right); \right. \\ \left. \sum_{M=1}^{M=m} \left( \sum_{r=1}^{r=n} \frac{\partial \Pi^{(M)}}{\partial K_{kl}^{(r)}} dK_{kl}^{(r)} + \sum_{s=1}^{s=m} \frac{\partial \Pi^{(M)}}{\partial K^{(s)}} dK^{(s)} \right) \right\};$$

It is seen that summation must be carried out over all  $r$  from 1 to  $n$  and over all  $s$  from 1 to  $m$  for each  $N$  and  $M$ , respectively.

## 1. Introduction

FIELD equations of the generalized theory of thermoplasticity will be derived making use of the notions of classical thermodynamics of irreversible processes. The concept of macroscopic internal parameters will be used to describe the thermodynamic state of an elastic-plastic body (cf., for instance, Refs. [1–11]). The former papers [1–6, 8, 12, 13], concerning the same problems will be used as a model, the essential difference consisting in the fact that the free energy is no more represented by a sum of two energies, one of which characterizes the thermoelastic properties of the body and the other — the strain-hardening process independent of the thermoelastic properties of the body. As a result it is possible to describe the effects of elastic-plastic coupling observed in non-metal bodies such as, for instance, rocks; nor the usual postulate of existence of a dissipation potential (the GYARMATI postulate [14]) is used to derive the velocity equation. The use of that postulate does not necessarily lead to associated laws of plastic flow. It turns out, however (cf. Sect. 4.2), that, as a consequence of that assumption, the direction of the sum of the plastic strain rate tensor and the irreversible plastic strain tensor is normal to the instantaneous surface of plasticity. This important fact does not appear to be sufficiently emphasized in the existing literature. In Refs. [15–16], in which it was attempted to derive non-associated laws of plastic flow, making use of the KADASHEV.CH-NOVOZH LOV theory [17], this effect has not been observed. Making use of the postulates and assumptions of rational thermodynamics and making use of the concept of macroscopic internal parameters, the effect of elastic-plastic coupling was studied by DEFALIAS (Refs. [24] and [25]) assuming a general form of the free-energy function for the yield condition formulated in deformation spaces. The limitations on the elastic-plastic coupling effect by the second law of thermodynamics were studied, it being shown that the plastic strain-rate tensor is not normal to the yield surface.

Section 2 is devoted to the fundamental assumptions concerning the equation of state (the Gibbs equations) and the source of entropy.

It will be assumed throughout the entire paper that the displacement and velocity gradients are small, which means that the assumptions of the infinitesimal theory of the continuum are used. These assumptions enable more lucid analysis of the physical aspects of the theory, the analysis of any thermodynamic couplings, in particular (cf., for instance, Refs. [1] to [5]).

It is assumed for simplicity that all the equations and the entire description are expressed in rectangular Cartesian coordinates.

## 2. Fundamental assumptions. The Gibbs equation and the source of entropy

Let us assume that the local thermodynamic state is described by the following parameters of state [1]:  $\epsilon^e$  — the tensor of small elastic strain,  $s$  — specific entropy (per unit mass),  $\kappa^{(N)}$  — the set of symmetric internal tensor parameters of second order ( $N = 1, \dots, n$ ),  $\kappa^{(N)} = \kappa^{(N)T}$ , that is  $\kappa_{ij}^{(N)} = \kappa_{ji}^{(N)}$ ,  $\kappa^{(M)}$  — the set of internal scalar parameters ( $M = 1, \dots, m$ ).

The symbol  $K$  will denote the set of internal parameters in the form of a „vector of pairs”

$$K \Leftrightarrow \{\kappa^{(N)}, \kappa^{(M)}\}, \quad N = 1 \dots n, \quad M = 1 \dots m.$$

Let us assume that the fundamental equation of state, that is the Gibbs equation, has the following form of a differential of the specific internal energy  $u$

$$(2.1) \quad du(s, \epsilon^e, K) = Tds + \frac{1}{\rho_0} \boldsymbol{\sigma} \cdot d\boldsymbol{\epsilon}^e + \frac{1}{\rho_0} \Pi \cdot dK,$$

where  $\Pi$  is the „vector of pairs” of the internal forces associated with the set of internal parameters

$$\Pi \Leftrightarrow \{\Pi^{(N)}, \Pi^{(M)}\}, \quad N = 1, \dots, n, \quad M = 1, \dots, m,$$

where  $\Pi^{(N)} = \Pi^{(N)T}$ , that is  $\Pi_{ij}^{(N)} = \Pi_{ji}^{(N)}$ . The local approach to the principle of conservation of energy is as follows:

$$(2.2) \quad \dot{u} = \frac{1}{\rho_0} \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\epsilon}} - \frac{1}{\rho_0} \operatorname{div} \mathbf{q},$$

where  $\mathbf{q}$  denotes the heat flux exchanged with the neighbourhood per unit time across a unit area,  $T$  — the absolute temperature,  $\boldsymbol{\sigma}$  — the stress,  $\rho_0$  — the mass density of the body in the natural state,  $\operatorname{div} \mathbf{q} = \partial q_i / \partial x_i$ , and the orthogonal coordinates  $x_i$  express the initial location of the particle.

The equation of local entropy balance per unit volume of the body has the form

$$(2.3) \quad \rho_0 \dot{s} = -\operatorname{div} \left( \frac{\mathbf{q}}{T} \right) + \sigma^{(s)},$$

where  $(\mathbf{q}/T)$  is the entropy flux and  $\sigma^{(s)}$  is the entropy produced in a definite particle per unit time and volume. The other (local) formulation of the second law of thermodynamics is given by the inequality

$$(2.4) \quad \sigma^{(s)} \geq 0.$$

The entropy production can be evaluated by solving a set of three equations (2.1) to (2.3) for  $\dot{u}$ ,  $\dot{s}$  and  $\sigma^{(s)}$

$$(2.5) \quad T\sigma^{(s)} = D - \frac{1}{T} \mathbf{q} \cdot \nabla T, \quad \nabla T = \operatorname{grad} T,$$

$$(2.6) \quad D = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\epsilon}}^p - \Pi \cdot \dot{K},$$

$$(2.7) \quad \dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^p,$$

$D$  expresses the dissipation of mechanical energy per unit time and volume. The set of forces involved in (2.5)

$$(2.8) \quad X^\sigma = \left\{ \boldsymbol{\sigma}; -\Pi; \frac{1}{T} \nabla T \right\}$$

is referred to as a set of dissipation forces or a set of thermodynamic impulses and

$$(2.9) \quad \dot{x}^\sigma = \{ \dot{\boldsymbol{\epsilon}}^p; \dot{K}; \mathbf{q} \}$$

is a set of measures of thermodynamic flow rates ([2, 10, 14, 18 and 19]).

**3. Discussion of the thermostatic properties of an elastic-plastic body**

Some different thermodynamic potentials may be used for the description of the thermostatic properties of a material. This leads, of course, to equivalent descriptions. Let us denote the sets of independent parameters of state as follows:

$$(3.1) \quad Y_K^{s\epsilon} = \{s, \epsilon^e, K\}; \quad Y_K^{T\epsilon} = \{T, \epsilon^e, K\}, \quad \text{and} \quad Y_K^{T\sigma} = \{T, \sigma, K\}.$$

The relevant thermodynamic potentials will be denoted

$$(3.2) \quad \begin{aligned} u &= u(Y_K^{s\epsilon}) \quad \text{internal energy,} \\ A(Y_K^{T\epsilon}) &= [u - Ts]_{s=s(Y_K^{T\epsilon})} \quad \text{free energy,} \\ G(Y_K^{T\sigma}) &= \left[ A - \frac{1}{\varrho_0} \sigma \cdot \epsilon^e \right]_{\epsilon^e = \epsilon^e(Y_K^{T\sigma})} \quad \text{Gibbs function.} \end{aligned}$$

Their total differentials are

$$(3.3) \quad \begin{aligned} dA &= -s dT + \frac{1}{\varrho_0} \sigma \cdot d\epsilon^e + \frac{1}{\varrho_0} \Pi \cdot dK, \\ dG &= -s dT - \frac{1}{\varrho_0} \epsilon^e \cdot d\sigma + \frac{1}{\varrho_0} \Pi \cdot dK, \end{aligned}$$

respectively (cf. (2.1)), and the resulting thermal equations of state are given in Table 1.

**Table 1**

	internal energy $u = u(Y_K^{s\epsilon})$	free energy $A = A(Y_K^{T\epsilon})$	Gibbs function $G = G(Y_K^{T\sigma})$
(3.4)	$T(Y_K^{s\epsilon}) = \frac{\partial u(Y_K^{s\epsilon})}{\partial s}$	$-s(Y_K^{T\epsilon}) = \frac{\partial A(Y_K^{T\epsilon})}{\partial T}$	$-s(Y_K^{T\sigma}) = \frac{\partial G(Y_K^{T\sigma})}{\partial T}$
(3.5)	$\frac{1}{\varrho_0} \sigma(Y_K^{s\epsilon}) = \frac{\partial u(Y_K^{s\epsilon})}{\partial \epsilon^e}$	$\frac{1}{\varrho_0} \sigma(Y_K^{T\epsilon}) = \frac{\partial A(Y_K^{T\epsilon})}{\partial \epsilon^e}$	$-\frac{1}{\varrho_0} \epsilon^e = \frac{\partial G(Y_K^{T\sigma})}{\partial \sigma}$
(3.6)	$\frac{1}{\varrho_0} \Pi(Y_K^{s\epsilon}) = \frac{\partial u(Y_K^{s\epsilon})}{\partial K}$	$\frac{1}{\varrho_0} \Pi(Y_K^{T\epsilon}) = \frac{\partial A(Y_K^{T\epsilon})}{\partial K}$	$\frac{1}{\varrho_0} \Pi(Y_K^{T\sigma}) = \frac{\partial G(Y_K^{T\sigma})}{\partial K}$

Differentiation with respect to the "vector of pairs" in Table 1 is thus defined as follows

$$(3.7) \quad \frac{\partial(\cdot)}{\partial K} \Leftrightarrow \left\{ \frac{\partial(\cdot)}{\partial \kappa^{(N)}}, \frac{\partial(\cdot)}{\partial \kappa^{(M)}} \right\}, \quad N = 1 \dots n, \quad M = 1 \dots m.$$

Similarly we have

$$(3.8) \quad \frac{\partial(\cdot)}{\partial \Pi} \Leftrightarrow \left\{ \frac{\partial(\cdot)}{\partial \Pi^{(N)}}, \frac{\partial(\cdot)}{\partial \Pi^{(M)}} \right\}, \quad N = 1 \dots n, \quad M = 1 \dots m.$$

the symbol  $(\cdot)$  denoting the function to be differentiated. The fundamental physical quantities describing the thermostatic properties of solids are defined as follows

$$(3.9) \quad C_s(Y_K^{T^e}) = T \frac{\partial s(Y_K^{T^e})}{\partial T} = \frac{\partial u(Y_K^{T^e})}{\partial T},$$

$$(3.10) \quad C_\sigma(Y_K^{T^\sigma}) = T \frac{\partial s(Y_K^{T^\sigma})}{\partial T};$$

$$L(Y_K^{T^\sigma}) = \frac{\partial \epsilon^e(Y_K^{T^\sigma})}{\partial \sigma}, \quad M(Y_K^{T^e}) = \frac{\partial \sigma(Y_K^{T^e})}{\partial \epsilon^e};$$

$$(3.11) \quad \alpha(Y_K^{T^\sigma}) = \frac{\partial \epsilon^e(Y_K^{T^\sigma})}{\partial T};$$

$$(3.12) \quad Z(Y_K^{T^\sigma}) = \frac{\partial \epsilon^e(Y_K^{T^\sigma})}{\partial K}, \quad N(Y_K^{T^e}) = \frac{\partial \sigma(Y_K^{T^e})}{\partial K};$$

where  $C_s$  and  $C_\sigma$  are specific heats with constant  $(\epsilon^e, K)$  and  $(\sigma, K)$ , respectively,  $M$  is the tensor of isothermal moduli of elasticity in the state  $Y_K^{T^e}$  and  $L$  — the tensor of isothermal elasticity in the state  $Y_K^{T^\sigma}$ . The symbol  $\alpha$  denotes the tensor of thermal expansion of the material.

$N$  is the "vector of pairs" of tensors of orders 4 and 2 representing the isothermal variation the state of stress due to the internal processes accompanying plastic deformation in the state  $Y_K^{T^e}$ .

$Z$  is the "vector of pairs" of tensors of orders 4 and 2 representing isothermal variation of elastic deformation due to internal processes in the state  $Y_K^{T^\sigma}$ .

The quantities (3.9) to (3.12) are not independent. They satisfy the following identities resulting from the existence of thermodynamic potentials [2]

$$(3.13) \quad M = (L^{-1})_{\sigma=\sigma(Y_K^{T^e})} \quad \text{or} \quad 2[M_{ijklmn}L_{mnpq}]_{\sigma=\sigma(Y_K^{T^e})} = \delta_{is}\delta_{jr} + \delta_{ir}\delta_{js},$$

$$(3.14) \quad \alpha_{ij} = \alpha_{ji}, \quad M_{ijklmn} = M_{mnlj} = M_{jlmn} = M_{ijnm},$$

$$(3.15) \quad C_\sigma = \left[ C_s + \frac{T}{\varrho_0} \alpha \cdot M \alpha \right]_{\epsilon^e = \epsilon^e(Y_K^{T^\sigma})}.$$

In what follows use will be made of the following thermodynamical identities, the proof of which can be found in Ref. [2]

$$(3.16) \quad \frac{\partial s(Y_K^{T^e})}{\partial \epsilon^e} = -\frac{1}{\varrho_0} \frac{\partial \sigma(Y_K^{T^e})}{\partial T} = \frac{1}{\varrho_0} (M \alpha)_{\sigma=\sigma(Y_K^{T^e})},$$

$$-\varrho_0 \frac{\partial s(Y_K^{T^\sigma})}{\partial K} = \frac{\partial \Pi(Y_K^{T^\sigma})}{\partial T}, \quad -\varrho_0 \frac{\partial s(Y_K^{T^e})}{\partial K} = \frac{\partial \Pi(Y_K^{T^e})}{\partial T}$$

and

$$(3.17) \quad \frac{\partial s(Y_K^{T^\sigma})}{\partial \sigma} = \frac{1}{\varrho_0} \alpha,$$

$$\frac{1}{\varrho_0} \frac{\partial \Pi(Y_K^{T^\sigma})}{\partial s} = \frac{\partial T(Y_K^{T^\sigma})}{\partial K},$$

$$\frac{\partial \epsilon^e(Y_K^{T^\sigma})}{\partial T} = \varrho_0 \frac{\partial s(Y_K^{T^\sigma})}{\partial \sigma},$$

The thermodynamic potentials being not expressed in an additive form (Refs. [1] to [4]), but in the most general form, we have the following additional identities of thermostatic couplings, which will be used in a further part of the paper

$$(3.18) \quad \begin{aligned} -LN &= -L(Y_K^{T\sigma})N(Y_K^{T\epsilon}) = Z, \\ -MZ &= -M(Y_K^{T\epsilon})Z(Y_K^{T\sigma}) = N, \end{aligned}$$

$$(3.19) \quad \frac{\partial \Pi(Y_K^{T\epsilon})}{\partial \epsilon^e} = N, \quad - \frac{\partial \Pi(Y_K^{T\sigma})}{\partial \sigma} = Z$$

and

$$(3.20) \quad \frac{\partial s(Y_K^{T\sigma})}{\partial K} = \frac{\partial s(Y_K^{T\epsilon})}{\partial K} + \frac{1}{\varrho_0} \alpha \odot (MZ),$$

$$(3.21) \quad \frac{\partial \Pi(Y_K^{T\sigma})}{\partial T} = \frac{\partial \Pi(Y_K^{T\epsilon})}{\partial T} - \alpha \odot (MZ).$$

The identities expressing thermostatic couplings (3.19) have the following physical interpretation: A variation in internal forces as a result of elastic strain results in a process of hardening (softening) of the material and a variation in internal forces as a result of stress is connected with a variation in the moduli of elasticity as a result of a variation in internal parameters.

The identities (3.20) and (3.21) are complex thermostatic identities and are sometimes referred to as identities of the second kind.

The most important thermostatic properties of elastic-plastic materials can be discussed by assuming consecutively  $Y_K^{T\epsilon}$  and  $Y_K^{T\sigma}$  as a set of independent parameters of state and evaluating the increments in the dependent parameters [1] (cf. columns 2 and 3 of Table 1 and the formulae (3.9) to (3.21)).

Thus

$$(3.22) \quad Tds(Y_K^{T\epsilon}) = \underbrace{C_\epsilon dT}_{\text{heat capacity}} + \underbrace{\frac{T}{\varrho_0} d\epsilon^e \cdot M\alpha}_{\text{heat of elastic strain}} - \underbrace{\frac{T}{\varrho_0} \frac{\partial \Pi}{\partial T} \cdot dK}_{\text{heat of internal processes}}$$

$$(3.23) \quad Tds(Y_K^{T\sigma}) = \underbrace{C_\sigma dT}_{\text{thermostatic piezo-calorific effect}} + \underbrace{\frac{T}{\varrho_0} \alpha \cdot d\sigma}_{\text{thermostatic piezo-calorific effect}} - \underbrace{\frac{T}{\varrho_0} \frac{\partial \Pi}{\partial T} \cdot dK}_{\text{heat of internal processes}}$$

$$(3.24) \quad d\epsilon^e(Y_K^{T\sigma}) = \underbrace{Ld\sigma}_{\text{isothermal elastic deformability}} + \underbrace{\alpha dT}_{\text{thermal expansion}} + \underbrace{Z \square dK}_{\text{isothermal variation in elastic strain as a result of internal processes}}$$

$$(3.25) \quad d\sigma(Y_K^{T\sigma}) = \underbrace{M d\epsilon^e}_{\text{isothermal elasticity}} + \underbrace{M\alpha dT}_{\text{elastic stress}} + \underbrace{N \square dK}_{\text{isothermal variation in stress due to internal processes}}$$

$$(3.26) \quad d\Pi(Y_K^{T\sigma}) = \underbrace{\frac{\partial\Pi}{\partial K} * dK}_{\text{variation in internal forces accompanying internal processes}} + \underbrace{\frac{\partial\Pi}{\partial T} dT}_{\text{internal thermal forces}} + \underbrace{\frac{\partial\Pi}{\partial \epsilon^e} \odot d\epsilon^e}_{\text{variation in internal forces due to elastic strain}}$$

$$(3.27) \quad d\Pi(Y_K^{T\sigma}) = \underbrace{\frac{\partial\Pi}{\partial K} * dK}_{\text{variation in internal forces accompanying internal processes}} + \underbrace{\frac{\partial\Pi}{\partial T} dT}_{\text{internal thermal forces}} + \underbrace{\frac{\partial\Pi}{\partial \sigma} \odot d\sigma}_{\text{variation in internal forces due to a change in the state of stress}}$$

The dimensionless coefficients  $\gamma_1, \gamma_3, \gamma_{12}, \gamma_{21}, \gamma_{12}^*, \gamma_{21}^*, \gamma_{23}, \gamma_{32}^*, \gamma_{13}, \gamma_{31}^*$  and  $\gamma_{31}$ , involved in the formulae (3.28) to (3.29) below and further equations, have no physical sense. Making use of the idea of Ref. [1], we introduce them to facilitate the interpretation of various terms and coupling effects in the equations. They are also useful for making certain simplifications; they take the value 1 in the general case and value zero if any of the coupling effects in the Eqs. (3.22) to (3.27) is disregarded. If, for instance, we set  $\gamma_{12}^* = 0$ , the rejected quantity is the elastic strain heat. If  $\gamma_{23} = 0$ , the rejected influence is that of internal processes on the elastic strain, etc. The object of our interest being only these internal processes which are due to plastic strain and which, on macroscopic grounds, are manifested in the form of strain hardening, the effects denoted by the numbers ( $\gamma_{23}, \gamma_{23}^*, \gamma_{13}, \gamma_{13}^*$ ) may be termed "effects of elastic-plastic coupling". Such a name has already been used in the literature Refs. [20] to [25]. The number  $\gamma_1$  will represent the dissipation heat, which does not belong to thermostatic effects and is not specified in the description of the formulae (3.22) to (3.27).

On eliminating  $\dot{s}$  from the Eqs. (2.3), (3.22) and (3.23) we obtain two alternative equations for the temperature. By finding  $\dot{\sigma}$  from the Eq. (3.5)<sub>2</sub> and  $\dot{\epsilon}_i^e$  from the Eq. (3.5)<sub>3</sub> and making use of the Eqs. (3.9) to (3.12), respectively, we obtain the following two alternative sets of equations composed of the equation for the temperature and the relation between the rates of elastic strain, elastic stress and temperature

$$(3.28) \quad \varrho_0 C_\sigma \dot{T} = \gamma_1 D - \gamma_{12} T \alpha \cdot M \dot{\epsilon}^e + \gamma_3 T \frac{\partial\Pi}{\partial T} \cdot \dot{K} + q_0,$$

$$\text{and} \quad \dot{\epsilon}^e = L \dot{\sigma} + \gamma_{21} \alpha \dot{T} + \gamma_{23} Z \square \dot{K}$$

$$(3.29) \quad \varrho_0 C_\sigma \dot{T} = \gamma_1 D - \gamma_{12}^* T \alpha \cdot M \dot{\epsilon}^e + \gamma_3 T \frac{\partial\Pi}{\partial T} \cdot \dot{K} + q_0,$$

$$\dot{\sigma} = M \dot{\epsilon}^e - \gamma_{21}^* \dot{T} M \alpha + \gamma_{23}^* N \square \dot{K},$$



where

$$(3.30) \quad \begin{aligned} q_0 &= -\operatorname{div} \mathbf{q} \\ \theta &= T - T_0, \end{aligned}$$

$T_0$  being the temperature of reference.

Let us observe that the elastic strain rate involved in (3.28)<sub>2</sub> can be represented in the form

$$(3.31) \quad \dot{\epsilon}^e = \dot{\epsilon}^{e'} + \dot{\epsilon}^{e''},$$

where  $\dot{\epsilon}^{e'} = \mathbf{L}\dot{\sigma} + \gamma_{21}\alpha\dot{T}$  is what is termed the reversible part of the elastic strain rate, and  $\dot{\epsilon}^{e''} = \gamma_{23}\mathbf{Z} \square \dot{K}$  is the irreversible, coupled part of the strain rate connected with the internal processes accompanying plastic strain. Such a separation of the tensor  $\dot{\epsilon}^e$  into two parts was adopted and interpreted in Refs. [20] to [23] in the case of an isothermal process.

By confronting (3.31) with (2.7) it is found that

$$(3.32) \quad \dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p = \dot{\epsilon}^{e'} + \dot{\epsilon}^{e''} + \dot{\epsilon}^p.$$

Let us introduce another thermodynamic potential — the enthalpy [18 to 19], [26 to 28]

$$(3.33) \quad \chi(Y_K^{T\sigma}) = \gamma_1 G(Y_K^{T\sigma}) + \gamma_3 Ts(Y_K^{T\sigma}).$$

Then, making use of (3.28)<sub>1</sub> and (3.33), the equation for the temperature can be expressed in terms of plastic strain power and the variation in enthalpy due to a variation in the internal parameters as follows

$$(3.34) \quad \varrho_0 C_\sigma \dot{T} = \gamma_1 \sigma \cdot \dot{\epsilon}^p - \gamma_{12} T \alpha \cdot \dot{\sigma} - \varrho_0 \frac{\partial \chi(Y_K^{T\sigma})}{\partial K} \cdot \dot{K} + q_0.$$

In this form the Eq. (3.34) can be useful for the analysis of the energy stored in the body as a result of plastic deformation in a closed cycle of stress ([1, 2] and [12]). It shows that the variation in the energy stored in the body in the course of an isothermal process will be equal to that in enthalpy due to a variation in the internal parameters.

## 4. Rate equations

### 4.1. General form of the rate equations

The thermal equations of state appearing in any column of Table 1, represent the first group of the constitutive equations of thermoelasticity. The second group are the rate equations relating thermodynamic impulses (2.8) with the thermodynamic flow rates (2.9).

The rate equations will be assumed in the following general form

$$(4.1) \quad \mathbf{q} = \boldsymbol{\varphi}^q(\nabla T, Y_K^T), \quad \boldsymbol{\varphi}^q(\mathbf{0}, Y_K^T) = \mathbf{0},$$

where

$$(4.2) \quad \begin{aligned} Y_K^T &= \{T, K\}, \\ \dot{\epsilon}^p &= A \frac{\partial f_1(X^p, Y_K^T)}{\partial \sigma}, \quad \dot{K} = \Lambda d(X^p, Y_K^T) = \Lambda d; \end{aligned}$$

if  $f_1 = 0$  and  $\Lambda \geq 0$

$$(4.3) \quad \dot{\epsilon}^p = 0, \quad \dot{K} = 0 \quad \text{if} \quad f_1 < 0 \quad \text{or} \quad f_1 = 0 \quad \text{and} \quad \Lambda < 0,$$

where

$$X^D = (\boldsymbol{\sigma}, -II) \quad \text{and} \quad f_1 = f_1(\boldsymbol{\sigma}, -II, K, T)$$

is the generalized function of plastic flow defined in the space of thermodynamic forces  $X^D$  and such that  $f_1 = 0$  determines the yield surface in that space.

Since  $\boldsymbol{\varphi}^q$  is independent of the dissipative forces  $\boldsymbol{\sigma}$  and  $-II$ , a condition necessary for the inequality (2.4) to be satisfied is that the following two independent inequalities should be satisfied

$$(4.4) \quad -\boldsymbol{\varphi}^q(\nabla T, Y_K^T) \cdot \nabla T \geq 0 \quad \text{and} \quad \frac{\partial f_1}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma} - II \cdot d \geq 0.$$

If  $II$  is replaced by the relevant equation of state (3.6)<sub>3</sub>, we shall obtain the flow function and the relevant yield condition  $f$  in the space of stresses

$$(4.5) \quad f_1(\boldsymbol{\sigma}, -II, K, T)|_{II = II(\boldsymbol{\sigma}, K, T)} = f(Y_K^{T\sigma}).$$

The factor  $\Lambda$  in (4.2)<sub>1,2</sub> can be eliminated by making use of the "association condition"  $\dot{f}_1 = \dot{f} = 0$ , if  $f_1 = f = 0$

$$(4.6) \quad \mathbf{f}_\sigma \cdot \dot{\boldsymbol{\sigma}} + f_T \dot{T} - \Lambda h = 0 \quad \text{whence} \quad \Lambda = (\mathbf{f}_\sigma \cdot \dot{\boldsymbol{\sigma}} + f_T \dot{T}) \frac{1}{h},$$

where

$$(4.7) \quad h = -\frac{\partial f}{\partial K} \cdot d$$

is what is termed the strain hardening function and

$$(4.8) \quad \mathbf{f}_\sigma = \frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad f_T = \frac{\partial f}{\partial T}.$$

By assuming the classical condition for plastic loading ( $\dot{\epsilon}^p \neq 0$ ), if and only if  $\mathbf{f}_\sigma \cdot \dot{\boldsymbol{\sigma}} + f_T \dot{T} \geq 0$  we find, by virtue of (4.6)<sub>2</sub>, that  $h \geq 0$ . The Eqs. (4.2)<sub>1</sub> can be expressed in the form

$$(4.9) \quad \dot{\epsilon}^p = \begin{cases} \frac{1}{h} \mathbf{f}_{1,\sigma} (\mathbf{f}_\sigma \cdot \dot{\boldsymbol{\sigma}} + f_T \dot{T}), & \text{if } f = 0 \text{ and } \mathbf{f}_\sigma \cdot \dot{\boldsymbol{\sigma}} + f_T \dot{T} \geq 0, \\ 0 & \text{if } f < 0 \quad \text{or} \quad f = 0 \text{ and } \mathbf{f}_\sigma \cdot \dot{\boldsymbol{\sigma}} + f_T \dot{T} < 0, \end{cases}$$

where, by virtue of (4.5)

$$(4.10) \quad \mathbf{f}_{1,\sigma} = \frac{\partial f_1}{\partial \boldsymbol{\sigma}} = \mathbf{f}_\sigma - \frac{\partial II}{\partial \boldsymbol{\sigma}} \square \frac{\partial f_1}{\partial II}.$$

#### 4.2. The particular form of rate equations resulting from the Gyarmati postulate

A more special form of rate equations can be obtained by making use of the Gyarmati postulate ([1, 14]). Let us assume that the dissipative (thermodynamic) forces  $X^D$  have a potential  $\psi^D(\dot{x}^D, Y_K^T)$  ([1, 2] and [14])

$$(4.11) \quad X^D = \frac{\partial \psi^D}{\partial \dot{x}^D}, \quad \dot{x}^D = \{\dot{\epsilon}^p, \dot{K}\},$$

which is differentiable everywhere except at the point  $\dot{x}^D = 0$ .

Since the relations (4.11) are to describe a plastic material insensitive to thermodynamic flow rate, they must be invariant under a change of time scale. It can be easily seen that this is the case, provided that  $\psi^D$  is a homogeneous function of the first order in  $\dot{x}^D$  [2].

The homogeneity of the function  $\psi^D$  implies in turn the dependence of the function  $X^D(\dot{x}^D, Y_K^T)$  [1, 2] and [29]. This means that there exists a function  $f_1(X^D, Y_K^T)$  termed the generalized flow function such that

$$(4.12) \quad f_1(X^D, Y_K^T) = 0 \quad \text{if} \quad \dot{x}^D \neq 0.$$

Then, the inverse relations to (4.11) can be represented in the form

$$(4.13) \quad \dot{x}^D = \Lambda \frac{\partial f_1}{\partial X^D} \quad \text{if} \quad \dot{x}^D \neq 0 \quad \text{and} \quad f_1 = 0.$$

Assuming that the states of  $f_1 > 0$  cannot be attained and taking the usual loading and unloading criterion (4.9)<sub>1,2</sub> we shall obtain generalized W. PRAGER's non-isothermal laws of plastic flow [8] in a form which is identical with (4.9) except, that the function  $d$  involved in (4.2)<sub>2</sub> and (4.7) has now the form

$$(4.14) \quad d = -\frac{\partial f_1}{\partial II}.$$

Making use of (4.14) and (4.10), (3.12) and (3.19), we find

$$(4.15) \quad \frac{\partial \epsilon^e}{\partial K} \square \dot{K} + \dot{\epsilon}^p = \Lambda \frac{\partial f}{\partial \sigma},$$

where  $\Lambda$  is defined by the formula (4.6)<sub>2</sub>.

On substituting (3.31)<sub>3</sub> into (4.15), we find

$$(4.16) \quad \dot{\epsilon}^{e''} + \dot{\epsilon}^p = \Lambda \frac{\partial f}{\partial \sigma}.$$

In the general formulation (cf. Sect. 4.1) tensor  $\dot{\epsilon}^p$  is normal to the surface  $f_1 = 0$  but not normal to the surface  $f = 0$ . As regards the sum of tensors  $\dot{\epsilon}^{e''} + \dot{\epsilon}^p$  it is normal to neither of them. As a consequence of the Gyarmati postulate, the sum of tensors  $\dot{\epsilon}^{e''} + \dot{\epsilon}^p$  (cf. Eq. (4.16)) is normal to the yield surface  $f = 0$ . It appears that this fact has not yet been observed in the literature. It is illustrated in Fig. 1, in which

$\Pi_f$  denotes the hyperplane tangent to the instantaneous yield surface  $f = 0$  at the point  $p$  determined by the unit normal vector  $\mathbf{n}_f$ ,  $\mathbf{n}_f = \mathbf{f}_\sigma / (\mathbf{f}_\sigma \cdot \mathbf{f}_\sigma)^{1/2}$ ;

$\Pi_g$  — the hyperplane tangent to the yield surface  $f_1 = 0$  having the character of a plastic potential and determined by the normal unit vector

$$\mathbf{n}_g = \mathbf{f}_{1,\sigma} / (\mathbf{f}_{1,\sigma} \cdot \mathbf{f}_{1,\sigma})^{1/2};$$

$\alpha_0$  — the angular measure of deviation due to the variation in internal forces  $II$  (cf. Eq. (3.19)) depending on the state of stress, which is equivalent to a change in moduli of elasticity as a result of plastic deformations (cf. Eqs. (3.12) and (3.26)).

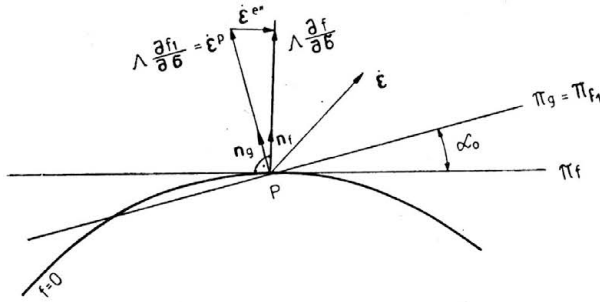


FIG. 1.

In the outline of the theory of materials intensive to thermodynamic flow rate presented in Sect. 4.2, all the equations take a definite form, once the thermodynamic function  $G$  (or any other thermodynamic potential),  $\psi^D$  and  $\varphi^q$  or  $G, f_1$  and  $\varphi^q$  are known. In the most general formulation given in Sect. 4.1 it is also necessary to know the function  $d = d(X^D, Y_K^I)$  for the equations of evolution of the internal parameters  $K$ .

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