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Generalized coupled thermoplasticity
Part. II. On the uniqueness and bifurcation criteria

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IN THIS PAPER the fundamental incremental boundary-value problem of the generalized coupled thermoplasticity is formulated. Furthermore, the local and global criteria excluding the possibility of appearance of the bifurcation are derived. These criteria were derived by analyzing the uniqueness of solution of the incremental boundary-value problem formulated in Sect. 3. The nonassociated plastic flow laws are assumed, and the effects of thermomechanical and elastic-plastic couples are taken into account. Due to such assumptions the mathematical problem considered is not self-adjoint. This (II) part of the paper is a direct continuation of the previous (I) part [2].

W pracy sformułowano podstawowy przyrostowy problem brzegowy sprzężonej uogólnionej termoplastyczności. Następnie wyprowadzono lokalne i globalne kryterium wykluczające możliwość wystąpienia stanu bifurkacji. Kryteria te wyprowadzono, analizując problem jednoznaczności rozwiązania sformułowanego w punkcie 3 przyrostowego (prędkościowego) problemu brzegowego. Oryginalnym elementem jest przyjęcie niestowarzyszonych praw plastycznego płynięcia oraz uwzględnienie wpływu odkształceń plastycznych na własności termosprężyste ciał. Z powodu takiego przyjęcia, rozpatrywany tutaj problem matematyczny nie jest problemem samosprzężonym. Niniejsza II część pracy jest bezpośrednim rozwinięciem części I pracy [2].

В работе сформулирована в приростах основная краевая задача сопряженной обобщенной термопластичности. Затем выведены локальный и глобальный критерия, исключающие возможность выступления состояния бифуркации. Эти критерия выведены, анализируя проблему единственности решения, сформулированной в пункте 3, краевой задачи в приростах (скоростной). Оригинальным элементом является принятие неассоциированных законов пластического течения, а также учет влияния пластических деформаций на термоупругие свойства тел. Из-за такого принятия рассматриваемая здесь математическая проблема не является самосопряженной проблемой. Настоящая II часть работы является непосредственным развитием I части работы [2].

1. Introduction

THE INCREMENTAL boundary-value problem of generalized coupled thermoplasticity will be formulated. This will be followed by an interpretation of the uniqueness conditions for the solution of that problem. The necessary uniqueness conditions will be derived as well as the sufficient local condition and the sufficient global uniqueness criterion, and the sufficient global uniqueness condition. A similar incremental boundary-value problem of coupled thermoplasticity has already been studied in Refs. [1, 3, 5]. The method used in those references will be taken as a model, an original feature of the present paper being that non-associated laws of plastic flow are assumed and the influence of plastic deformation on the thermoelastic properties of the body is taken into consideration. Such a requirement leads to a more difficult problem than those hitherto considered. Besides, the sufficient local uniqueness condition of coupled thermoplasticity derived by RANIECKI and MRÓZ [3, 4] is not the optimum condition.

The procedure for obtaining the optimum condition from the one-parameter family of local uniqueness conditions introduced in the present paper is explained in Appendix B.

The present Part II of the work is directly connected with Part I [2].

2. Uniqueness of solution of incremental problems for homogeneous processes

Let us assume that the thermodynamic state of the body at a certain moment t_0 of a homogeneous process is known and such that the condition $f_1 = f = 0$ is satisfied. The following incremental problems can be formulated for such a type of processes. Satisfying the set of equations of Ref. [2] (Eqs. (2.7), (3.24), (3.25), (3.28), (3.29), (4.2) and (4.6)), we must find, for the time t_0, the values

a$_1$)	$\dot{\boldsymbol{\epsilon}}$ and q_0	assuming that	$\dot{\boldsymbol{\sigma}}(t_0)$	and	$\dot{T}(t_0)$	are prescribed
a$_2$)	$\dot{\boldsymbol{\sigma}}$ and q_0	,, ,,	$\dot{\boldsymbol{\epsilon}}(t_0)$	and	$\dot{T}(t_0)$	,, ,,
b$_1$)	$\dot{\boldsymbol{\epsilon}}$ and $\dot{T}$	,, ,,	$\dot{\boldsymbol{\sigma}}(t_0)$	and	$q_0(t_0)$	,, ,,
b$_2$)	$\dot{\boldsymbol{\sigma}}$ and $\dot{T}$	,, ,,	$\dot{\boldsymbol{\epsilon}}(t_0)$	and	$q_0(t_0)$	,, ,,

where

$$q_0 = -\operatorname{div}\mathbf{q}.$$

It is easy to see that if a solution of the problems (a$_1$) and (a$_2$) is to be unique, it is necessary that the following respective conditions known from the isothermal theory of plasticity should be satisfied

$$h > 0 \quad \text{and} \quad h+\mathbf{g}_4\cdot\mathbf{M}\mathbf{f}_\sigma > 0, \tag{2.1}$$

where

$$\mathbf{g}_4 = \mathbf{f}_{1,\sigma}+\gamma_{23}\mathbf{Z}\,\Box\, d \tag{2.2}$$

and h is the isothermal strain-hardening function obtained in Ref. [2]. Those conditions are also sufficient but it turns out, however, that two solutions of the problems (b$_1$) and (b$_2$) may exist, even if the inequalities (2.1) are satisfied. The uniqueness conditions for the problem (b$_1$) and (b$_2$) have the following forms (cf. Appendix A):

Problem (b$_1$)

$$h_1 = h - m_\sigma f_T > 0. \tag{2.3}$$

Problem (b$_2$)

$$H = h+\mathbf{g}_4\cdot\mathbf{M}\mathbf{f}_\sigma - \frac{1}{p}(m_\sigma+\gamma_{12}^{*}\xi\mathbf{g}_4\cdot\mathbf{M}\mathbf{f}_\sigma)(f_T-\gamma_{21}^{*}\boldsymbol{\alpha}\cdot\mathbf{M}\mathbf{f}_\sigma) > 0, \tag{2.4}$$

where

$$m_\sigma = \frac{1}{\varrho_0 C_\sigma}\left[\gamma_1(\boldsymbol{\sigma}\cdot\mathbf{f}_{1,\sigma} - \Pi\cdot d) - \gamma_3 T\left(\frac{\partial\Pi(Y_{\mathbf{K}}^{T\sigma})}{\partial T}\cdot d\right)\right], \tag{2.5}$$

$$\xi = \frac{1-p}{M_\alpha^2} = \frac{T}{\varrho_0 C_\sigma}, \quad p = \frac{C_\epsilon}{C_\sigma}, \quad M_\alpha^2 = \boldsymbol{\alpha}\cdot\mathbf{M}\boldsymbol{\alpha}. \tag{2.6}$$

In the case of associated laws of plastic flow ($\mathbf{f}_{1,\sigma} = \mathbf{f}_\sigma$) the quantity $m_\sigma = m$ was analysed in Refs. [1] and [3], all the elastic-plastic coupling effects being rejected ($\gamma_{23} = \gamma_{23}^* = \gamma_{13} = \gamma_{13}^* = 0$). By analysing the cyclic isothermal process in the space of stresses the authors of those works observed that for metals m_σ is in general positive

$$m_\sigma > 0. \tag{2.7}$$

The inequalities (2.3) and (2.4) are a generalization of the uniqueness conditions derived by MRÓZ and RANIECKI [1, 3, 4] and KAMIENIARZ [11]. This generalization consists in the non-associated laws of plastic flow being taken into account as well as the influence of plastic deformations on the thermoelastic properties of the body. The conditions obtained in Refs. [1] and [3] can also be obtained from Eq. (2.3) and (2.4) by rejecting all the effects of elastic-plastic coupling ($\gamma_{23} = \gamma_{23}^* = \gamma_{13} = \gamma_{13}^* = 0$) and assuming associated laws of plastic flow ($\mathbf{f}_{1,\sigma} = \mathbf{f}_\sigma$).

It is worthwhile to observe that in the case of metals the satisfaction of the condition (2.3) implies, in general, that of the condition (2.4) (cf. [1] and [3]).

Let us assume that the conditions (2.3) and (2.4) are both satisfied. The solutions of the incremental problems (b_1) and (b_2) can be expressed in the following forms:

Problem (b_1)

Making use of Eqs. (4.2), (3.28) and (2.7) of Ref. [2], we obtain the following relations between $\dot{\boldsymbol{\epsilon}}$ and $\dot{\boldsymbol{\sigma}}$, and $\dot{T}$ and $\dot{\boldsymbol{\sigma}}$:

$$\dot{\boldsymbol{\epsilon}} = \mathbf{L}^{(a)}\dot{\boldsymbol{\sigma}} + \frac{j}{h_1}\overset{(p)}{\mathbf{K}}\dot{\boldsymbol{\sigma}} + \frac{j}{h_1} q f_T[(\mathbf{f}_{1,\sigma} + \gamma_{23}\mathbf{Z}\Box d) + \gamma_{21} m_\sigma \boldsymbol{\alpha}] - \gamma_{21} q \boldsymbol{\alpha}, \tag{2.8$_1$}$$

$$\dot{T} = -\gamma_{12}\xi\boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}} + \frac{j m_\sigma}{h_1}[(\mathbf{f}_\sigma - \gamma_{12}\xi f_T \boldsymbol{\alpha})\cdot\dot{\boldsymbol{\sigma}} - f_T q] - q, \tag{2.8$_2$}$$

$$\dot{\boldsymbol{\epsilon}}^p = \frac{j}{h_1}[(\mathbf{f}_\sigma - \gamma_{12}\xi f_T \boldsymbol{\alpha})\cdot\dot{\boldsymbol{\sigma}} - f_T q], \tag{2.8$_3$}$$

$$\dot{\boldsymbol{\epsilon}}^e = \mathbf{L}\dot{\boldsymbol{\sigma}} + \gamma_{21}\dot{T}\alpha + \frac{j}{h_1}[(\mathbf{f}_\sigma - \gamma_{12} f_T \xi \boldsymbol{\alpha})\cdot\dot{\boldsymbol{\sigma}} - q f_T](\gamma_{23}\mathbf{Z}\Box d), \tag{2.8$_4$}$$

where

$$\overset{(p)}{\mathbf{K}} = (\mathbf{f}_{1,\sigma} + \gamma_{23}\mathbf{Z}\Box d + \gamma_{21} m_\sigma \boldsymbol{\alpha}) \otimes (\mathbf{f}_\sigma - \gamma_{12}\xi f_T \boldsymbol{\alpha}),$$

$$\mathbf{L}^{(a)} = \mathbf{L} - \gamma_{21}\gamma_{12}\xi(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}), \quad q = \frac{1}{\varrho_0 C_\sigma}\operatorname{div}\mathbf{q}, \tag{2.9}$$

$$j = \begin{cases} 1 & \text{if} \quad f = 0 \quad \text{and} \quad (\mathbf{f}_\sigma - \gamma_{12}\xi f_T \boldsymbol{\alpha})\cdot\dot{\boldsymbol{\sigma}} - q f_T \geqslant 0, \\ 0 & \text{if} \quad f < 0 \quad \text{or} \quad f = 0 \quad \text{and} \quad (\mathbf{f}_\sigma - \gamma_{12}\xi f_T \boldsymbol{\alpha})\cdot\dot{\boldsymbol{\sigma}} - q f_T < 0; \end{cases}$$

$$\Lambda = \frac{1}{h_1}[(\mathbf{f}_\sigma - \gamma_{12}\xi f_T \boldsymbol{\alpha})\cdot\dot{\boldsymbol{\sigma}} - q f_T]. \tag{2.10}$$

The symbol $\mathbf{L}^{(a)}$ denotes the tensor of adiabatic elasticity. Let us observe (cf. [1]) that the second right-hand term of $(2.8)_1$ is not equal to the plastic strain rate, but may be considered as representing the adiabatic plastic strain rate. The tensor $\overset{(p)}{\mathbf{K}}$ is asymmetric, $\overset{(p)}{K}_{ijmn} \neq \overset{(p)}{K}_{mnij}$. The equations for the thermodynamic flow rates can also be expressed in terms of $\dot{\boldsymbol{\sigma}}$ and q.

They have the form

$$\dot{K} = \frac{j}{h_1}[(\mathbf{f}_\sigma - \gamma_{12}\xi f_T \boldsymbol{\alpha}) \cdot \dot{\boldsymbol{\sigma}} - q f_T] d(X^D, Y_K^T). \tag{2.11}$$

Taking into consideration the Gyarmati postulate and the resulting condition (4.14) of Ref. [2], the above relation takes the form

$$-\dot{K} = \frac{j}{h_1}[(\mathbf{f}_\sigma - \gamma_{12}\xi f_T \boldsymbol{\alpha}) \cdot \dot{\boldsymbol{\sigma}} - q f_T] \frac{\partial f_1(X^D, Y_K^T)}{\partial \Pi}. \tag{2.12}$$

Problem (b$_2$)

The alternative equations (corresponding to Eq. (2.8)) are obtained by making use of the relations (4.2), (3.29), (2.7) given in Ref. [2]

$$\dot{\boldsymbol{\sigma}} = \mathbf{M}^{(a)}\dot{\boldsymbol{\epsilon}} + \frac{\gamma_{21} q}{p}\mathbf{M}\boldsymbol{\alpha} - \frac{j_1}{H}\overset{(p)}{\mathbf{K}}{}^1\dot{\boldsymbol{\epsilon}} - \frac{j_1 q}{pH}(\gamma_{21}^* \boldsymbol{\alpha} \cdot \mathbf{M}\mathbf{f}_\sigma - f_T)(\tilde{\varphi}_\epsilon \mathbf{M}\boldsymbol{\alpha} + \mathbf{B}_N), \tag{2.13$_1$}$$

$$\dot{T} = \frac{1}{p}(\gamma_{12}\xi\boldsymbol{\alpha} \cdot \mathbf{M}\dot{\boldsymbol{\epsilon}} + q) + \frac{j_1}{pH}(m_\sigma + \gamma_{12}^* \xi \mathbf{g}_4 \cdot \mathbf{M}\mathbf{f}_\sigma)\left[\mathbf{B} \cdot \dot{\boldsymbol{\epsilon}} + \frac{q}{p}(\gamma_{21}^* \boldsymbol{\alpha} \cdot \mathbf{M}\mathbf{f}_\sigma - f_T)\right], \tag{2.13$_2$}$$

$$\dot{\boldsymbol{\epsilon}}^p = \frac{j}{H}\left[\mathbf{B} \cdot \dot{\boldsymbol{\epsilon}} + \frac{q}{p}(\gamma_{21}^* \boldsymbol{\alpha} \cdot \mathbf{M}\mathbf{f}_\sigma - f_T)\right]\mathbf{f}_{1,\sigma}, \tag{2.13$_3$}$$

where

$$j_1 = \begin{cases} 1 & \text{if} \quad f = 0 \quad \text{and} \quad \mathbf{B} \cdot \dot{\boldsymbol{\epsilon}} + \frac{q}{p}(\gamma_{21}^* \boldsymbol{\alpha} \cdot \mathbf{M}\mathbf{f}_\sigma - f_T) \geqslant 0, \\ 0 & \text{if} \quad f < 0 \quad \text{or} \quad f = 0 \quad \text{and} \quad \mathbf{B} \cdot \dot{\boldsymbol{\epsilon}} + \frac{q}{p}(\gamma_{21}^* \boldsymbol{\alpha} \cdot \mathbf{M}\mathbf{f}_\sigma - f_T) < 0, \end{cases}$$

$$\begin{aligned} &\mathbf{M}^{(a)} = \mathbf{M} + \frac{\gamma_{12}^* \gamma_{21}^*}{p}\xi(\mathbf{M}\boldsymbol{\alpha}) \otimes (\mathbf{M}\boldsymbol{\alpha}), \qquad \mathbf{B}_N = \mathbf{B} - \mathbf{N}_2, \\ &\mathbf{B} = \mathbf{M}\mathbf{f}_\sigma + \gamma_{12}^* \frac{\xi}{p}(\gamma_{21}^* \boldsymbol{\alpha} \cdot \mathbf{M}\mathbf{f}_\sigma - f_T)\mathbf{M}\boldsymbol{\alpha}; \; \overset{(p)}{\mathbf{K}}{}^1 = [\tilde{\varphi}_\epsilon(\mathbf{M}\boldsymbol{\alpha}) - \mathbf{N}_2] \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{B}. \end{aligned} \tag{2.14}$$

The following additional quantities are involved in the tensor $\overset{(p)}{\mathbf{K}}{}^1$:

$$\tilde{\varphi}_\epsilon = \gamma_{21}^* \frac{m_\sigma}{p} + \gamma_{12}^* \frac{\xi}{p} f_T - \gamma_{12}^* \gamma_{23}^* \frac{\xi}{p}\{\boldsymbol{\alpha} \cdot [\mathbf{N} \square (\gamma_3 d + \gamma_{21}^* f_{1,\Pi})]\} \tag{2.15}$$

and

$$\mathbf{N}_2 = \gamma_{23}^* \mathbf{N} \square d + \gamma_{13} \mathbf{N} \square f_{1,\Pi} = \mathbf{N} \square (\gamma_{23}^* d + \gamma_{13} f_{1,\Pi}), \tag{2.16}$$

where

$$f_{1,\Pi} = \frac{\partial f_1}{\partial \Pi}. \tag{2.17}$$

The Eqs. (2.15) have been derived making use of the expression $(A.2)_2$ of Appendix A and Eq. $(3.18)_2$ of Ref. [2]. Symbol $\mathbf{M}^{(a)}$ denotes the tensor of adiabatic moduli of elasticity. Similarly to the former case, the tensor interrelating the stress rate and the strain rate is asymmetric because $\overset{(p)}{K}{}^1_{ijmn} \neq \overset{(p)}{K}{}^1_{mnij}$. Let us observe that, if the conditions (2.3) and (2.4) are both satisfied, the Eqs. (2.13) are equivalent to Eq. (2.8). They can be obtained by solving Eq. $(2.13)_1$ for $\dot{\boldsymbol{\sigma}}$ and substituting the result into Eqs. $(2.13)_2$ and $(2.13)_3$. If all the thermodynamic coupling effects in the Eqs. (2.8) to (2.12) and (2.13) to (2.17) are rejected ($\gamma_1 = \gamma_3 = \gamma_{12} = \gamma^*_{23} = \gamma_{13} = \gamma^*_{12} = \gamma_{23} = 0$) and if ($\mathbf{f}_{1,\sigma} = \mathbf{f}_\sigma$), those equations will constitute two equivalent sets of fundamental equations of the theory of thermal stresses in an elastic-plastic body (Then $q = \dot{T}$ or $-\operatorname{div}\mathbf{q} = \varrho_0 C_\sigma \dot{T}$ and $p = 1$).

3. Formulation of the incremental boundary value problem

If the condition (2.3) is satisfied, the set of equations (2.8) to (2.12) is equivalent to the fundamental set of equations (2.7), (3.19), (3.20), (3.24), (3.25), (3.28), (3.29), (4.2), (4.6) and (4.7) of Ref. [2],

$$q = \frac{1}{\varrho_0 C_\sigma}\operatorname{div}\mathbf{q}. \tag{3.1}$$

As regards the Eqs. (2.13), (2.14) (together with the relevant equations for $\dot{\boldsymbol{\varkappa}}^{(N)}$ and $\dot{\boldsymbol{\varkappa}}^{(M)}$) there is a similar equivalence, provided that $H > 0$. The set of those equations, together with the law of heat conduction (4.1) of Ref. [2], with the equation of motion and the kinematic relations

$$\begin{aligned} \operatorname{div}\boldsymbol{\sigma} + \varrho_0 \mathbf{b}_m &= \varrho_0 \dot{\mathbf{v}}, \\ 2\dot{\varepsilon}_{ij} &= v_{i,j} + v_{j,i}, \end{aligned} \tag{3.2}$$

where $\mathbf{v}$ is the vector of velocity of particles and $\mathbf{b}_m$ the body force, constitute a set of fundamental field equations of coupled thermoplasticity. Together with the boundary conditions and the initial conditions it may be used as a basis for analysis of many problems of generalized thermoplasticity, both dynamic and quasi-static.

The following static incremental boundary-values problem can be formulated [1]. Let the body occupy, at a time t_0, a region D in space. Let us denote by $\bar{D}$ the closure of D and by the symbol S — the boundary of $\bar{D}$. S is the closure of the sum of non-intersecting regular open surfaces S_v and S_T. Let the thermodynamic state of the body

$$T, \boldsymbol{\sigma}, K \tag{3.3}$$

and the rates of body forces $\dot{\mathbf{b}}_m$ be known, at a time t_0 and at every point $\mathbf{x}$ of the closure $\bar{D}$. It is assumed that the functions (3.3) satisfy the condition $f \leqslant 0$. It is also assumed that the values of the surface forces $\dot{\mathbf{t}}^{(0)}$ and the velocities of material points $\mathbf{v}^{(0)}$ are known at the time t_0 over the parts S_T and S_v of the boundary, that is

$$\begin{aligned} \dot{\boldsymbol{\sigma}}\mathbf{n} &= \dot{\mathbf{t}}^{(0)} \quad \text{for} \quad \mathbf{x} \in S_T, \\ \mathbf{v} &= \mathbf{v}^{(0)} \quad \text{for} \quad \mathbf{x} \in S_v, \end{aligned} \tag{3.4}$$

where $\mathbf{n}$ is a unit vector normal to S, directed towards the outside of D ([1, 3, 5]). Our task is to find the functions $\dot{\boldsymbol{\sigma}}$, $\dot{\boldsymbol{\epsilon}}$, v defined in $\bar{D}$ and the function $\dot{T}$ defined in D, which satisfy, in the region D, the Eqs. (2.1) to (2.4), (3.1) (3.2) and (4.1) of Ref. [2] and the incremental equations of equilibrium

$$\operatorname{div}\dot{\boldsymbol{\sigma}}+\varrho_0\dot{\mathbf{b}}_m = 0. \tag{3.5}$$

Let us observe that, knowing the functions (3.3), we can determine q at every point of the region D, directly from the Eq. (4.1) of Ref. [2], by differentiating T and $\mathbf{q}$ with respect to the variables $\mathbf{x}$.

4. Discussion of uniqueness conditions

4.1. Local uniqueness condition

The following theorem is proved in Appendix B of the present paper.

THEOREM. *If the inequality*

$$h_1 = h-m_\sigma f_T > \frac{1}{2}\left[\sqrt{(\mathbf{g}\cdot\mathbf{M}^{(a)}\mathbf{g})(\bar{\mathbf{f}}_\sigma\cdot\mathbf{M}^{(a)}\bar{\mathbf{f}}_\sigma)}-\mathbf{g}\cdot\mathbf{M}^{(a)}\bar{\mathbf{f}}_\sigma\right] = h_1^*, \tag{4.1}$$

where

$$\begin{aligned}\mathbf{g} &= (\mathbf{f}_{1,\sigma}+\gamma_{23}\mathbf{Z}\Box d+\gamma_{21}m_\sigma\boldsymbol{\alpha}),\\ \bar{\mathbf{f}}_\sigma &= (\mathbf{f}_\sigma-\gamma_{12}\xi f_T\boldsymbol{\alpha})\end{aligned} \tag{4.2}$$

is satisfied at every point of the plastic portion of the body $D_f = \{\mathbf{x}:f=0\}$, there can exist only one set of functions $\{\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\epsilon}}, \dot{T}\}$ of class C^1 at least, which is a solution of the incremental boundary value problem of generalized coupled thermoplasticity, which was formulated in 3.

The inequality (4.1) is the sufficient local uniqueness condition. Each thermodynamic state, for which the condition (4.1) is satisfied, is secure against bifurcation. Since in the course of a deformation process of the body the value of the strain-hardening function (the modulus) decreases, in general, therefore the value of h_1^* may be treated as an upper estimation of the unknown critical value h corresponding to the critical state.

Some particular cases of the expression (4.1) have already been mentioned in the literature. A similar condition was obtained by HUECKEL and MAIER (Refs. [6, 7]) in their analysis of the stability of material defined as a condition of half the product of the stress rate tensor and the strain rate tensor being positive. Their study was confined to the case of the isothermal theory of plasticity (with no thermo-mechanical couplings), the elastic plastic coupling effects and non-associated laws of plastic flow being preserved. An expression of this type was also obtained by MRÓZ [9] who analysed the sufficient local uniqueness condition in the case of isothermal uncoupled theory of plasticity and non--associated laws of plastic flow. The condition obtained in Ref. [9] was given in a "normalized" form for an elastically isotropic compressible material.

4.2. The global uniqueness condition and the global uniqueness criterion

Let us assume that there exist two sets of functions $\{\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\epsilon}}, \dot{T}, \dot{\mathbf{u}}\}$ and $\{\dot{\boldsymbol{\sigma}}^*, \dot{\boldsymbol{\epsilon}}^*, \dot{T}^*, \dot{\mathbf{u}}^*\}$ which are solutions of the incremental boundary-value problem of generalized coupled thermoplasticity, which was formulated in 3. Then, the following equality must be satisfied

$$\Lambda^* = \int_D (\dot{\boldsymbol{\sigma}}-\dot{\boldsymbol{\sigma}}^*)\cdot(\dot{\boldsymbol{\epsilon}}-\dot{\boldsymbol{\epsilon}}^*)dV = 0 \tag{4.3}$$

due to the fact that both solutions satisfy the same boundary conditions (3.4). Let us denote by J the integrand in the expression (4.3), which depends on $\dot{\boldsymbol{\epsilon}}$ and $\dot{\boldsymbol{\epsilon}}^*$, for an elastic-plastic body, as follows

$$J(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^*, j_1, j_1^*) = [\dot{\boldsymbol{\sigma}}(\dot{\boldsymbol{\epsilon}})-\dot{\boldsymbol{\sigma}}(\dot{\boldsymbol{\epsilon}}^*)]\cdot \Delta\dot{\boldsymbol{\epsilon}}, \tag{4.4}$$

where

$$\Delta\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}-\dot{\boldsymbol{\epsilon}}^*, \quad \dot{\boldsymbol{\sigma}}^* = \dot{\boldsymbol{\sigma}}(\dot{\boldsymbol{\epsilon}}^*)$$

and $j_1 = j_1(\dot{\boldsymbol{\epsilon}})$ and $j_1^* = j_1(\dot{\boldsymbol{\epsilon}}^*)$ are defined by the Eq. $(2\ 14)_1$. The quantities $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\epsilon}}$ and $\dot{\boldsymbol{\sigma}}^*$ and $\dot{\boldsymbol{\epsilon}}^*$ are interrelated by the Eq. $(2.13)_1$, which can be rewritten in a more compact form as follows

$$\dot{\boldsymbol{\sigma}} = \mathbf{M}_1\dot{\boldsymbol{\epsilon}}-\mathbf{M}_1\mathbf{d}_1-\frac{j_1}{H_1}\mathbf{g}^*[\bar{\mathbf{f}}_\sigma^*\cdot(\dot{\boldsymbol{\epsilon}}-\mathbf{d}_1)+z_1] \tag{4.5}$$

where

$$\begin{gathered} \mathbf{g}^* \equiv \mathbf{M}_1\mathbf{g} = \tilde{\varphi}_\varepsilon(\mathbf{M}\boldsymbol{\alpha})-\mathbf{N}_2+\mathbf{B}, \\ \bar{\mathbf{f}}_\sigma^* \equiv \mathbf{M}_1\bar{\mathbf{f}}_\sigma = \mathbf{B}, \quad \mathbf{d}_1 = \gamma_{21}^* q\boldsymbol{\alpha}, \\ z_1 \equiv -qf_T, \quad \mathbf{M}_1 = \mathbf{M}^{(a)}, \quad H_1 \equiv H. \end{gathered} \tag{4.6}$$

Let us introduce the following function J', depending on $\dot{\boldsymbol{\epsilon}}$ and $\dot{\boldsymbol{\epsilon}}^*$

$$J'(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^*) = \Delta\dot{\boldsymbol{\epsilon}}\cdot\mathbf{M}_1\Delta\dot{\boldsymbol{\epsilon}}-\frac{1}{4x^2H}[(\mathbf{g}^*+x^2\bar{\mathbf{f}}_\sigma^*)\cdot\Delta\dot{\boldsymbol{\epsilon}}]^2, \tag{4.7}$$

where x^2 is a scalar quantity; therefore this expression represents a one-parameter family of expressions J', with respect to the parameter x^2. The functions J and J' depend in addition to the variables $\dot{\boldsymbol{\epsilon}}$ and $\dot{\boldsymbol{\epsilon}}^*$, on the thermodynamic state of the body (cf. Sect. 3.3). It will be shown that if the same thermodynamic state is prescribed for J and J', then for each pair ($\dot{\boldsymbol{\epsilon}}$ and $\dot{\boldsymbol{\epsilon}}^*$) the following inequality holds

$$J(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^*, j_1, j_1^*)-J'(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^*) \geqslant 0. \tag{4.8}$$

Let us introduce the following notations for the function $J(\boldsymbol{\epsilon}, \dot{\boldsymbol{\epsilon}}^*, j_1, j_1^*)$:

$$\begin{aligned} J_1 &= J(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^*, 1, 1) && \text{if} \quad j_1(\dot{\boldsymbol{\epsilon}}) = 1 \quad \text{and} \quad j_1(\dot{\boldsymbol{\epsilon}}^*) = 1, \\ J_2 &= J(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^*, 1, 0) && \text{if} \quad j_1(\dot{\boldsymbol{\epsilon}}) = 1 \quad \text{and} \quad j_1(\dot{\boldsymbol{\epsilon}}^*) = 0, \\ J_3 &= J(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^*, 0, 1) && \text{if} \quad j_1(\dot{\boldsymbol{\epsilon}}) = 0 \quad \text{and} \quad j_1(\dot{\boldsymbol{\epsilon}}^*) = 1, \\ J_4 &= J(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^*, 0, 0) && \text{if} \quad j_1(\dot{\boldsymbol{\epsilon}}) = 0 \quad \text{and} \quad j_1(\dot{\boldsymbol{\epsilon}}^*) = 0. \end{aligned} \tag{4.9}$$

Then, by evaluating the difference (4.8) for all the possible four cases (4.9), we obtain, by virtue of (4.5), (4.4) and (4.7)

$$(J_1-J')H = \frac{1}{4x^2}(\nu_g-x^2\nu_f)^2 \geqslant 0,$$

$$(J_2-J')H = -x^2A_\varepsilon A_\varepsilon^* + \left[xA_\varepsilon^* - \frac{1}{2x}(\nu_g - x^2\nu_f)\right]^2 \geqslant 0,$$
$$A_\varepsilon \geqslant 0 \quad \text{and} \quad A_\varepsilon^* < 0,$$

because

$$(J_3-J')H = -x^2A_\varepsilon A_\varepsilon^* + \left[xA_\varepsilon + \frac{1}{2x}(\nu_g - x^2\nu_f)\right]^2 \geqslant 0, \tag{4.10}$$
$$A_\varepsilon < 0 \quad \text{and} \quad A_\varepsilon^* \geqslant 0,$$

because

$$(J_4-J')H = \left[x\nu_f + \frac{1}{2x}(\nu_g - x^2\nu_f)\right]^2 \geqslant 0,$$

where

$$\nu_g = \mathbf{g}^* \cdot \Delta\dot{\boldsymbol{\epsilon}}, \quad \nu_f = \bar{\mathbf{f}}_\sigma^* \cdot \Delta\dot{\boldsymbol{\epsilon}} \tag{4.11$_1$}$$

and

$$A_\varepsilon = \bar{\mathbf{f}}_\sigma^* \cdot (\dot{\boldsymbol{\epsilon}} - \mathbf{d}_1) + z_1, \quad A_\varepsilon^* = \bar{\mathbf{f}}_\sigma^* \cdot (\dot{\boldsymbol{\epsilon}}^* - \mathbf{d}_1) + z_1. \tag{4.11$_2$}$$

It follows that the inequality (4.8) is valid.

Let us now formulate a sufficient global uniqueness criterion (that is a criterion which excludes bifurcation). Let $H > 0$ at every point $\mathbf{x} \in D_f$. If, for every non-zero kinematically admissible and integrable velocity field $\mathbf{v}$, which vanishes over the part S_v of the surface, the inequality

$$\int_D J_2'(\mathbf{v})dV - \int_{D_f} J_1'(\mathbf{v})dV > 0 \tag{4.12}$$

is satisfied, there exists only one pair $\{\dot{\boldsymbol{\sigma}}, T\}$ constituting a solution of the incremental boundary-value problem in coupled thermoplasticity. The integrands in Eq. (4.12) are

$$J_1'(\dot{\boldsymbol{\epsilon}}) = \frac{1}{4x^2H}[(\mathbf{g}^* + x^2\bar{\mathbf{f}}_\sigma^*) \cdot \dot{\boldsymbol{\epsilon}}]^2, \tag{4.13}$$
$$J_2'(\dot{\boldsymbol{\epsilon}}) = \dot{\boldsymbol{\epsilon}} \cdot \mathbf{M}_1 \dot{\boldsymbol{\epsilon}}.$$

This criterion can easily be demonstrated.

Since the expression (4.3) with a zero right-hand term admits the existence of two sets of functions, which are solution of the boundary-value problem stated, therefore the conditions of nonexistence of a state of bifurcation will be that the expression (4.3) should be positive, that is $\Lambda^* > 0$ [5], [12] to [15]. This inequality is a sufficient global uniqueness condition.

The validity of the sufficient global uniqueness criterion (4.12) follows from the inequalities (4.8) and $\Lambda^* > 0$. The integral condition (4.12) is, in this particular form, of essential practical importance. If, for a prescribed state (3.3) it is impossible to find such a field $\mathbf{v}$ that the sum of integrals at the left-hand side of the expression is zero, we are assured that this state is secure against bifurcation.

The idea of deriving such a criterion was conceived as early as in HILL's works ([12–14]) for elastic-plastic bodies under considerable strain, for the isothermal incremental boundary-value problem. For the incremental boundary-value problem in coupled thermoplasticity such a criterion has been derived by MRÓZ and RANIECKI (cf. Refs. [1] and [3] to [5]) in the case of associated laws of plastic flow. Another sufficient global uniqueness criterion for incremental problems of isothermal plasticity of elastic-plastic bodies with non-associated laws of plastic flow has been given by MAIER [16].

It will be shown in Appendix C that the sufficient local uniqueness condition following from the requirement that the integrand J' should be definite positive is the same as for an elastic-plastic body Eq. $(2.13)_1$ or Eq. (4.5), provided that the parameter x^2 takes its optimum form

$$x_0^2 = \left(\frac{\mathbf{g}^* \cdot \mathbf{L}_1 \mathbf{g}^*}{\mathbf{f}_\sigma^* \cdot \mathbf{L}_1 \mathbf{f}_\sigma^*}\right)^{1/2} \tag{4.14}$$

(cf. C.10). A procedure for obtaining the optimum parameter x_0^2 is also discussed in Appendix C.

For the parameter x^2 the sufficient local uniqueness condition becomes the optimum (strongest) condition for the entire one-parameter family of sufficient uniqueness conditions. Now, by substituting the optimum value of the parameter ($x^2 = x_0^2$) into the expression (4.7) we shall obtain the optimum (strongest) integrand which will be denoted by the symbol J_0'.

Appendix A

The procedure used here for deriving the conditions below has been modelled after Refs. [3] and [5].

I. To derive the condition $H > 0$ (problem b_2) let us assume first that for prescribed values of the strain rate $\dot{\boldsymbol{\epsilon}}$ and div $\mathbf{q}$ the plastic loading process is active ($\dot{\boldsymbol{\epsilon}}^p \neq \mathbf{0}$), and let us denote by $\Lambda^{(p)}$ and $\dot{T}^{(p)}$ the corresponding values of Λ and T. Then, a set of algebraic equations can be obtained by substituting the expressions $(3.29)_2$, (2.7) and (4.2) of Ref. [2] into the association condition (4.6) and the Eqs. $(3.29)_1$. Thus

$$\begin{aligned} &(f_T - \gamma_{21}^* \boldsymbol{\alpha} \cdot \mathbf{M}\mathbf{f}_\sigma)\dot{T}^{(p)} - (h + \mathbf{g}_4 \cdot \mathbf{M}\mathbf{f}_\sigma)\Lambda^{(p)} = -\mathbf{f}_\sigma \cdot \mathbf{M}\dot{\boldsymbol{\epsilon}}, \\ &\dot{T}^{(p)} - \frac{1}{p}(m_\sigma + \gamma_{12}^* \xi \boldsymbol{\alpha} \cdot \mathbf{M}\mathbf{f}_\sigma)\Lambda^{(p)} = -\gamma_{12}^* \frac{\xi}{p} \boldsymbol{\alpha} \cdot \mathbf{M}\dot{\boldsymbol{\epsilon}} - \frac{q}{p}, \end{aligned} \tag{A.1}$$

where the following relations have been used for the derivation of $(\text{A.1})_2$

$$\begin{aligned} &m_\sigma = m_\epsilon + \gamma_3 \gamma_{12}^* \gamma_{23}^* \xi \boldsymbol{\alpha} \cdot (\mathbf{N} \square d), \qquad q = \frac{1}{\varrho_0 C_\sigma} \operatorname{div} \mathbf{q}, \\ &m_\epsilon = \frac{1}{\varrho_0 C_\sigma}\left[\gamma_1(\boldsymbol{\sigma} \cdot \mathbf{f}_{1,\sigma} - \Pi \cdot d) - \gamma_3 T \frac{\partial \Pi(Y_K^{T\epsilon})}{\partial T} \cdot d\right]. \end{aligned} \tag{A.2}$$

The set of algebraic equations (A.1) has a unique solution in the form

$$\begin{aligned} &H\Lambda^{(p)} = \mathbf{f}_\sigma \cdot \mathbf{M}\dot{\boldsymbol{\epsilon}} - \frac{1}{p}(f_T - \gamma_{21}^* \boldsymbol{\alpha} \cdot \mathbf{M}\mathbf{f}_\sigma)(\gamma_{12}^* \xi \boldsymbol{\alpha} \cdot \mathbf{M}\dot{\boldsymbol{\epsilon}} + q), \\ &H\dot{T} = \frac{1}{p}(m_\sigma + \gamma_{12}^* \xi \boldsymbol{\alpha} \cdot \mathbf{M}\mathbf{f}_\sigma)(\mathbf{f}_\sigma \cdot \mathbf{M}\dot{\boldsymbol{\epsilon}}) - \frac{h + \mathbf{g}_4 \cdot \mathbf{M}\mathbf{f}_\sigma}{p}(\gamma_{12}^* \xi \boldsymbol{\alpha} \cdot \mathbf{M}\mathbf{f} + q) \end{aligned} \tag{A.3}$$

provided that the condition $H \neq 0$ is satisfied. Now it will be assumed that unloading takes place for the prescribed values of $\dot{\boldsymbol{\epsilon}}$ and div $\mathbf{q}$ ($\dot{\boldsymbol{\epsilon}}^p = 0$). Let us denote by $\dot{T}^{(e)}$ and $\dot{\boldsymbol{\sigma}}^{(e)}$ the relevant values for that process. From the conditions (3.29), of Ref. [2], we obtain

$$
\text{(A.4)}\qquad
\begin{aligned}
\dot{T}^{(e)} &= \frac{1}{p}(\gamma_{12}^{*}\xi\boldsymbol{\alpha}\cdot\mathbf{M}\dot{\boldsymbol{\epsilon}}+q),\\
\dot{\boldsymbol{\sigma}}^{(e)} &= \mathbf{M}\dot{\boldsymbol{\epsilon}}+\frac{\gamma_{21}^{*}}{p}(\gamma_{12}^{*}\xi\boldsymbol{\alpha}\cdot\mathbf{M}\dot{\boldsymbol{\epsilon}}+q)(\mathbf{M}\boldsymbol{\alpha}).
\end{aligned}
$$

Making use of the Eqs. (A.3), we can establish the following relation between $L^{(e)} = \mathbf{f}_\sigma\cdot\dot{\boldsymbol{\sigma}}^{(e)}+\mathbf{f}_T\dot{T}$ and $\Lambda^{(p)}$.

$$
\text{(A.5)}\qquad H\Lambda^{(p)} = L^{(e)}.
$$

In agreement with the unloading criterion assumed, which is that of the expression of $L^{(e)}$ being negative, we have the relation sign $\Lambda^{(p)}$ = sign $L^{(e)}$, which ensures the existence and uniqueness of $\dot{\boldsymbol{\sigma}}$. Then, from (A.5) we find that

$$
\text{(A.6)}\qquad H > 0.
$$

II. The procedure of deriving the condition $h_1 > 0$ (problem b_1) is analogous to that for the first problem, except that, in the present case, it is more convenient to use the Eqs. (3.28) of Ref. [2]. We are interested only in those points of the body where $f = 0$. The quantities $\Lambda^{(p)}$, $\dot{T}^{(p)}$ and $\dot{T}^{(e)}$ are now functions of $\dot{\boldsymbol{\sigma}}$. Let us assume that for given values of $\dot{\boldsymbol{\sigma}}$ and div $\mathbf{q}$ the loading process occurs ($\dot{\boldsymbol{\epsilon}}^p \neq \mathbf{0}$). From Eqs. (2.7), $(3.28)_1$, $(3.28)_2$ and (4.2) of Ref. [2], we find

$$
\text{(A.7)}\qquad
\begin{aligned}
f_T\dot{T}^{(p)}-h_1\Lambda^{(p)} &= -\mathbf{f}_\sigma\cdot\dot{\boldsymbol{\sigma}},\\
\dot{T}^{(p)}-m_\sigma\Lambda^{(p)} &= -\gamma_{12}\xi\boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}-q.
\end{aligned}
$$

This alternative set of algebraic equations has a unique solution in the form

$$
\text{(A.8)}\qquad
\begin{aligned}
h_1\Lambda^{(p)} &= \mathbf{f}\cdot\dot{\boldsymbol{\sigma}}-\gamma_{12}f_T\xi\boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}-qf_T,\\
h_1\dot{T}^{(p)} &= m_\sigma\mathbf{f}_\sigma\cdot\dot{\boldsymbol{\sigma}}-\gamma_{12}h\xi\boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}-hq
\end{aligned}
$$

provided that

$$
h_1 = h-m_\sigma f_T \neq 0.
$$

Let us assume, for prescribed $\dot{\boldsymbol{\sigma}}$ and div $\mathbf{q}$, that the process of unloading takes place ($\dot{\boldsymbol{\epsilon}}^p = 0$). Then from $(3.28)_1$ — Ref. [2], we obtain immediately

$$
\text{(A.9)}\qquad
\begin{aligned}
\dot{T}^{(e)} &= -\gamma_{12}\xi\boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}-q,\\
\dot{\boldsymbol{\epsilon}}^{(e)} &= \mathbf{L}\dot{\boldsymbol{\sigma}}-\gamma_{21}(\gamma_{12}\xi\boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}-q)\boldsymbol{\alpha},\\
\dot{\boldsymbol{\epsilon}} &= \dot{\boldsymbol{\epsilon}}^{(e)}.
\end{aligned}
$$

By evaluating $L^{(e)} = \mathbf{f}_\sigma\cdot\dot{\boldsymbol{\sigma}}+f_T\dot{T}^{(e)}$ it is easily observed that

$$
\text{(A.10)}\qquad h_1\Lambda^{(p)} = L^{(e)}.
$$

Hence, the condition sought for is

$$
\text{(A.11)}\qquad h_1 > 0.
$$

Appendix B

The derivation of the sufficient local uniqueness condition for the solution of an incremental boundary problem has been modelled after the method described in Refs. [3, 4] and [20]. Let us assume that there are two sets of functions $\{\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\epsilon}}, \dot{\mathbf{u}}, \dot{T}\}$ and $\{\dot{\boldsymbol{\sigma}}^*, \dot{\boldsymbol{\epsilon}}^*, \dot{\mathbf{u}}^*, \dot{T}^*\}$, which are solutions of the incremental (rate type) boundary value problem. Our task is to find the condition for $\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}^*$ and the functions $\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^*$, $\dot{\mathbf{T}} = \dot{\mathbf{T}}^*$ are obtained in a unique manner from Eqs. $(2.8)_1$ and $(2.8)_2$. The satisfaction of this condition implies that $h_1 > 0$. By applying the Gauss–Ostrogradsky theorem we have

$$\int_D (\dot{\boldsymbol{\sigma}}-\dot{\boldsymbol{\sigma}}^*)\cdot(\dot{\boldsymbol{\epsilon}}-\dot{\boldsymbol{\epsilon}}^*)dV = 0. \tag{B.1}$$

According to Eq. $(2.8)_1$, $\dot{\boldsymbol{\epsilon}}$ and $\dot{\boldsymbol{\epsilon}}^*$ are functions of $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\sigma}}^*$, respectively. Let us denote the integrand Eq. (B.1) as follows

$$I(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\sigma}}^*, j, j^*) = \Delta\dot{\boldsymbol{\sigma}}\cdot\Delta\dot{\boldsymbol{\epsilon}}, \tag{B.2}$$

where

$$\Delta\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}-\dot{\boldsymbol{\sigma}}^*, \qquad \Delta\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}-\dot{\boldsymbol{\epsilon}}^*$$

and

$$j = j(\dot{\boldsymbol{\sigma}}) \quad \text{and} \quad j^* = j(\dot{\boldsymbol{\sigma}}^*)$$

are determined by Eq. (2.9). Bearing this in mind we must find a condition for I to be a definite positive function (that is $I > 0$ for $\dot{\boldsymbol{\sigma}} \neq \boldsymbol{\sigma}^*$ and $I = 0$ for $\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}^*$). Let us observe that the Eq. (2.8) can be written in the form

$$\dot{\boldsymbol{\epsilon}} = \mathbf{L}_1\dot{\boldsymbol{\sigma}} + \frac{j}{h_1}\mathbf{g}(\bar{\mathbf{f}}_\sigma\cdot\dot{\boldsymbol{\sigma}}+Z_1)+\mathbf{d}_1, \tag{B.3}$$

where

$$\begin{aligned} \mathbf{g} &= (\mathbf{f}_{1,\sigma}+\gamma_{23}\mathbf{Z}\square d+\gamma_{21}m_\sigma\boldsymbol{\alpha}),\\ \bar{\mathbf{f}}_\sigma &= (\mathbf{f}_\sigma-\gamma_{12}\xi f_T\boldsymbol{\alpha}), \quad \mathbf{d}_1 = -\gamma_{21}q\boldsymbol{\alpha},\\ Z_1 &= -qf_T, \quad \mathbf{L}_1 = \mathbf{L}^{(a)}, \quad \mathbf{M}_1 = \mathbf{M}^{(a)}. \end{aligned} \tag{B.4}$$

a) Let $j = j^* = 1$. On substituting Eq. (B.3) into Eq. $(\mathrm{B.2})_1$ we find

$$I(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\sigma}}^*, 1, 1) = I_1 = \Delta\dot{\boldsymbol{\sigma}}\cdot\mathbf{L}_1\Delta\dot{\boldsymbol{\sigma}}+\frac{1}{h_1}[(\mathbf{g}\cdot\Delta\dot{\boldsymbol{\sigma}})(\bar{\mathbf{f}}_\sigma\cdot\Delta\dot{\boldsymbol{\sigma}})]. \tag{B.5}$$

Let us now resolve the vector $\Delta\dot{\boldsymbol{\sigma}}$ into components, the directions of which are those of $\mathbf{M}_1\mathbf{g}$, $\mathbf{M}_1\bar{\mathbf{f}}_\sigma$ and the direction t normal to the other two in a 9-dimensional space with a metric $\mathbf{L}_1$. By evaluating the product $(\Delta\dot{\boldsymbol{\sigma}}\cdot L_1\Delta\dot{\boldsymbol{\sigma}})$ we find that the expression for $\mathbf{L}_1$ takes the form

$$I_1 = \begin{cases} \mathbf{t}\cdot\mathbf{L}_1\mathbf{t}+\dfrac{1}{h_1(1-m_{gf}^2)}[A(\bar{\mathbf{f}}_\sigma\cdot\Delta\dot{\boldsymbol{\sigma}})^2+2B(\bar{\mathbf{f}}_\sigma\cdot\Delta\dot{\boldsymbol{\sigma}})(\mathbf{g}\cdot\Delta\dot{\boldsymbol{\sigma}})+C(\mathbf{g}\cdot\Delta\dot{\boldsymbol{\sigma}})^2], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad m_{gf}^2 < 1, \\ \mathbf{t}\cdot\mathbf{L}_1\mathbf{t}+\dfrac{A_0}{M_f^2h_1}(\bar{\mathbf{f}}_\sigma\cdot\Delta\dot{\boldsymbol{\sigma}})^2 \quad \text{if} \quad m_{gf}^2 = 1, \end{cases} \tag{B.6}$$

where

$$\mathbf{t} = \Delta\dot{\boldsymbol{\sigma}} - \frac{1}{1-m_{gf}^2}\left[\frac{1}{M_f^2}(\bar{\mathbf{f}}_\sigma \cdot \Delta\dot{\boldsymbol{\sigma}}) - \frac{m_{gf}}{M_f M_g}(\mathbf{g} \cdot \Delta\dot{\boldsymbol{\sigma}})\right]\mathbf{M}_1\bar{\mathbf{f}}_\sigma - \frac{1}{1-m_{gf}^2}\left[\frac{1}{M_g^2}(\mathbf{g} \cdot \Delta\dot{\boldsymbol{\sigma}}) - \frac{m_{gf}}{M_g M_f}(\bar{\mathbf{f}}_\sigma \cdot \Delta\dot{\boldsymbol{\sigma}})\right]\mathbf{M}_1\mathbf{g} \quad \text{if} \quad m_{gf}^2 < 1,$$

$$\mathbf{t} = \Delta\dot{\boldsymbol{\sigma}} - \frac{\bar{\mathbf{f}}_\sigma \cdot \Delta\dot{\boldsymbol{\sigma}}}{M_f^2}\mathbf{M}_1\bar{\mathbf{f}}_\sigma \quad \text{if} \quad m_{gf}^2 = 1$$

and

$$M_f^2 = \bar{\mathbf{f}}_\sigma \cdot \mathbf{M}_1\bar{\mathbf{f}}_\sigma, \quad M_g^2 = \mathbf{g} \cdot \mathbf{M}_1\mathbf{g}, \quad m_{gf} = \frac{\mathbf{g} \cdot \mathbf{M}_1\bar{\mathbf{f}}_\sigma}{M_f M_g},$$

$$\begin{aligned} A &= \frac{h_1}{M_f^2}, \\ B &= \frac{1}{2}(1-m_{gf}^2) - \frac{m_{gf}}{M_f M_g}h_1, \quad C = \frac{h_1}{M_g^2}, \\ A_0 &= h_1 + M_f^2 + m_{gf}M_f M_g. \end{aligned} \tag{B.7}$$

Since $h_1 > 0$, therefore I_1 (Eq. (B.6)) is a symmetric quadratic form, which is definite positive, if

$$AC - B^2 > 0. \tag{B.8}$$

On substituting Eq. (B.7) into Eq. (B.8) we find, in a similar manner,

$$a_1 h_1^2 + b_1 h_1 + c_1 > 0,$$

where

$$\begin{aligned} a_1 &= 1, \\ b_1 &= m_{gf}M_f M_g, \\ c_1 &= -(1-m_{df}^2)\frac{M_f^2 M_g^2}{4}. \end{aligned} \tag{B.9}$$

On solving the above inequality we find the condition

$$h_1 > \frac{1}{2}M_g M_f(1-m_{gf}). \tag{B.10}$$

On substituting Eq. (B.7) in the above expression we obtain easily the condition (4.1). Let us add that in the case of $\gamma_{12} = \gamma_{21} = \gamma_{23} = \gamma_{13} = 0$ the condition (B.10) or (4.1) takes the form

$$h_1 > 0. \tag{B.11}$$

b) Let $j = 1$, $j^* = 0$. Then,

$$I(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\sigma}}^*, 1, 0) = I_2 = \Delta\dot{\boldsymbol{\sigma}} \cdot \mathbf{L}_1 \Delta\dot{\boldsymbol{\sigma}} + \frac{1}{h_1}(\mathbf{g} \cdot \Delta\dot{\boldsymbol{\sigma}})[\bar{\mathbf{f}}_\sigma \cdot \dot{\boldsymbol{\sigma}} + z_1]. \tag{B.12}$$

The study of the expression (B.12) for positive definiteness will be as follows. Let us resolve $\mathbf{g}$ in the directions $\bar{\mathbf{f}}_\sigma$ and $\boldsymbol{\beta}$ in the following manner

$$\mathbf{g} = c\bar{\mathbf{f}}_\sigma + \boldsymbol{\beta}, \tag{B.13}$$

where c is a parameter to be used for optimizing the uniqueness condition in the case of simultaneous loading and unloading (b) and (c). Then, from Eq. $(\mathrm{B}.4)_{1,2}$ and Eq. (B.13) we find

$$\boldsymbol{\beta} = \bar{\mathbf{f}}_\sigma(1-c) + (\gamma_{21} m_\sigma + c\gamma_{12}\xi f_T)\boldsymbol{\alpha} + \mathbf{Z}\Box(\gamma_{13} f_{1,\Pi} + \gamma_{23}\dot{d}). \tag{B.14}$$

(B.12) must be expressed as a quadratic form. Let us estimate therefore I_2 as follows

$$I_2 \geqslant I_2' = \Delta\dot{\boldsymbol{\sigma}} \cdot \mathbf{L}_1 \Delta\dot{\boldsymbol{\sigma}} + \frac{1}{h_1}[c\bar{A}_\sigma^2 + \bar{A}_\sigma(\boldsymbol{\beta} \cdot \Delta\dot{\boldsymbol{\sigma}})], \tag{B.15}$$

where

$$I_2 = I_2' - \frac{c}{h_1}\bar{A}_\sigma \bar{A}_\sigma^* \tag{B.16}$$

and

$$\bar{A}_\sigma = \bar{\mathbf{f}}_\sigma \cdot \dot{\boldsymbol{\sigma}} + z_1 \geqslant 0, \quad \bar{A}_\sigma^* = \bar{\mathbf{f}}_\sigma \cdot \dot{\boldsymbol{\sigma}}^* + z_1 < 0.$$

Let us resolve, as before, the vector $\Delta\dot{\boldsymbol{\sigma}}$ in the directions $\mathbf{M}_1\boldsymbol{\beta}$ and the direction $\mathbf{t}_1$ normal to $\mathbf{M}_1\boldsymbol{\beta}$ in a 9-dimensional space with a metric $\mathbf{L}_1$. On substituting the result thus obtained into (B.15) we find

$$I_2' = \mathbf{t}_1 \cdot \mathbf{L}_1 \mathbf{t}_1 + \frac{1}{h_1}[\bar{A}_1(\boldsymbol{\beta} \cdot \Delta\dot{\boldsymbol{\sigma}})^2 + 2\bar{B}_1 \bar{A}_\sigma(\boldsymbol{\beta} \cdot \Delta\dot{\boldsymbol{\sigma}}) + \bar{C}_1 \bar{A}_\sigma^2], \tag{B.17}$$

where

$$\begin{gathered} \mathbf{t}_1 = \Delta\dot{\boldsymbol{\sigma}} - \frac{\boldsymbol{\beta} \cdot \Delta\boldsymbol{\sigma}}{M_\beta^2}\mathbf{M}_1\boldsymbol{\beta}, \\ M_\beta^2 = \boldsymbol{\beta} \cdot \mathbf{M}_1 \boldsymbol{\beta}, \\ \bar{A}_1 = \frac{h_1}{M_\beta^2}, \quad \bar{B}_1 = \frac{1}{2}, \quad \bar{C}_1 = c. \end{gathered} \tag{B.18}$$

The expression (B.17) is a definite positive quadratic form, if

$$\bar{A}_1 \bar{C}_1 - \bar{B}_1^2 > 0. \tag{B.19}$$

On substituting Eq. (B.18) into Eq. (B.19) we find

$$h_1 > \frac{M_\beta^2}{4c}. \tag{B.20}$$

The form of Eq. (B.20) shows that we have a one-parameter family of uniqueness conditions, the parameter being c. We want to determine the condition of minimum with respect to c, in order that the bifurcation states should be estimated as closely as possible.

On substituting Eq. (B.13) in Eq. (B.20), we obtain

$$h_1 > \frac{1}{4c}[(\mathbf{g} - c\bar{\mathbf{f}}_\sigma) \cdot \mathbf{M}_1(\mathbf{g} - c\bar{\mathbf{f}}_\sigma)] = \frac{1}{4c}[M_g^2 - 2cM_{gf} + c^2 M_f^2], \tag{B.21}$$

where

$$M_{gf} = \mathbf{g} \cdot \mathbf{M}_1 \bar{\mathbf{f}}_\sigma = \bar{\mathbf{f}}_\sigma \cdot \mathbf{M}_1 \mathbf{g}.$$

The right-hand side of Eq. (B.21) must, therefore, become minimum in the scalar parameter c. Let

(B.22)
$$F(c) = \frac{1}{4}\left(\frac{M_g^2}{c} - 2M_{gf} + cM_f^2\right).$$

Then

(B.23)
$$\frac{\partial F(c)}{\partial c} = 0.$$

From the condition (B.23) we find that

(B.24)
$$c^0 = \frac{M_g}{M_f}, \quad F(c^0) = F^0, \quad c > 0.$$

On substituting Eq. $(B.24)_1$ into the inequality (B.21) we obtain, after some manipulation,

(B.25)
$$h_1 > \frac{1}{2}(M_g M_f - M_{gf}).$$

By introducing the notations (B.7) we obtain easily the Eq. (4.1) sought-for. It can be shown that the right-hand member is zero, if

(B.26)
$$\bar{A}_1(\boldsymbol{\beta} \cdot \Delta\dot{\boldsymbol{\sigma}}) = -\bar{B}_1 \bar{A}_\sigma.$$

Hence $I_2 = I_2'$, if $\bar{A}_\sigma = 0$. Then, it follows from Eq. (B.16) to (B.26) that

$$I_2 > 0.$$

(c) Let $j = 0$, $j^* = 1$. The procedure of demonstrating that $I(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\sigma}}^*, 0, 1) = I_3 > 0$ is analogous to that in the case (b). We have, therefore,

(B.27)
$$I_3' = I_3 + \frac{c}{h_1}\bar{A}_\sigma \bar{A}_\sigma^*,$$
$$\bar{A}_\sigma = \bar{\mathbf{f}}_\sigma \cdot \dot{\boldsymbol{\sigma}} + z_1 < 0, \quad \bar{A}_\sigma^* = \bar{\mathbf{f}}_\sigma \cdot \dot{\boldsymbol{\sigma}}^* + z_1 \geqslant 0.$$

Hence

(B.28)
$$I_3' \geqslant I_3 = \mathbf{t}_1 \cdot \mathbf{L}_1 t_1 + \frac{1}{h_1}[\bar{A}_1(\boldsymbol{\beta} \cdot \Delta\dot{\boldsymbol{\sigma}})^2 - 2\bar{B}_1 \bar{A}_\sigma(\boldsymbol{\beta} \cdot \Delta\dot{\boldsymbol{\sigma}}) + \bar{C}_1(A_\sigma^*)^2],$$

where $\mathbf{t}_1$, $\bar{A}_1$, $\bar{B}_1$ and $\bar{C}_1$ are defined by Eq. (B.18). Similarly to the case (b) it follows that $I_3 > 0$.

d) Let $j = j^* = 0$. Then, the integrand

(B.29)
$$I(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\sigma}}^*, 0, 0) = I_4 = \Delta\dot{\boldsymbol{\sigma}} \cdot \mathbf{L}_1 \Delta\dot{\boldsymbol{\sigma}}$$

is positive definite because $\mathbf{L}_1 = \mathbf{L}^{(a)}$ is definite positive. Then from (B.1) it follows directly that $\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}^*$.

Appendix C

It will now be shown that J' is definite positive if Eq. (4.1) is satisfied. This will show that the sufficient local uniqueness condition for the integrand J' is the same as in the case of an elastic-plastic body J. It constitutes also a criterion enabling us to confirm the reason for introducing the expression J'. Let us denote

$$(\mathbf{g}^* + x^2 \overline{\mathbf{f}}_\sigma^*) = \mathbf{M}_1 \mathbf{W}. \tag{C.1}$$

From Eq. (4.6) and Eq. (C.1) it follows that

$$\mathbf{M}_1 \mathbf{W} = \tilde{\varphi}_\varepsilon (\mathbf{M}\boldsymbol{\alpha}) - \mathbf{N}_2 + \mathbf{B}(1+x^2) \tag{C.2}$$

and

$$\mathbf{W} = \tilde{\varphi}_\varepsilon \mathbf{L}_1 (\mathbf{M}\boldsymbol{\alpha}) - \mathbf{L}_1 \mathbf{N}_2 + \mathbf{L}_1 \mathbf{B}(1+x^2).$$

On substituting Eq. (C.1) into Eq. (4.7), we obtain

$$J' = \Delta\dot{\boldsymbol{\varepsilon}} \cdot \mathbf{M}_1 \Delta\dot{\boldsymbol{\varepsilon}} - \frac{1}{4x^2 H_1'} [(\mathbf{M}_1 \mathbf{W}) \cdot \Delta\dot{\boldsymbol{\varepsilon}}]^2. \tag{C.3}$$

On resolving $\Delta\dot{\boldsymbol{\varepsilon}}$ onto the directions $\mathbf{W}$ and $\mathbf{t}^*$ normal to $\mathbf{W}$ in a 9-dimensional space with a metric $\mathbf{M}_1$, we obtain the expressions

$$\mathbf{t}^* = \Delta\dot{\boldsymbol{\varepsilon}} - \frac{\Delta\dot{\boldsymbol{\varepsilon}} \cdot (\mathbf{M}_1 \mathbf{W})}{M_W^2} \mathbf{W}, \tag{C.4}$$

where

$$M_W^2 = \mathbf{W} \cdot \mathbf{M}_1 \mathbf{W}.$$

Then

$$J' = \mathbf{t}^* \cdot \mathbf{M}_1 \mathbf{t}^* + \frac{1}{(4x^2 H_1') M_W^2} (4x^2 H_1' - M_W^2)(\Delta\dot{\boldsymbol{\varepsilon}} \cdot \mathbf{M}_1 \mathbf{W})^2. \tag{C.5}$$

From Eq. (C.5) it follows that J' is positive definite, if

$$H_1' > \frac{1}{4x^2} (\mathbf{W} \cdot \mathbf{M}_1 \mathbf{W}) = \frac{\mathscr{P}(x^2)}{4}. \tag{C.6}$$

The condition (C.6) must now be optimized by finding the minimum in the parameter x^2. Condition (C.6) yields a one-parameter family of uniqueness conditions for a comparative body.

Let

$$y = x^2 \quad \text{hence} \quad y > 0,$$

$$\mathscr{P}(y) = \frac{1}{y} (\mathbf{W} \cdot \mathbf{M}_1 \mathbf{W}). \tag{C.7}$$

On substituting Eq. $(C.7)_1$ in the expression (C.2) and calculating the derivative, we find

$$\frac{\partial \mathscr{P}(y)}{\partial y} = 0,$$

therefore

$$\text{(C.8)} \qquad \frac{\partial \mathscr{P}(y)}{\partial y} = -\frac{1}{y^2}(\mathbf{W}\cdot\mathbf{M}_1\mathbf{W}) + \frac{2}{y}(\mathbf{W}\cdot\mathbf{M}_1\mathbf{L}_1\mathbf{B}) = 0.$$

Hence

$$2y(\mathbf{W}\cdot\mathbf{B}) = (\mathbf{W}\cdot\mathbf{M}_1\mathbf{W}).$$

The expression $(C.8)_3$ results from the condition of the function $\mathscr{P}(y)$ taking an extremum value in the scalar parameter y. The expression $(C.8)_3$ must be transformed to obtain the desired result, the expressions for $\mathbf{W}$ (cf. Eq. $(C.2)_1$) being taken into account. Let us rewrite the expression $(C.2)_2$ in the form

$$\text{(C.9)} \qquad \mathbf{W} = \mathbf{W}_1 + x^2\mathbf{L}_1\mathbf{B},$$

where

$$\mathbf{W}_1 = \tilde{\varphi}_\varepsilon\mathbf{L}_1(\mathbf{M}\boldsymbol{\alpha}) - \mathbf{L}_1\mathbf{N}_2 + \mathbf{L}_1\mathbf{B}.$$

On substituting Eq. (C.9) into Eq. (C.8), we find, after rearrangement

$$\text{(C.10)} \qquad y_0 = x_0^2 = \sqrt{\frac{\mathbf{g}^*\cdot\mathbf{L}_1\mathbf{g}^*}{\bar{\mathbf{f}}_\sigma^*\cdot\mathbf{L}_1\bar{\mathbf{f}}_\sigma^*}},$$

where $\mathbf{g}^*$ and $\bar{\mathbf{f}}_\sigma^*$ are defined by the relations $(4.6)_1$ and $(4.6)_2$, respectively.

It can easily be shown that, by substituting Eq. (C.10) into Eq. (C.6) and taking into consideration $(4.6)_{1,2}$ and (C.9), we shall obtain, after rearrangement,

$$\text{(C.11)} \qquad H > \frac{1}{2}\left[\sqrt{(\mathbf{W}_1\cdot\mathbf{M}_1\mathbf{W}_1)(\mathbf{B}\cdot\mathbf{L}_1\mathbf{B})} + \mathbf{W}_1\cdot\mathbf{B}\right].$$

It can also be shown, that

$$\text{(C.12)} \qquad \mathbf{W}_1 = \mathbf{g}, \qquad \mathbf{B} = \mathbf{M}_1\bar{\mathbf{f}}_\sigma.$$

Then, from Eqs. (2.3), (2.4), (4.2) and Eq. (C.12) it follows that

$$\text{(C.13)} \qquad H = h_1 + \mathbf{g}\cdot\mathbf{M}_1\bar{\mathbf{f}}_\sigma.$$

On substituting Eq. (C.12) and Eq. (C.13) into Eq. (C.11) we find easily, after rearrangement, the condition (4.1), our proof thus being accomplished.

References

1. B. RANIECKI, *Problems in applied thermoplasticity*, IFTR-Reports, 29, 1977 [in Polish].
2. Z. ŚLODERBACH, *Generalized coupled thermoplasticity, Part I. Fundamental equations and identities*, Arch. Mech., **35**, 3, [Submitted],
3. Z. MRÓZ, B. RANIECKI, *On the uniqueness problem in coupled thermoplasticity*, Int. J. Eng. Sci., **14**, 211–221, 1976.
4. Z. MRÓZ, B. RANIECKI, *A derivation of the uniqueness conditions in coupled thermoplasticity*, Int. J. Engn. Sci., **14**, 395–401, 1976.
5. B. RANIECKI, *Uniqueness problems and variational principles in thermoplasticity*, in: Thermoplasticity, Ossolineum, 175–222, Wrocław 1973 [in Polish].

6. T. HUECKEL and G. MAIER, *Incremental boundary value problems in the presence of coupling of elastic, and plastic deformations. A rock mechanics oriented theory*, Int. J. Solids Structures, **13**, 1–15, 1977.
7. T. HUECKEL and G. MAIER, *Nonassociated and coupled flow rules of elastoplasticity for geotechnical media*, Presented on the Proc. 9-th JCSMFE, Tokyo 1977.
8. Z. MRÓZ, *Non-associated flow laws in plasticity*, J. de Mecanique, II, 1, Mars 1963.
9. Z. MRÓZ, *On forms of constitutive laws for elastic-plastic solids*, Arch. Mech., **18**, 1, 1966.
10. W. OLSZAK, P. PERZYNA, A. SAWCZUK *et al.*, *Theory of plasticity*, PWN, Warszawa 1965.
11. JA. A. KAMIENIARZ, *Some properties of model equations in coupled thermoplasticity*, PMM, **36**, 6, 1100–1108, 1972 [in Russ.].
12. R. HILL, *A general theory of uniqueness and stability in elastic-plastic solids*, J. Mech. Phys. Solids, **6**, 236–249, 1958.
13. R. HILL, *Some basic principles in the mechanics of solids without a natural time*, J. Mech. Phys. Solids, **7**, 209–225, 1959.
14. R. HILL, *Uniqueness in general boundary-value problems for elastic or inelastic solids*, J. Mech. Phys. Solids, **9**, 114–130, 1961.
15. R. HILL, *Uniqueness criteria and extremum principles in self-adjoint problems of continuum mechanics*, J. Mech. Phys. Solids, **10**, 185-194, 1962.
16. G. MAIER, *A minimum principle for incremental elastoplasticity, with non-associated flow laws*, J. Mech. Phys. Solids, **18**, 319–330, 1970.
17. Z. ŚLODERBACH, *Criteria of bifurcation of equilibrium states in the generalized theory of plasticity*, Doctor's Thesis, IFTR-Reports, **37**, 1980 [in Polish].
18. T. HUECKEL, *Coupling of elastic and plastic deformations of bulk solids*, Meccanica, **11** 227–235, 1976.
19. T. HUECKEL, J. A. KÖNIG, *Some problems in elastoplasticity*, Academia Polacca Delle Scienze, Conferenze 74, Ossolineum 1979.
20. B. RANIECKI, *Uniqueness criteria in solids with non-associated plastic flow laws at finite deformations*, Bull. Acad. Polon. Sci., Série Sci. Techn., **27**, 8–9, 391–399, 1979.
21. B. RANIECKI, and O. T. BRUHNS, *Bounds to bifurcations stresses in solids with non-associated flow law at finite strain*, J. Mech. Phys. Solids, **29**, 2, 153–172, 1981.

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