371.

ON A FORMULA FOR THE INTERSECTIONS OF A LINE AND CONIC, AND ON AN INTEGRAL FORMULA CONNECTED THEREWITH.

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In a letter to me, dated 15 May, 1862, Mr Spottiswoode has extracted from an unpublished Memoir, and he has kindly permitted me to communicate, the following formula for the points of intersection of a line and conic; viz. if the equations of the line and conic are

$$\xi x+\eta y+\zeta z=0,$$

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2=0\,;$$

and if

$$\theta^2=\begin{vmatrix} & \xi, & \eta, & \zeta\ , \\ \xi, & a, & h, & g \\ \eta, & h, & b, & f \\ \zeta, & g, & f, & c \end{vmatrix}$$

or, what is the same thing, if

$$\theta^2=-(A,\ B,\ C,\ F,\ G,\ H \between \xi,\ \eta,\ \zeta)^2,$$

where $A=bc-f^2$, &c. as usual; then the coordinates (x, y, z) of a point of intersection of the line and conic are found from the linear equations

$$\begin{aligned}
&(g\eta-h\zeta-\theta)\,x+(f\eta-b\zeta\qquad)\,y+(c\eta-f\zeta\qquad)\,z=0,\\
&(a\zeta-g\xi\qquad)\,x+(h\zeta-f\xi-\theta)\,y+(g\zeta-c\xi\qquad)\,z=0,\\
&(h\xi-a\eta\qquad)\,x+(b\xi-h\eta\qquad)\,y+(f\xi-g\eta-\theta)\,z=0,
\end{aligned}$$

equivalent of course to two equations, and giving by the elimination of (x, y, z), the equation

$$\theta\,[-(A, \ldots \between \xi, \eta, \zeta)^2 - \theta^2] = 0,$$

that is, giving for θ the foregoing value. And the linear equations then give

$$\begin{aligned} &x &&: y &&: z,\\ &= \xi\frac{d\theta}{d\xi} + g\eta - h\zeta + \theta &&: \xi\frac{d\theta}{d\eta} + a\zeta - g\xi &&: \xi\frac{d\theta}{d\zeta} + h\xi - a\eta,\\ &= \eta\frac{d\theta}{d\xi} + f\zeta - b\xi &&: \eta\frac{d\theta}{d\eta} + h\xi - f\eta + \theta &&: \eta\frac{d\theta}{d\zeta} + b\eta - h\zeta,\\ &= \zeta\frac{d\theta}{d\xi} + c\xi - f\eta &&: \zeta\frac{d\theta}{d\eta} + g\eta - c\zeta &&: \zeta\frac{d\theta}{d\zeta} + f\zeta - g\xi + \theta, \end{aligned}$$

where obviously

$$-\theta\frac{d\theta}{d\xi} = A\xi + H\eta + G\zeta,\quad -\theta\frac{d\theta}{d\eta} = H\xi + B\eta + F\zeta,\quad -\theta\frac{d\theta}{d\zeta} = G\xi + F\eta + C\zeta.$$

By changing the sign of θ, we have of course the coordinates of the other point of intersection. The formulæ which, singularly enough, have since been given incidentally by M. Aronhold [1], may be easily obtained as follows.

Writing for shortness

$$\begin{aligned} P &= ax + hy + gz,\\ Q &= hx + by + fz,\\ R &= gx + fy + cz, \end{aligned}$$

then the equation of the conic gives

$$Px + Qy + Rz = 0,$$

and combining with this the equation

$$\xi x + \eta y + \zeta z = 0,$$

we have

$$x : y : z = Q\zeta - R\eta : R\xi - P\zeta : P\eta - Q\xi,$$

or what is the same thing, taking an indeterminate multiplier θ,

$$\begin{aligned} -\theta x + R\eta - Q\zeta &= 0,\\ -\theta y + P\zeta - R\xi &= 0,\\ -\theta z + Q\xi - P\eta &= 0, \end{aligned}$$

[1] In his interesting Memoir "Ueber eine neue algebraische Behandlungsweise der Integrale irrationaler Differentiale von der Form $\Pi(x, y)\,dx$, in welcher $\Pi(x, y)$ eine beliebige rationale Function ist, und zwischen x und y eine allgemeine Gleichung zweiter Ordnung besteht." *Crelle*, t. LXI. (1862).

which are, in fact, Mr Spottiswoode's linear equations, and which lead, as before, to the value

$$\theta^2 = -(A, \ldots \between \xi, \eta, \zeta)^2.$$

But this value of θ is obtained in a different manner by expressing (x, y, z) as linear functions of P, Q, R; viz. putting as usual $K = abc - af^2 - bg^2 - ch^2 + 2fgh$, the linear equations thus become

$$AP + HQ + GR + \frac{K}{\theta}(\zeta Q - \eta R) = 0,$$

$$HP + BQ + FR + \frac{K}{\theta}(\xi R - \zeta P) = 0,$$

$$GP + FQ + CR + \frac{K}{\theta}(\eta P - \xi Q) = 0,$$

or eliminating (P, Q, R), we have

$$\begin{vmatrix} A & , & H + \frac{K\zeta}{\theta}, & G - \frac{K\eta}{\theta} \\ H - \frac{K\zeta}{\theta}, & B & , & F + \frac{K\xi}{\theta} \\ G + \frac{K\eta}{\theta}, & F - \frac{K\xi}{\theta}, & C & \end{vmatrix} = 0,$$

that is

$$ABC - A\left(F^2 - \frac{K^2\xi^2}{\theta^2}\right) - B\left(G^2 - \frac{K^2\eta^2}{\theta^2}\right) - C\left(H^2 - \frac{K^2\zeta^2}{\theta^2}\right)$$

$$+ \left(F + \frac{K\xi}{\theta}\right)\left(G + \frac{K\eta}{\theta}\right)\left(H + \frac{K\zeta}{\theta}\right)$$

$$+ \left(F - \frac{K\xi}{\theta}\right)\left(G - \frac{K\eta}{\theta}\right)\left(H - \frac{K\zeta}{\theta}\right) = 0;$$

or, reducing,

$$ABC - AF^2 - BG^2 - CH^2 + 2FGH + \frac{K^2}{\theta^2}(A, \ldots \between \xi, \eta, \zeta)^2 = 0,$$

that is

$$\theta^2 + (A, \ldots \between \xi, \eta, \zeta)^2 = 0,$$

as before.

I reproduce, as follows, a fundamental formula of Aronhold's Memoir. Consider the function

$$\varpi = \frac{1}{\sqrt{\{-(A, \ldots \between u, v, w)^2\}}} \log \frac{(a, \ldots \between x_1, y_1, z_1 \between x, y, z)}{ux + vy + wz},$$

where x_1, y_1, z_1 (corresponding to x, y, z in the former part of this paper) are determined by the conditions

$$(a, \ldots \between x_1, y_1, z_1)^2 = 0,$$

$$ux_1 + vy_1 + wz_1 = 0,$$

so that putting

$$\theta = \sqrt{\{-(A, \ldots \between u, v, w)^2\}},$$

and

$$P_1 = ax_1 + hy_1 + gz_1,$$
$$Q_1 = hx_1 + by_1 + fz_1,$$
$$R_1 = gx_1 + fy_1 + cz_1,$$

we now have

$$-\theta x_1 + R_1 v - Q_1 w = 0,$$
$$-\theta y_1 + P_1 w - R_1 u = 0,$$
$$-\theta z_1 + Q_1 u - P_1 v = 0,$$

and the value of ϖ is

$$\varpi = \frac{1}{\theta} \log \frac{P_1 x + Q_1 y + R_1 z}{ux + vy + wz}.$$

Treating (x, y, z) as independent variables and differentiating, we have

$$d\varpi = \frac{1}{\theta}\left\{\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 x + Q_1 y + R_1 z} - \frac{udx + vdy + wdz}{ux + vy + wz}\right\}$$

$$= \frac{1}{\theta}\frac{(xdy - ydx)(Q_1 u - P_1 v) + (ydz - zdy)(R_1 v - Q_1 w) + (zdx - xdz)(P_1 w - R_1 u)}{(P_1 x + Q_1 y + R_1 z)(ux + vy + wz)}$$

$$= \frac{x_1(ydz - zdy) + y_1(zdx - xdz) + z_1(xdy - ydx)}{(P_1 x + Q_1 y + R_1 z)(ux + vy + wz)},$$

or, what is the same thing, if

$$P = ax + hy + gz,$$
$$Q = hx + by + fz,$$
$$R = gx + fy + cz,$$

so that

$$P_1 x + Q_1 y + R_1 z = (a, \ldots \between x, y, z \between x_1, y_1, z_1) = Px_1 + Qy_1 + Rz_1,$$

then we have

$$d\varpi = \frac{(ydz - zdy)\, x_1 + (zdx - xdz)\, y_1 + (xdy - ydx)\, z_1}{(ux + vy + wz)(Px_1 + Qy_1 + Rz_1)}.$$

Suppose now that (x, y, z) are connected by the equation

$$(a, \ldots \between x, y, z)^2 = 0,$$

we have

$$P\ \ x + Q\ \ y + R\ \ z = 0,$$
$$Pdx + Qdy + Rdz = 0,$$

and thence

$$ydz - zdy = \Theta P,$$
$$zdx - xdz = \Theta Q,$$
$$xdy - ydx = \Theta R,$$

and consequently

$$d\varpi = \frac{\Theta}{ux+vy+wz} = \frac{ydz - zdy}{(ux+vy+wz)\,P} = \frac{zdx - xdz}{(ux+vy+wz)\,Q} = \frac{xdy - ydx}{(ux+vy+wz)\,R};$$

or selecting the value

$$d\varpi = \frac{zdx - xdz}{(ux+vy+wz)\,Q} = \frac{zdx - xdz}{(ux+vy+wz)(hx+by+fz)},$$

and writing

$$\frac{x}{z} = X, \quad \frac{y}{z} = Y,$$

we have

$$d\varpi = \frac{z^2 dX}{(ux+vy+wz)(hx+by+fz)}$$
$$= \frac{dX}{(uX+vY+w)(hX+bY+f)},$$

where X and Y are connected by the equation

$$(a, \ldots \between X, Y, 1)^2 = 0,$$

that is, Y is a given quadric radical function of X. Hence integrating and restoring for ϖ its original value, but writing therein $\frac{x}{z} = X$ and $\frac{y}{z} = Y$, we have

$$\int \frac{dX}{(uX+vY+w)(hX+bY+f)} = \frac{1}{\sqrt{\{-(A, \ldots \between u, v, w)^2\}}} \log \frac{(a, \ldots \between x_1, y_1, z_1 \between X, Y, 1)}{uX+vY+w},$$

where, as just mentioned, Y is a given quadric radical function of X determined by the equation

$$(a, b, c, f, g, h \between X, Y, 1)^2 = 0,$$

and the constants x_1, y_1, z_1 are such that

$$(a, \ldots \between x_1, y_1, z_1)^2 = 0,$$
$$ux_1 + vy_1 + wz_1 = 0,$$

the ratios of these quantities being therefore determinate; there would, it is clear, be no loss of generality in assuming $z_1 = 1$. This is Aronhold's Theorem I.

2, *Stone Buildings, W.C., October* 23, 1862.