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ON A THEOREM RELATING TO FIVE POINTS IN A PLANE.

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Two triangles, ABC, $A'B'C'$ which are such that the lines AA', BB', CC' meet in a point, are said to be in perspective; and a triangle $A'B'C'$, the angles A', B', C' of which lie in the sides BC, CA, AB respectively, is said to be inscribed in the triangle ABC; hence, if A', B', C' are the intersections of the sides by the lines AO, BO, CO respectively (where O is any point whatever), the triangle $A'B'C'$ is said to be perspectively inscribed in the triangle ABC, viz. it is so inscribed by means of the point O.

We have the following theorem, relating to any triangle ABC, and two points O, O'. If in the triangle ABC, by means of the point O, we inscribe a triangle $A'B'C'$, and in the triangle $A'B'C'$, by means of the point O', we inscribe a triangle $\alpha\beta\gamma$, then the triangles ABC, $\alpha\beta\gamma$ are in perspective, viz. the lines $A\alpha$, $B\beta$, $C\gamma$ will meet in a point.

This is very easily proved analytically; in fact, taking $x=0$, $y=0$, $z=0$ for the equations of the lines $B'C'$, $C'A'$, $A'B'$ respectively, and (X, Y, Z) for the coordinates of the point O, then the coordinates of (A, B, C) are found to be $(-X, Y, Z)$, $(X, -Y, Z)$, $(X, Y, -Z)$ respectively. Moreover, if (X', Y', Z') are the coordinates of the point O', then the coordinates of (α, β, γ) are found to be

$$(0, Y', Z'), (X', 0, Y), (X', Y', 0)$$

respectively. Hence the equations of the lines $A\alpha$, $B\beta$, $C\gamma$ are respectively

$$\begin{vmatrix} x, & y, & z \\ -X, & Y, & Z \\ 0, & Y', & Z' \end{vmatrix} = 0, \quad \begin{vmatrix} x, & y, & z \\ X, & -Y, & Z \\ X', & 0, & Z' \end{vmatrix} = 0, \quad \begin{vmatrix} x, & y, & z \\ X, & Y, & -Z \\ X', & Y', & 0 \end{vmatrix} = 0;$$

that is

$$\begin{aligned}
x(\ YZ' - Y'Z) + y(\ \ Z'X) + z(-XY'\ ) &= 0,\\
x(-YZ'\ ) + y(\ ZX' - Z'X) + z(\ X'Y) &= 0,\\
x(\ +Y'Z) + y(-ZX'\ ) + z(\ XY' - X'Y) &= 0,
\end{aligned}$$

which are obviously the equations of three lines which meet in a point.

But the theorem may be exhibited as a theorem relating to a quadrangle 1234 and a point O'; for writing 1, 2, 3, 4 in place of A, B, C, O, the triangle $A'B'C'$ is in fact the triangle formed by the three centres 41.23, 42.31, 43.12 of the quadrangle 1234, hence the triangle in question must be similarly related to each of the four triangles 423, 431, 412, 123; or, forming the diagram

	P	Q	R	S
41.23	4	3	2	1
42.31	3	4	1	2
43.12	2	1	4	3,

we have the following form of the theorem: viz. the lines

$\alpha 4,\ \beta 3,\ \gamma 2$ meet in a point P,

$\alpha 3,\ \beta 4,\ \gamma 1$ " " Q,

$\alpha 2,\ \beta 1,\ \gamma 4$ " " R,

$\alpha 1,\ \beta 2,\ \gamma 3$ " " S,

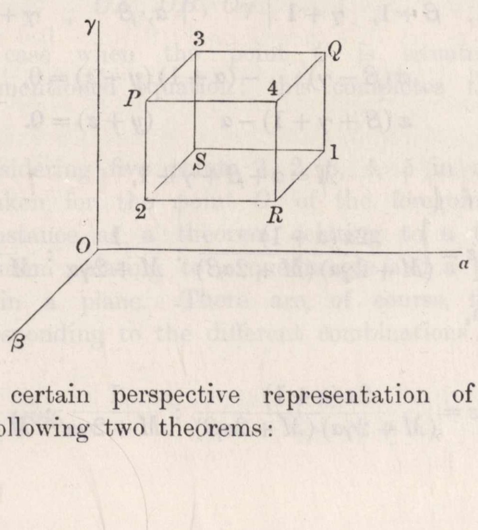
or, what is the same thing, we have with the points 1, 2, 3, 4 and the point O' constructed the four points P, Q, R, S such that

$1S,\ 2R,\ 3Q,\ 4P$ meet in a point α,

$2S,\ 1R,\ 4Q,\ 3\beta$ " " β,

$3S,\ 4R,\ 1Q,\ 2P$ " " γ.

The eight points 1, 2, 3, 4, P, Q, R, S form a figure such as the perspective representation of a parallelopiped, or, if we please, a cube; and not only so, but the



plane figure is really a certain perspective representation of the cube; this identification depends on the following two theorems:

1. Considering the four summits 1, 2, 3, 4, which are such that no two of them belong to the same edge, then, if through any point O we draw

the line OA' meeting the lines 41, 23,
" OB' " " 42, 31,
" OC' " " 43, 12,

and the lines $O\alpha$, $O\beta$, $O\gamma$ parallel to the three edges of the cube respectively, the three planes $(OA', O\alpha)$, $(OB', O\beta)$, $(OC', O\gamma)$ will meet in a line.

2. For a properly selected position of the point O,

the lines OB', OC', $O\alpha$ will lie in a plane,
" OC', OA', $O\beta$ " " ,
" OA', OB', $O\gamma$ " " .

In fact for such a position of O, projecting the whole figure on any plane whatever, the lines $O1$, $O2$, $O3$, $O4$, OP, OQ, OR, OS, $O\alpha$, $O\beta$, $O\gamma$, OA', OB', OC' meet the plane of projection in the points 1, 2, 3, 4, P, Q, R, S, α, β, γ, A', B', C' related to each other as in the last-mentioned form of the plane theorem. To prove the two solid theorems, take O for the origin, $O\alpha$, $O\beta$, $O\gamma$ for the axes, (α, β, γ) for the coordinates of the summit S, and 1 for the edge of the cube,

the coordinates of 1 are $\alpha+1, \beta, \gamma$,
" 2 " $\alpha, \beta+1, \gamma$,
" 3 " $\alpha, \beta, \gamma+1$,
" 4 " $\alpha+1, \beta+1, \gamma+1$.

The equations of the line OA', or say of the line $O\,(41, 23)$, are those of the planes $O\,41$, $O\,23$, viz. these are

$$\begin{vmatrix} x, & y, & z \\ \alpha+1, & \beta, & \gamma \\ \alpha+1, & \beta+1, & \gamma+1 \end{vmatrix} = 0, \quad \begin{vmatrix} x, & y, & z \\ \alpha, & \beta+1, & \gamma \\ \alpha, & \beta, & \gamma+1 \end{vmatrix} = 0\,;$$

that is

$$x(\beta-\gamma) \quad -(\alpha+1)(y-z)=0,$$
$$x(\beta+\gamma+1)-\alpha \quad (y+z)=0.$$

Writing for shortness

$$M=\alpha+\beta+\gamma+1,$$

these equations give

$$x : y : z = \frac{2\alpha(\alpha+1)}{(M+2\gamma\alpha)(M+2\alpha\beta)} : \frac{1}{M+2\gamma\alpha} : \frac{1}{M+2\alpha\beta}\,;$$

or, completing the system,

for line OA' we have

$$x : y : z = \frac{2\alpha(\alpha+1)}{(M+2\gamma\alpha)(M+2\alpha\beta)} : \frac{1}{M+2\gamma\alpha} : \frac{1}{M+2\alpha\beta}\,;$$

for line OB' we have

$$x : y : z = \frac{1}{M+2\beta\gamma} : \frac{2\beta(\beta+1)}{(M+2\alpha\beta)(M+2\beta\gamma)} : \frac{1}{M+2\alpha\beta};$$

for line OC' we have

$$x : y : z = \frac{1}{M+2\beta\gamma} : \frac{1}{M+2\gamma\alpha} : \frac{2\gamma(\gamma+1)}{(M+2\beta\gamma)(M+2\gamma\alpha)}.$$

The equations of the lines $O\alpha$, $O\beta$, $O\gamma$ are of course $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$ respectively; and we therefore see at once that the planes $(OA',\ O\alpha)$, $(OB',\ O\beta)$, $(OC',\ O\gamma)$ meet in a line, viz. in the line which has for its equations

$$x : y : z = \frac{1}{M+2\beta\gamma} : \frac{1}{M+2\gamma\alpha} : \frac{1}{M+2\alpha\beta}.$$

The lines OB', OC', $O\alpha$ will lie in a plane, if only

$$1 = \frac{4\beta\gamma(\beta+1)(\gamma+1)}{(M+2\beta\gamma)^2},$$

that is

$$(M+2\beta\gamma)^2 = 4\beta\gamma(\beta+1)(\gamma+1),$$

or, as this may be written,

$$M^2 + 4\beta\gamma(\alpha+\beta+\gamma+1+\beta\gamma) = 4\beta\gamma(\beta\gamma+\beta+\gamma+1);$$

that is

$$M^2 + 4\alpha\beta\gamma = 0,$$

or, what is the same thing,

$$(\alpha+\beta+\gamma+1)^2 + 4\alpha\beta\gamma = 0;$$

and from the symmetry of this equation we see that, when it is satisfied,

the lines OB', OC', $O\alpha$ will lie in a plane,
„ OC', OA', $O\beta$ „ „ ,
„ OA', OB', $O\gamma$ „ „ ;

viz. this will be the case when the point O is situate in the cubic surface represented by the last-mentioned equation; this completes the demonstration of the solid theorems.

It is clear that considering five points 1, 2, 3, 4, 5 in a plane, then, since any one of these may be taken for the point O' of the foregoing theorem, the theorem exhibited in the first instance as a theorem relating to a triangle and two points, and afterwards as a theorem relating to a quadrangle and a point, is really a theorem relating to five points in a plane. There are, of course, five different systems of points (P, Q, R, S), corresponding to the different combinations of four out of the five points.

Cambridge, March 6, 1865.