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Micromechanics of discrete systems

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It is the main purpose of this paper to develop a general theory of stochastic deformations of discrete media. From the onset of the analysis the discrete arrangement of individual elements forming the macroscopic body as well as their short range interaction effects are taken into account. The analysis is based on the mathematical theory of probability and employs functional analysis in the representation of the deformational behaviour of discrete systems. The analysis further uses an operator formalism in order to represent the material characteristics during the steady-state and transient behaviour of the material.

Większość materiałów konstrukcyjnych wykazuje pewną mikrostrukturę, ich zachowanie badać można również za pomocą probabilistycznej mikromechaniki układów dyskretnych. Głównym celem tej pracy jest krótkie omówienie podstawowych założeń takiego podejścia, jak również sformułowanie ogólnej teorii odkształcenia układów dyskretnych w ramach mikromechaniki.

Большинство конструкционных материалов обладает некоторой микроструктурой. Их поведние можно исследовать тоже при помощи пробабилистической микромеханики дискретных систем. Главной целью данней работы является краткое обсуждение основных предположений такого подхода, как тоже формулировка общей теории деформации дискретных систем в рамках микромеханики.

1. Introduction

Most engineering materials have a distinct microstructure and hence their deformational behaviour should be analyzed with the inclusion of the effects of such microstructures. Although there exists a large class of structured materials encompassing those employed in biomechanics, the present paper will consider only two groups of materials, which are important representatives of discrete systems, e.g. crystalline solids and fibrous materials. Whilst it appears that these types of materials are largely different in nature, it is this author's belief that a general deformation theory can be developed that is applicable to both these systems.

It is the main purpose of this paper to show the development of such a theory within the framework of "probabilistic micromechanics" [1]. Hence from the onset of the analysis the discrete arrangement of individual elements forming the macroscopic material body as well as the short range interaction effects between these elements are taken into account. The analysis is based on the mathematical theory of probability and statistical micromechanics. Furthermore, the theory employs "material operators", that include specific properties of a given material and replace the conventional constitutive relations of continuum mechanics. However, this approach necessitates the introduction of "functional analysis" in the study of the deformational behaviour of discrete systems. For this purpose, the deformations of a discrete system occurring in the physical domain are represented by an "abstract dynamical system" generally characterized by a quadruplet $[X, \mathcal{F}, \mathcal{P}, \mathbf{Q}]$ where X is a probabilistic function space with an appropriate topological structure, $\mathcal{F}$ the σ-algebra of Borel sets defined later, $\mathcal{P}$ a probability measure and $\mathbf{Q}$ a transition function expressing the transition of an individual element of the microstructure or an ensemble of such elements, from one state to another during a general deformation process. The latter, for the completely reversible and steady-state response of the material, is seen as a Markov process. However, for the intermediary state or the transient behaviour of the material, it will be shown later that the process is of the non-Markovian type.

In general, direct notation will be used in the presentation, but when necessary, general tensor notation will also be used. Bold face symbols represent vectors or tensors of the appropriate rank.

2. Basic concepts of the micromechanics theory

In order to develop a general deformation theory of discrete systems, it is necessary to review briefly the basic concepts on which this theory is founded (see also references [1–3]).

2.1. Basic concepts

The first concept in probabilistic micromechanics is concerned with the use of three measuring scales. Thus the smallest scale refers to a "microelement" of the discrete system, which is given a distinct volume ${}^{\alpha}v$ and surface ${}^{\alpha}s$. The next scale is an intermediary one, called a "mesodomain" and contains a statistical ensemble of microelements ($\alpha = 1 \ldots N$; N large). Finally, the macroscopic material body is formed by a denumerable number of non-intersecting mesodomains such that:

(2.1)
$$\begin{gathered} {}^{M}V \cap {}^{L}V = \phi, \quad L \neq M, \\ V = \bigcup_{M=1}^{P} {}^{M}V, \quad (M = 1 \ldots P), \quad V \gg {}^{M}V \gg {}^{\alpha}v, \end{gathered}$$

in which V is the volume of the macroscopic material body. The second concept postulates that all microscopic field quantities, whether they are of a geometrical or physical nature, are random variables or stochastic functions of such variables. These quantities will be more specifically defined in subsequent sections. The third fundamental concept of this theory is concerned with a generalization of the stress principle so as to permit the inclusion of surface tractions caused by boundary effects between contiguous microelements. Further, a generalization is required in the deformation analysis, since a significant parameter in the theory is the "relative displacement vector ${}^{\alpha\beta}\mathbf{d}$" associated with the displacements of two adjacent microelements α, β. It is to be noted that the approach taken by using generalized quantities permits to deal with seemingly different types of discrete systems in terms of the same formalism. Although the systems have different properties, the common feature of the occurring overall deformations is the response of individual elements and the additional effects associated with the existing binding potentials between the elements. Finally, the fourth concept is that of "material operators" as mentioned previously. These operators form the link between stress and deformation and are of a special form depending on the characteristics of a discrete system under consideration.

2.2. Kinematics of discrete systems

In continuum mechanics, the motion of a material point is usually described by a vector $\mathbf{u}(\mathbf{X}, t)$ as a function of the position vector $\mathbf{X}$ from its centre of mass in the undeformed configuration of the material body and time. In accordance with the concepts given above, in probabilistic micromechanics, two random vectors ${}^{\alpha}\mathbf{w}$ and ${}^{\alpha}\boldsymbol{\mu}$ ($\alpha = 1 \dots N$) are used to represent the motion of a microelement. Thus the first random vector ${}^{\alpha}\mathbf{w}$ refers to the motion of an arbitrary point within the element, whilst the second random vector ${}^{\alpha}\boldsymbol{\mu}$ describes the motion of a surface point at the interface of two adjacent microelements. It becomes necessary in this description of the kinematics of deformation to use an external fixed reference frame as well as an internal body frame that is attached to the centre of mass of the individual element. Following the more detailed study of deformation kinematics of discrete systems as given in Ref. [3], the total deformation of an element can be expressed in general by the following random vector function:

$$ {}^{\alpha}\mathbf{u}({}^{\alpha}\mathbf{X}, t) = f\{{}^{\alpha}\mathbf{w}({}^{\alpha}\mathbf{X}, t); {}^{\alpha}\boldsymbol{\mu}({}^{\alpha}\mathbf{X}, t)\}, \tag{2.2} $$

or, equivalently, by using the relative displacement vector ${}^{\alpha\beta}\mathbf{d}$ introduced in the preceding paragraph:

$$ {}^{\alpha}\mathbf{u}({}^{\alpha}\mathbf{X}, t) = g\{{}^{\alpha}\mathbf{w}; {}^{\alpha\beta}\mathbf{d}\}. \tag{2.3} $$

The form of the functions f, g in (2.2) and (2.3) have been investigated in detail in references [4, 5] for the elastic response of crystalline solids and fibrous structures, respectively. It has also been shown in Refs. [6] concerned with the mechanical relaxation of crystalline solids, that the vector ${}^{\alpha}\mathbf{u}$ can be considered in terms of a linear function of ${}^{\alpha}\mathbf{w}$ and ${}^{\alpha\beta}\mathbf{d}$ such that:

$$ {}^{\alpha}\mathbf{u} = H_1(|\cdot|)\,{}^{\alpha}\mathbf{w} + H_2(|\cdot|)\,{}^{\alpha\beta}\mathbf{d}, \tag{2.4} $$

in which $H_1(|\cdot|)$ and $H_2(|\cdot|)$ are the Heaviside functions. The arguments of these functions indicated in (2.4) by $(|\cdot|)$ contain the absolute value of the distance vector of the centre of mass and that of an arbitrary point within the element with respect to the external frame for the deformed configuration, as well as the surface coordinates of a point at the interface and the corresponding orientations with respect to the same external frame. In the derivation of this relation, it has been implicitly assumed that the random vectors ${}^{\alpha}\mathbf{w}$ and ${}^{\alpha\beta}\mathbf{d}$ are at least piecewise continuous functions of ${}^{\alpha}\mathbf{X}(t)$ within a compact support, e.g. that the occurring deformations are locally continuous. On the basis of this assumption, it is possible to define a "microdeformation gradient" and the conventional forms of Lagrangian and Eulerian strains. However, one obtains two infinitesimal strain definitions, i.e. one which is valid for the domain within a microelement and another that can be used for the representation of the strain field in the boundary zone between elements. It is to be noted that in such a formulation, the gradient operators of ${}^{\alpha}\mathbf{w}$ and ${}^{\alpha\beta}\mathbf{d}$ must be taken with respect to the internal body frame and the surface coordinate frame, respectively. Since the present analysis is concerned with a more general formulation of the micromechanics of discrete systems, it is possible without changing any of the physical characteristics of such systems to introduce generalized quantities. Thus, one can define a generalized microstrain for a microelement as follows:

$$ {}^{\alpha}\mathscr{E} = f(\nabla^{\alpha}\mathbf{u}) \tag{2.5} $$

and in an analogous form to that of the microdeformation (2.4) one can also write the microstrain as:

$$ {}^{\alpha}\mathscr{E} = H_1(|\cdot|)\nabla^{\alpha}\mathbf{w}+H_2(|\cdot|)\nabla^{\alpha\beta}\mathbf{d}, \tag{2.6} $$

where the arguments of the Heaviside functions H_1, H_2 are the same as before and the gradient operators must be taken in the sense mentioned above. Although generalized strains have been defined in (2.5) and (2.6), it is convenient for the subsequent analysis to consider generalized stresses and deformations, in particular, since the latter are experimentally accessible in terms of their corresponding distribution functions.

3. General deformation theory of discrete systems

It has been stated previously that for the description of the overall response behaviour of a discrete system, a general deformation theory is required, which is developed in terms of an abstract dynamical system to which the notion of continuity can be applied. For this purpose, some topological aspects are considered below.

3.1. Topological considerations

The mechanical states of individual microelements of the discrete system can be described by a set of r-dimensional state vectors ${}^{\alpha}\mathbf{v}(t)$: ${}^{\alpha}v_i(t)$, $\alpha = 1 \dots N$; $i = 1 \dots r$, $\alpha \in Z^+$. These vectors will be in general stochastic functions of random geometric and thermomechanical parameters that characterize the microstructure of the material. The "state space" formed by the vectors is identified in the present theory with the "probabilistic function space X". Since the material body is regarded as a collection of a finite number of mesodomains ($M = 1 \dots P$), each of which contains a statistical ensemble of microelements ($\alpha = 1 \dots N$), two other sets of state vectors can be written as follows:

$$ \begin{aligned} {}^{M}X &= \{{}^{\alpha}\mathbf{v}(t);\ \alpha = 1 \dots N\}, \\ X &= \{{}^{M}X;\ M = 1 \dots P\}. \end{aligned} \tag{3.1} $$

Whilst it is possible to use the general space X and the abstract dynamical system $[X, \mathscr{F}, \mathscr{P}]$ in the sense of Jancel [7], it is more convenient for the subsequent analysis to deal with subspaces of this general state space. Thus two subspaces are of immediate significance, e.g. the "stress space" and the "deformation space". It is to be noted that the "material operator" mentioned earlier connects these spaces under the conditions and restrictions imposed on the operator by the requirements of a proper mapping between the spaces.

As an example of the application of subspaces by considering the kinematics of deformations only, one can assign a finite dimensional subspace Ω such that ${}^{\alpha}\boldsymbol{\omega}: {}^{\alpha}\omega_i$ $(i = 1 \dots r) \in \Omega$. Thus the set ${}^{\alpha}\boldsymbol{\omega}$ can be associated with the change of kinematic parameters related to the undeformed and deformed configurations of the discrete system. It must be emphasized however, that due to experimental constraints, the kinematic state vector ${}^{\alpha}\boldsymbol{\omega}$ can only be specified within a certain range in which observations can be made with a certain accuracy. Thus a particular range can be represented by an open set or sphere such that:

$$ {}^{\ni}E = \{ {}^{\ni}\omega_i < {}^{\alpha}\omega_i < {}^{\ni}\omega_i + \Delta {}^{\ni}\omega_i;\ i = 1 \dots r \}, \tag{3.2} $$

where ${}^{\ni}E \in \Omega$, the superscript $\ni$ designates this range, $\Delta {}^{\ni}\boldsymbol{\omega}$ the accuracy with which observations can be made and $\ni \in Z^+$ is the set of all integers. Hence, it is possible to define a class $\mathscr{F}$ of these spheres ${}^{\ni}E \in \mathscr{F}$ with the following properties:

$$ \begin{aligned} &\text{(i)} \quad {}^{\ni}E, {}^{\text{я}}E \in \mathscr{F} \Rightarrow {}^{\text{я}}E \bigcap {}^{\ni}E \in \mathscr{F}, \\ &\text{(ii)} \quad {}^{\ni}E \Delta {}^{\text{я}}E \in \mathscr{F}, \\ &\text{(iii)} \quad {}^{\ni}E(\ni \in Z^+) \Rightarrow \bigcup_{\ni=0}^{\infty} {}^{\ni}E \in \mathscr{F} \Leftrightarrow \bigcap_{\ni=0}^{\infty} {}^{\ni}E \in \mathscr{F}, \\ &\text{(iv)} \quad \Omega \in \mathscr{F}, \end{aligned} \tag{3.3} $$

which indicate that the class $\mathscr{F}$ forms a σ-algebra in the subspace Ω. The latter together with $\mathscr{F}$ is called a "measurable space" in the sense of HALMOS [8] and is written as $[\Omega, \mathscr{F}]$. One can further describe the subspace Ω in terms of the triplet $[\Omega, \mathscr{F}, \mathscr{P}]$ by using an appropriate probability measure given by:

$$ \mathscr{P}\{ {}^{\alpha}\boldsymbol{\omega} \in {}^{\ni}E \} = \mathscr{P}\{ {}^{\ni}\omega_i < {}^{\alpha}\omega_i < {}^{\ni}\omega_i + \Delta {}^{\ni}\omega_i \}. \tag{3.4} $$

More specifically, if a mapping of $\mathscr{F}$ into itself is considered, and on the assumption that each element ${}^{\alpha}\omega_i \in {}^{\ni}E$ has equal probability to be in the set ${}^{\ni}E \in \mathscr{F}$, whereby $\mathscr{F}$ has the properties given in (3.3), then the probability measure for the discrete system can be written as:

$$ \mathscr{P}\{ {}^{\alpha}\boldsymbol{\omega} \in {}^{\ni}E \} \, [\equiv] \, \mathscr{P}\{ {}^{\alpha}\boldsymbol{\omega} : {}^{\alpha}\omega_i = {}^{\ni}\eta_i \} \tag{3.5} $$

in which ${}^{\ni}\eta_i$ refers to a measurable value of ${}^{\alpha}\omega_i$ in the set ${}^{\ni}E$.

As pointed out earlier by considering subspaces of the more general function space generated by the state vectors ${}^{\alpha}\mathbf{v}(t)$, the latter may be regarded to be formed essentially by:

$$ {}^{\alpha}\mathbf{v} = \begin{pmatrix} {}^{\alpha}\boldsymbol{\omega} \\ {}^{\alpha}\boldsymbol{\sigma} \end{pmatrix} \tag{3.6} $$

in which ${}^{\alpha}\boldsymbol{\omega}$ is the deformation and ${}^{\alpha}\boldsymbol{\sigma}$ the corresponding stress vector. The vector ${}^{\alpha}\boldsymbol{\omega}$ itself can be considered to be composed of:

$$ {}^{\alpha}\boldsymbol{\omega} = \begin{bmatrix} {}^{\alpha}\mathbf{w} \\ {}^{\alpha}\boldsymbol{\epsilon} \\ {}^{\alpha\beta}\mathbf{d} \\ {}^{\alpha\beta}\boldsymbol{\epsilon} \end{bmatrix}, \tag{3.7} $$

where the quantities ${}^{\alpha}\mathbf{w}$ and ${}^{\alpha\beta}\mathbf{d}$ have been defined previously ${}^{\alpha}\boldsymbol{\epsilon}$ and ${}^{\alpha\beta}\boldsymbol{\epsilon}$ designate infinitesimal strains in the microelement and within the boundary zone between structural elements (see also Refs. [1, 3]). Designating the subspaces belonging to the general deformation space, i.e. ${}^{\alpha}\mathbf{w} \in {}^{\alpha}U$ and ${}^{\alpha\beta}\mathbf{d} \in {}^{\alpha\beta}U$, then:

(3.8)
$$ {}^{\alpha}U \bigcup {}^{\alpha\beta}U = U \subset \Omega. $$

In accordance with the statements given in (3.2), (3.3) and (3.4), the open sets of ${}^{\alpha}U$ and ${}^{\alpha\beta}U$ can be expressed as follows:

(3.9)
$$ \begin{aligned} {}^{\alpha}E &= \{{}^{\ni}w_i < {}^{\alpha}w_i < {}^{\ni}w_i + \Delta{}^{\ni}w_i\}, \\ {}^{\alpha\beta}E &= \{{}^{\ni}d_i < {}^{\alpha\beta}d_i < {}^{\ni}d_i + \Delta{}^{\ni}d_i\}. \end{aligned} $$

Hence, two classes of open sets can be defined, namely $\mathfrak{F}^{\alpha_{\mathbf{w}}}$ and $\mathfrak{F}^{\alpha\beta_{\mathbf{d}}}$, which form σ-algebras in the subspaces ${}^{\alpha}U$ and ${}^{\alpha\beta}U$, respectively. Based on these definitions, the probability measures associated with these sets will be:

(3.10)
$$ \mathscr{P}^{\alpha_{\mathbf{w}}} = \mathrm{Prob}\{{}^{\alpha}E\}, \qquad \mathscr{P}^{\alpha\beta_{\mathbf{d}}} = \mathrm{Prob}\{{}^{\alpha\beta}E\}. $$

Thus the deformation subspaces of the more general deformation space $[\Omega, \mathfrak{F}^{\omega}, \mathscr{P}^{\omega}]$ can be represented in the following manner:

(3.11)
$$ [{}^{\alpha}U, \mathfrak{F}^{\alpha_{\mathbf{w}}}, \mathscr{P}^{\alpha_{\mathbf{w}}}]; \ [{}^{\alpha\beta}U, \mathfrak{F}^{\alpha\beta_{\mathbf{d}}}, \mathscr{P}^{\alpha\beta_{\mathbf{d}}}] $$

and in a completely analogous manner the space $\Xi \subset X$ in terms of the stress subspaces can be characterized by:

(3.12)
$$ [{}^{\alpha}\Xi, \mathfrak{F}^{\alpha_{\xi}}, \mathscr{P}^{\alpha_{\xi}}]; \ [{}^{\alpha\beta}\Xi, \mathfrak{F}^{\alpha\beta_{\xi}}, \mathscr{P}^{\alpha\beta_{\xi}}]. $$

Since a subspace of a measurable space is also measurable (see HALMOS [8]), it may be concluded that the general deformation space as well as the subspace ${}^{\alpha}U, {}^{\alpha\beta}U$ are all measurable.

It is evident that the general deformation space $U \subset \Omega$ can be considered as a normed vector space, if it is complete, e.g. if every Cauchy sequence has a limit:

(3.13)
$$ ||u_m - u_n|| \to 0 \quad \text{as} \quad m, n \to \infty. $$

Thus, if this space is designated by the set of sequences of real or complex numbers u_j ($j = 1, \ldots, \infty$) with norm:

(3.14)
$$ ||\mathbf{u}|| = \left[\sum |u_i(X)|^p\right]^{1/p} \quad \text{for} \quad \forall p \geqslant 1. $$

It will be a normed vector space. However, by definition a normed vector space is considered as such if there is a non-negative function $||\cdot||$ such that:

$$ ||\mathbf{u}|| \geqslant 0, $$

and equal to zero if, and only if, $\mathbf{u} = 0$. It is apparent that this condition by the norm given in (3.14) is not satisfied. Hence, it follows that the subspaces ${}^{\alpha}U$ and ${}^{\alpha\beta}U$ must be specified in terms of "semi-norms". Spaces thus defined are a set of locally convex spaces. In this case every open set containing o has a convex open set also containing o. The topological structure of such a space is then given by a set of semi-norms. In general, a semi-norm p on a vector space x is a non-negative real function on X such that

(3.15)
$$ \begin{aligned} p(x+y) &\leqslant p(x)+p(y) \quad \text{for} \quad \forall x, y \in X, \\ p(ax) &= |a|p(x) \quad \text{for} \quad \forall a \in R \text{ and } x \in X. \end{aligned} $$

In particular, considering ${}^{\alpha}U$ to be the space of all continuous and bounded functions ${}^{\alpha}\mathbf{u}$ defined on a compact subset $c_n \in U$, $n = 1, 2, \dots$, U is known as a Frechet space with a topology induced by the semi-norm

$$(3.16) \qquad ||\mathbf{u}|| = \sup_{c_n} \left[\sum |u_i|^p\right]^{1/p}.$$

It is possible to define other semi-norms and to construct the corresponding function spaces (see for instance, BOURBAKI [9], TREVES [10] and YOSHIDA [11]). However, due to the probabilistic concepts used in the present theory, it is of interest to define the semi-norms in terms of the probability measures in the deformation space of all $p_{\mathbf{u}}$-regular measurable and bounded functions as follows:

$$(3.17) \qquad \langle \mathbf{u} \rangle = E\{\mathbf{u}\} = \sum {}^{\ni}\mathbf{u} p^{\mathbf{u}}({}^{\ni}\mathbf{u})$$

and

$$||\mathbf{u}|| = D\{\mathbf{u}\} = \left\{\sum [{}^{\ni}\mathbf{u} - \langle \mathbf{u} \rangle]^2 p^{\mathbf{u}}({}^{\ni}\mathbf{u})\right\}^{1/2},$$

in which $\langle \mathbf{u} \rangle$ or $E\{\mathbf{u}\}$ is the "expected value" of ${}^{\alpha}\mathbf{u}$ and $D\{\mathbf{u}\}$ the "standard deviation" (see also KAPPOS [12]).

3.2. Material operators

It is well known from statistical mechanics of continuous systems [13], when they are regarded as conservative ones, that they can be represented by a set of hyperbolic differential equations associated with a group of transformations T_t on the general state space (in the domain $-\infty < t < \infty$). Discrete systems however, lead to a set of differential equations of the parabolic type, that induce a semi-group of transformations T_t defined for $t \geqslant 0$ on the corresponding function space. Furthermore, since in the present theory the general deformation process is seen as a Markov process, the representation of the discrete system leads to the Chapman-Kolmogorov functional relations which connect the stochastic theory with that of semi-groups. Apart from the statistical point of view, it is equally necessary to establish a link between the induced stress field and the occurring deformation field by means of the concept of "material operators".

Before dealing with such operators that connect the stress and deformation fields in a particular mesodomain of the material body, the "transform operators" for individual elements must be defined first. Thus, denoting by ${}^{\alpha}\boldsymbol{\xi}(t)$ the "microstress" acting on an element and by $\hat{\mathbf{F}}({}^{\alpha\beta}\hat{\mathbf{d}}, t)$ a stochastic surface interaction force, which is assumed to act at discrete points at the interface between two adjacent elements, a generalized interaction force in the sense of YVON [14] can be written as follows:

$$(3.18) \qquad {}^{\alpha\beta}\boldsymbol{\tau}({}^{\alpha\beta}\mathbf{d}, t) = \langle \delta({}^{\alpha\beta}\mathbf{d} - {}^{\alpha\beta}\hat{\mathbf{d}}),\ \hat{\mathbf{F}}({}^{\alpha\beta}\hat{\mathbf{d}}, t)\rangle,$$

where δ is the three-dimensional Dirac-delta function, which is equal to unity, if ${}^{\alpha\beta}\hat{\mathbf{d}} = {}^{\alpha\beta}\mathbf{d}$ and equal to zero if ${}^{\alpha\beta}\hat{\mathbf{d}} \neq {}^{\alpha\beta}\mathbf{d}$. In terms of the above generalized force, the microstress acting at the boundary between the elements α, β, can be expressed by:

$$(3.19) \qquad {}^{\alpha\beta}\boldsymbol{\xi}(t) = {}^{\alpha\beta}\boldsymbol{\tau}({}^{\alpha\beta}\mathbf{d}, t)^{\alpha}\mathbf{n},$$

where ${}^{\alpha}\mathbf{n}$ is the unit normal vector in the outward direction to the interface between α, β and the quantity on the right-hand side of (3.19) is a dyadic product. The micro-stresses ${}^{\alpha}\boldsymbol{\xi}$ and ${}^{\alpha\beta}\boldsymbol{\xi}$ form ${}^{\alpha}\boldsymbol{\sigma}$ in relation (3.6). Together with these stresses, the deformation ${}^{\alpha}\mathbf{w}$, ${}^{\alpha\beta}\mathbf{d}$ and strains ${}^{\alpha}\boldsymbol{\epsilon}$, ${}^{\alpha\beta}\boldsymbol{\epsilon}$ forming ${}^{\alpha}\boldsymbol{\omega}$ in (3.7) represent the basic parameters during the deformation of a discrete system.

Using a system theory point of view, one can write operational relations between deformations and stresses for a microelement as follows:

$$(3.20)\qquad {}^{\alpha}\mathbf{u}(t) = \{H_1(|\cdot|)-H_2(|\cdot|)\}^{\alpha}\mathbf{A}(t)^{\alpha}\boldsymbol{\xi}(t)+\{H_3(|\cdot|)-H_4(|\cdot|)\}^{\alpha\beta}\mathbf{B}(t)^{\alpha\beta}\boldsymbol{\tau}^{\alpha}\mathbf{n}$$

in which the arguments of the Heaviside functions contain the same kinematic parameters as discussed in section (2.2). The operators $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are the "transform" and "interaction" operators, respectively, associated with a structural element [4]. For a particular mesodomain "material operators" are used analogously as above, that contain in their argument operators ${}^{M}a$, ${}^{M}B$ obtainable from $\mathbf{A}(t)$, $\mathbf{B}(t)$ by means of the relevant distribution functions over the mesodomain. The relation between mesoscopic stress and deformations can then be expressed by:

$$(3.21)\qquad {}^{M}\mathbf{u}(t) = {}^{M}\mathfrak{m}(t)^{M}\boldsymbol{\sigma}(t),$$

where

$${}^{M}\mathfrak{m}(t) = {}^{M}\mathfrak{m}({}^{M}A, {}^{M}B).$$

The role of the material operator ${}^{M}\mathfrak{m}$ will be further discussed in Sect. 4 of this paper.

3.3. Random theory of deformation

(i) In view of the statements made in the foregoing sections (3.1) and (3.2), it is now possible to outline the approach to a general deformation theory of discrete systems. Thus, considering a set of function spaces $[U, \mathfrak{F}^u, \mathscr{P}^u]$ representing the deformation space, each of these spaces will correspond to a point on the real line R^+. In particular, if $R^+ = [0, \infty)$ in which an element of $t \in R^+$, the occurring deformations at this instant of time t can be regarded as a random variable in $[U, \mathfrak{F}^u, \mathscr{P}^u]$. Following HALMOS [8], one can consider an infinite number of such spaces and construct an infinite product space so that the random process of deformations will be characterized by a measurable function $\mathbf{u}(\mathbf{X}, t)$ or briefly $\mathbf{u}_t(\mathbf{X})$. Alternatively, one may consider the function space $[U, \mathfrak{F}^u, \mathscr{P}^u]$ and a one-parameter family of mappings T_t such that:

$$(3.22)\qquad T_t\colon U \to U \quad \text{for} \quad \forall t \in R^+\colon \mathbf{u}_t(\mathbf{X}) \in U.$$

Hence, the stochastic function $\mathbf{u}_t(\mathbf{X})$ defines a general random deformation process. As mentioned earlier in such a general process, one can distinguish between the completely reversible, transient and steady-state behaviour of the system. Considering first the reversible or elastic range of the material undergoing a random deformation, the process is known to be strictly stationary, if the probability measures on all Borel sets $E_r \in \mathfrak{F}^u$ are equal, i.e.:

$$(3.23)\qquad \mathscr{P}^u\{E_{r+1}\} = \mathscr{P}^u\{E_r\}, \quad r = 1, 2, \ldots,$$

where

$$E_{r+1} = T_t E_r.$$

This form implies that the chosen probability measure is independent of time t. Hence the phenomenon is restricted to purely elastic deformations. In view of the introduction of the Markovian nature of the deformation process for the other two ranges of the material behaviour, it is convenient to use the concept of conditional probability measures as shown for instance in Ref. [16]. For this purpose, we consider two Borel sets E_i, E_j where $E_i \subset E_j$:

$$\mathscr{P}\{E_i|E_j\} = \frac{\mathscr{P}\{E_i\}}{\mathscr{P}\{E_j\}}, \tag{3.24}$$

since

$$\mathscr{P}\{E_i \cap E_j\} = \mathscr{P}\{E_i\}.$$

Thus a strictly stationary process in terms of the conditional probability measure is defined by:

$$\mathscr{P}\{E_r|E_{r+1}\} = \mathscr{P}\{E_{r+1}|E_r\} = 1. \tag{3.25}$$

It is seen that the sequence of deformations $\{\mathbf{u}_t(\mathbf{X})\}$ belonging to the space $[U, \mathfrak{F}^u, \mathscr{P}^u]$ together with the conditional probability

$$\mathscr{P}\{E_n|E_1, E_2, \ldots, E_{n-1}\} = \mathscr{P}\{E_n|E_{n-1}\} \tag{3.26}$$

defines *a* Markov process. Furthermore, a homogeneous Markov process is characterized by the "transition probability" $\mathbf{P}\{E_n, t_n; E_{n-1}, t_{n-1}\}$ or briefly $\mathbf{P}\{t_n, t_{n-1}\}$. Identifying t_n by s and t_{n-1} by t, further subdividing the interval $[t, s] \in R^+$ into smaller one, whereby a point $\tau \geqslant t$; $\tau \in [t, s]$, gives a relation for the "total probability" in the form of:

$$\mathbf{P}\{t, s\} = \int_U \mathbf{P}\{t, \tau) d\mathbf{P}\{\tau, s\} \tag{3.27}$$

which is the well-known Chapman-Kolmogorov functional equation in terms of the transition probability $\mathbf{P}\{t, s\}$. The importance of this transition probability and its corresponding matrix for the development of the general deformation theory is discussed below.

(ii) The transition matrix of the deformation process:
Relation (3.27) can also be expressed in matrix form as follows:

$$\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s), \tag{3.28}$$

where the *bar* sign indicates the matrix and the equation evidently shows the semi-group property of the system. From a statistical point of view, this property is also referred to as the "metric transitivity" [13]. Although t, s in the above relation refers to a time interval $[t, s]$, it is possible to establish the whole deformation history of a discrete system, but the reconstruction of the latter will only be valid for a class of materials for which the deformation process can be approximated by relation (3.26).

Considering now a "time-homogeneous" process in accordance with the assumed Markovian character of the process, the transition probability function $\mathbf{P}\{t, s\}$ depends only on the difference of time $t-s$. Thus, following the discussion given in Refs. [17, 18], the transition matrix representing the transition of the system from a state i to a state j is given by:

(3.29)
$$\mathbf{Q}_{ij}(t) = \lim_{\Delta t \to 0^+} \frac{\mathbf{P}_{ij}(t,s)}{\Delta t},$$
$$-\mathbf{Q}_{ii}(t) = \lim_{\Delta t \to 0^+} \frac{1-\mathbf{P}_{ij}(t,s)}{\Delta t}$$

for a σ-measurable function and for all $t \in R^+$, $\ni E \in \mathfrak{F}$. Thus, the Chapman-Kolmogorov equation (3.27) can be written as:

(3.30)
$$\frac{d}{dt}\mathbf{P}^u(t) = \mathbf{Q}^u(t)\mathbf{P}^u(t),$$

where

(3.31)
$$\mathbf{P}^u(0) = \mathbf{I} \text{ (Identity Matrix).}$$

It follows from relations (3.30), (3.31) that the transition probability $\mathbf{P}\{t, s\}$ has two important properties, viz:

(3.32)
$$\text{for all} \quad s, t \geqslant 0 \text{ and } E \in \mathfrak{F}^u, \lim_{t \to s} [(\mathbf{P}_t - \mathbf{P}_s)\,\mathscr{P}\{E\}] = 0,$$
$$\text{and for all} \quad s, t \geqslant 0, \ \mathbf{P}_t\mathbf{P}_s = \mathbf{P}_{t+s} \text{ with } \mathbf{P}(0) = \mathbf{I}.$$

So far as the elastic deformations are concerned, it is evident that such deformations are time-independent and hence the random deformation process can be regarded as a stationary one as mentioned in the preceding paragraph. In this case $\mathbf{u}(t)$ may be considered as belonging to the product space of countably finite copies of the function spaces $[U, \mathfrak{F}^u, \mathscr{P}^u]$ with corresponding measures on each of the spaces generated by the transition probabilities given by (3.26) and where the following properties of the transition probability measure will hold:

(3.33)
(i) the transition probability $\mathbf{P}(t, t+\Delta t)$ of the process $\mathbf{u}(t)$ changing from a state i to j in the interval Δt is zero;
(ii) the transition probability $\mathbf{P}(t, t+\Delta t)$ of no change in the interval Δt is one.

Hence, in that case, the transition probability will be determined by a constant matrix, i.e.

(3.34)
$$\mathbf{P}^u_{ij} = \text{Constant Matrix,}$$

so that $\mathbf{Q}^u_{ij} = 0$.

If the Markov assumption of the deformation process is maintained, it follows that the Kolmogorov differential equation reduces to:

(3.35)
$$\frac{d}{dt}\mathbf{P}^u_{ij}(t) = 0$$

which can be solved for an initial value of $\mathbf{P}^u_{ij}(0) = \delta_{ij}$ leading to the result:

(3.36)
$$\mathbf{P}^u_{ij}(t) = \delta_{ij}.$$

This relation suggests that the probability distribution for the reversible range of the material response remains constant with respect to time. This result is consistent with that shown subsequently, concerned with the two other stages of the material behaviour, i.e. the transient and steady-state deformation. It is further significant to note that the present formulation permits these two stages to be considered in terms of a single parameter such as $\mathbf{Q}^u_{ij}$.

Before dealing with the transient behaviour it may be convenient to study first the role of the transition matrix for the steady-state deformations. Thus for this type of deformations the probability distribution of the microdeformations $\mathscr{P}\{\mathbf{u}\}$ will also be a function of time and hence its evolution with time must be investigated. Introducing the concept of "probability intensities" as discussed for instance by BHARUCHA-REID [18], the relation between the intensity and the time-dependent transition probability $\mathbf{P}_{ij}(t)$ can be established in terms of the relative transition probability $\mathbf{Q}_{ij}$. For this purpose the latter is considered as a time-independent infinitesimal transition matrix as follows:

$$\tag{3.37} \begin{aligned} \mathbf{Q}_{ij} &= -\lambda \quad && \text{for} \quad i = 0, 1, \ldots, \\ \mathbf{Q}_{ij} &= \lambda && \text{for} \quad j = i+1, \\ \mathbf{Q}_{ij} &= 0 && \text{otherwise,} \end{aligned}$$

so that the transition matrix in the deformation space can be written as:

$$\tag{3.38} \mathbf{Q}^u_{ij} = \begin{bmatrix} -\lambda & \lambda & 0 & \ldots & 0 \\ 0 & -\lambda & \lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

The Chapman-Kolmogorov differential equation can, therefore, be expressed by:

$$\tag{3.39} \frac{d\mathbf{P}^u_{ij}(t)}{dt} = -\lambda \mathbf{P}^u_{ij}(t) + \lambda \mathbf{P}^u_{i,j-1}(t).$$

It is shown in Refs. [18, 19], that a solution of this matrix equation exists and is unique for a specified initial condition. Thus, taking $\mathbf{P}^u_{ij}(0) = \delta_{ij}$ as the initial condition, leads by employing an iteration procedure to the following solution:

$$\tag{3.40} \begin{aligned} \mathbf{P}_{ij}(t) &= \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t} \quad && \text{for} \quad j \geqslant i, \\ \mathbf{P}_{ij}(t) &= 0 && \text{for} \quad j < i. \end{aligned}$$

It is of interest to establish now the time-dependent probability, the properties of which have been investigated in detail in [19] leading to the following result:

$$\tag{3.41} \sum \mathbf{P}^u_{ik} \mathbf{P}^u_{kj} = \frac{[\lambda(t+s)]^{j-i}}{(j-i)!} e^{-\lambda(t+s)} = \mathbf{P}^u_{ij}(t+s),$$

in which $\mathbf{P}^u_{ik}$ and $\mathbf{P}^u_{kj}$ correspond to the transition probabilities $\mathbf{P}(t)$ and $\mathbf{P}(s)$ of Eq. (3.28), respectively. It is seen that the above relation again expresses the semi-group property of the transition probability of the system. It is possible to introduce now the "one-step" transition probability (see for instance, ROSENBLATT [20], PROHOROV and ROZANOV [21]), which can be expressed in the form of:

$$\tag{3.42} \mathbf{P}^u_{ij}(t) = \lambda t e^{-\lambda t},$$

where the index j has been replaced by $i+1$. If the initial probability distribution of the discrete system at $t = 0$ is known, then the time-dependent distribution is simply obtained from:

$$\mathcal{P}^u(t) = \lambda t e^{-\lambda t} \mathcal{P}^u(0). \tag{3.43}$$

It has been found from experimental observations that most discrete systems considered in micromechanics have an initial distribution of the Gaussian form [22]. Hence, considering that $\lambda t \ll 1$ and $e^{-\lambda t} \approx 1$, an approximate form for $\mathcal{P}^u(t)$ can be written as follows:

$$\mathcal{P}^u(t) = \frac{\lambda t}{\sqrt{2\pi}\,\sigma^u} e^{\frac{-(\mathbf{u}-\langle\mathbf{u}\rangle)^2}{2(\sigma^u)^2}}, \tag{3.44}$$

where σ^u denotes the standard deviation of the microdeformations. Hence, this form represents the time-dependent probability distribution for the steady-state deformation process from which the mean value and the variance of $\mathbf{u}(t)$ can readily be obtained. The above representation is based on a model of the material response analogous to that known as rigid-plastic in continuum mechanics. However, a more general consideration is required, if one wishes to characterize the transition from the purely elastic to the irreversible steady-state deformation of a discrete system, in particular, if considerations must be given to relaxation phenomena. Whilst such a model has been considered earlier in Ref. [6] concerned with the mechanical relaxation of crystalline solids, a more general exposition is given below.

Since the time-dependent transition probabilities of the transient deformation process do not have the semi-group property, the process is in general non-Markovian. However, one can consider two limiting processes approaching a fixed time interval $[t_1, t_2]$ of the general deformation process by splitting this interval into incremental times $\Delta\tau$ so that τ is a variable within $t_1 \leqslant \tau \leqslant t_2$. In this manner, the probability of changing from a state i to j in this range can be defined as follows:

(i) the probability of a change in the interval $(\tau, \tau+\Delta\tau)$ is:

$$\left\{\frac{-\alpha t_1(\tau-t_2)+\beta t_2(\tau-t_1)}{\tau(t_2-t_1)}\right\}\Delta\tau+o(\Delta\tau); \quad t_1 \leqslant \tau \leqslant t_2; \tag{3.45}$$

(ii) the probability of more than one change in

$$(\tau, \tau+\Delta\tau) \text{ is: } o(\Delta\tau); \tag{3.46}$$

(iii) the probability of no change in the interval $(\tau, \tau+\Delta\tau)$ is:

$$1-\left\{\frac{-\alpha t_1(\tau-t_2)+\beta t_2(\tau-t_1)}{\tau(t_2-t_1)}\right\}\Delta\tau-o(\Delta\tau) \tag{3.47}$$

in which the parameters α, β in (3.45), (3.47) are related to the probability intensity λ defined before. In this analysis, the fixed time t_1 is the upper bound of the completely reversible stage, whilst t_2 is considered as the time instant or lower bound of the irreversible steady-state deformation process. The quantity $o(\Delta\tau)$ in the above definition is the order of magnitude of the incremental time interval. The above forms are based on the more rigorous study of BHARUCHA-REID [18] and DYNKIN [23] concerned with the analysis of "discontinuous" Markov processes.

If the two limiting cases of the transient process are considered, then the following transition probabilities are obtained:

(i) at $\tau = t_1$: $\mathbf{P}^u_{ij}(\tau+\Delta\tau) = \mathbf{P}^u_{ij}(\tau)$
meaning that the transition probability is constant for completely reversible case;

(ii) at $\tau = t_2$: $\mathbf{P}^u_{ij}(\tau+\Delta\tau) = (1-\lambda\Delta\tau)\mathbf{P}^u_{ij}(\tau)+\lambda\Delta\tau^u_{i,j}\mathbf{P}_{-1}(\tau)$ where λ is the intensity that corresponds to a "Poisson process".

Letting

$$a = \frac{\beta t_2-\alpha t_1}{t_2-t_1}; \quad b = \frac{(\alpha-\beta)\,t_1 t_2}{t_2-t_1}, \tag{3.48}$$

permits to write the differential equation representing the change of the transition probability $\mathbf{P}^u(t)$ as follows:

$$\frac{d\mathbf{P}^u_{ij}(\tau)}{d\tau} = -b\,\mathbf{P}^u_{ij}(\tau)+\frac{a}{\tau}\mathbf{P}^u_{ij}(\tau)+b\,\mathbf{P}^u_{i,j-1}(\tau)-\frac{a}{\tau}\mathbf{P}^u_{i,j-1}(\tau). \tag{3.49}$$

Utilizing an iteration procedure, the solution of (3.49) has been studied in detail in [Ref. 19]. It has the following form:

$$\mathbf{P}^u_{ij}(\tau) = \frac{(b\tau-a\ln\tau)^{j-i}}{(j-i)!}\,\tau^a e^{-b\tau}. \tag{3.50}$$

In summarizing the above formulation, it is seen that the transient deformation process has the following properties:

(i) $|\mathbf{P}^u(t)| \leqslant 1$,

(ii) $\lim_{t\to s}\{[\mathbf{P}^u(t)-\mathbf{P}^u(s)]\mathscr{P}\{\mathbf{u}\} = 0, \quad \text{for} \quad \forall t, s \in R^+$,

(iii) $\mathbf{P}^u(t+s) \neq \mathbf{P}^u(t)\,\mathbf{P}^u(s)$,

(iv) $\lim_{a\to 0}\mathbf{P}^u_{ij}(t) = \frac{(bt)^{(j-i)}}{(j-i)!}e^{-bt}, \quad b = \frac{\lambda t_2}{t_2-t_1}$,

(v) $\lim_{\tau\to t_1}$ is a Poisson process with $\lambda = \alpha$, transient → (reversible state),

(vi) $\lim_{\tau\to t_2}$ is a Poisson process with $\lambda = \beta$, transient → (steady-state).

The non-Markovian nature of the transient process arises from the time-dependent transition probability during this stage, which is related to the change of the material behaviour. The latter following a transient deformation process, occurs rather abruptly in an unstable manner. The degree to which the material behaviour adjusts to the steady-state deformation process is closely related to the above introduced parameters a, b in relations (3.49) and (3.50). A more rigorous treatment of the transient behaviour of discrete systems is possible by using the concepts of "stability of random variables and processes" as discussed by PINSKER [24]. Finally, it is of interest to note that the experimental accessability of the significant quantity λ as an element of the transition probability matrix has already been pointed out in earlier work [25].

4. Response behaviour of discrete systems

In order to conclude the presentation of the general deformation theory of discrete systems and to formulate their response behaviour by the use of "material operators" (Sect. 3.2), a few remarks on the stress transition matrix associated with the evolution of the system in the stress space $\Xi \subset X$ may be indicated. Thus, in analogy to the considerations given in relation to the deformation subspaces ${}^{\alpha}U$, ${}^{\alpha\beta}U$, the stress subspaces ${}^{\alpha}\Xi$, ${}^{\alpha\beta}\Xi$ also belonging to the general function space X are as follows:

$$ {}^{\alpha}\Xi \bigcup {}^{\alpha\beta}\Xi = \Xi \subset X \tag{4.1} $$

and where the semi-norm of Ξ, again considered as a Fréchet space, is given by:

$$ \begin{aligned} \langle \boldsymbol{\sigma} \rangle &= E\{\boldsymbol{\sigma}\} = \Sigma \boldsymbol{\sigma} p^{\sigma}(\boldsymbol{\sigma});\ \sigma \in \Xi, \\ ||\boldsymbol{\sigma}|| &= D\{\boldsymbol{\sigma}\} = \{\Sigma[\boldsymbol{\sigma} - \langle \boldsymbol{\sigma} \rangle]^2 p^{\sigma}({}^{\beta}\boldsymbol{\sigma})\}^{1/2}, \end{aligned} \tag{4.2} $$

in which the probability measures of the subspaces are of the same form as those of the deformation subspaces. Hence, the change of the probability measure $\mathbf{P}^{\sigma}$ with time related to the stress space Ξ, will be represented in view of the duality of the stress and deformation spaces by the Chapman-Kolmogorov equation as before, i.e.:

$$ \frac{d\mathbf{P}^{\sigma}(t)}{dt} = \mathbf{Q}^{\sigma}(t)\,\mathbf{P}^{\sigma}(t) \tag{4.3} $$

with

$$ \mathbf{P}^{\sigma}(0) = \mathbf{I}. \tag{4.4} $$

It is to be noted that in contrast to previous work [26] in which the inverse of an average material operator has been used to relate these two subspaces, a more general view is taken here. As a consequence the relation between the stress and deformation space is expressed in terms of the distribution of the microelement material operator $\mathbf{M}$. Hence the following relation is assumed to hold:

$$ \mathscr{P}\{\boldsymbol{\sigma}, t\} = \mathscr{P}\{\mathbf{M}^{-1}, t\}\mathscr{P}\{\mathbf{u}, t\}. \tag{4.5} $$

In the above form, the inverse of the operator is used rather than the operator itself for convenience of the analysis. In this context it should be noted, that such an inverse will exist, if the operator is linear and satisfies the condition of a strictly monotone operator (see TONTI [27, 28]). On that basis, one can form a non-degenerate bilinear form or functional such that the invertible material operator connects the stress and deformation space. The domain of the operator is assumed to be dense. If this is not the case, one can still construct a bilinear form by considering dense subsets of the operator and apply a limit analysis [28], so that the condition of the functional to be positive definite is satisfied.

Since the distribution of the material operator can be established from the knowledge of the significant parameters characteristic of a discrete system by experimental observations in the "undeformed state" of the material, i.e. $\mathscr{P}\{\mathbf{M}^{-1}, 0\}$, it is possible to express the change of the distribution with time as follows:

$$ \mathscr{P}\{\mathbf{M}^{-1}, t\} = \psi(t)\mathscr{P}\{\mathbf{M}^{-1}, 0\}. \tag{4.6} $$

Using this relation and noting that in the stress space, analogously to the deformation space, the following relation holds, viz:

$$\mathscr{P}\{\boldsymbol{\sigma}, t\} = \mathbf{P}^{\sigma}(t)\mathscr{P}\{\boldsymbol{\sigma}, 0\} \tag{4.7}$$

and further, since (4.5) remains valid if $t = 0$, yields:

$$\mathbf{P}^{\sigma}(t)\mathscr{P}\{\mathbf{M}^{-1}, 0\} = \mathscr{P}\{\mathbf{M}^{-1}, t\}\mathbf{P}^{u}(t) \tag{4.8}$$

giving the relation between the transition probabilities in the stress and deformation space via the distribution of the inverse operator $\mathbf{M}^{-1}(0)$ and $\mathbf{M}^{-1}(t)$, respectively. It can readily be shown [19], that by utilizing relations (3.30) and (4.3) together with the above set of Eqs. (4.5)–(4.8) that for a Markov process in the stress space the distribution of the inverse operator must satisfy the following differential equation:

$$\frac{d\mathscr{P}\{\mathbf{M}^{-1}, t\}}{dt} + [\mathbf{Q}^{u} - \mathbf{Q}^{\sigma}]\mathscr{P}\{\mathbf{M}^{-1}, t\} = 0. \tag{4.9}$$

Its solution is of the form:

$$\mathscr{P}\{\mathbf{M}^{-1}, t\} = \exp\{-[\mathbf{Q}^{u} - \mathbf{Q}^{\sigma}]t\}\mathscr{P}\{\mathbf{M}^{-1}, 0\}, \tag{4.10}$$

if the transition matrices $\mathbf{Q}^{u}$, $\mathbf{Q}^{\sigma}$ are independent of time. It is seen that by comparing (4.10) with (4.6) it may be concluded that:

$$\psi(t) = \exp[-(\mathbf{Q}^{u} - \mathbf{Q}^{\sigma})t]. \tag{4.11}$$

Moreover, it can be shown from (4.10) that for the steady-state deformation process [19]:

$$\mathbf{Q}^{\sigma} = \mathbf{Q}^{u} + \frac{1}{t}[\ln\mathscr{P}\{\mathbf{M}^{-1}, t\} - \ln\mathscr{P}\{\mathbf{M}^{-1}, 0\}] \tag{4.12}$$

indicating the relationship between the transition matrices in both the stress and deformation spaces. It follows from the above considerations that during the steady-state deformation, the stress transition matrix $\mathbf{Q}^{\sigma}$ is zero-valued. This result has been obtained earlier (see Ref. [26]) by using the semi-group property of the deformation process.

For completion of the overall response behaviour of discrete systems, the corresponding equations for the transient state are given below. In this case the transition matrices $\mathbf{Q}^{\sigma}$ and $\mathbf{Q}^{u}$ are time-dependent and are obtained from a different form of Eq. (4.10) as follows:

$$\mathscr{P}\{\mathbf{M}^{-1}, t\} = \mathscr{P}\{\mathbf{M}^{-1}, 0\}\exp\left\{-\int[\mathbf{Q}^{u} - \mathbf{Q}^{\sigma}]dt\right\}, \tag{4.13}$$

where

$$\int[\mathbf{Q}^{u} - \mathbf{Q}^{\sigma}]dt = \ln\mathscr{P}\{\mathbf{M}^{-1}, 0\} - \ln\mathscr{P}\{\mathbf{M}^{-1}, t\}, \tag{4.14}$$

or

$$\int\mathbf{Q}^{\sigma}dt = \int\mathbf{Q}^{u}dt - \left\{\ln\mathscr{P}\{\mathbf{M}^{-1}, 0\} - \ln\mathscr{P}\{\mathbf{M}^{-1}, t\}\right\}. \tag{4.15}$$

Using the parameters a, b defined in (3.48) and letting

$$\mathbf{Q}^{u} = \frac{a}{t} - b$$

gives a relation that permits the evaluation of $\mathbf{Q}^{\sigma}$. Finally, the material operator or its inverse for a particular mesodomain can be obtained from the above developed relations as follows:

$$ {}^{M}\mathcal{M}^{-1}(t) = \Sigma \mathbf{M}^{-1}(t) p\{\mathbf{M}^{-1}, t\} \tag{4.16} $$

in which the distributions $p\{\mathfrak{m}^{-1}, t\}$ in the case of steady-state deformations are given by (4.10) and in the transient case by (4.13) and which together with the knowledge of the initial distribution of the operator determine the material operator ${}^{M}\mathcal{M}(t)$. Hence the stress-deformation relations for a general deformation process can be stated as follows:

$$ {}^{M}\mathbf{u}(t) = {}^{M}\mathcal{M}(t) {}^{M}\boldsymbol{\sigma}(t), \quad t > 0. \tag{4.17} $$

The use of the inverse of the material operator in the foregoing relations rather than the operator itself is due to the fact that the significant parameters of the analysis are only obtainable from experimental observations of microdeformations and their related distributions.

5. Concluding remarks

From the above presentation of the micromechanics of discrete systems, the following conclusions may be drawn:

(i) The assumptions made in the development of the theory are based on four fundamental concepts that concern the random geometrical and physical properties of the system. In order to include the elements of the microstructure and their interaction effects in the general deformation theory, it is important to use the distribution functions of the relevant quantities over a specific material domain.

(ii) The general deformation process in the reversible and steady-state range of the material response can be approximated by a Markov process, whilst for the transient state a quasi-Poisson process can be used.

(iii) In order to apply the notion of continuity in the analysis of the evolution of the deformations in the physical domain, the system can be represented by an abstract dynamical system $[X, \mathfrak{F}, \mathscr{P}, \mathbf{Q}]$ employing functional analysis and measure theory. The role of the transition matrix in the stress and deformation subspaces becomes significant for the determination of the "material operator". The latter is required to find the necessary stress-deformation relations of a discrete system.

(iv) The significant stochastic parameters in the formulation are expressed in terms of their distribution functions which in part are experimentally accessible.

As a final remark, it may be stated that the distribution functions of the significant parameters for two classes of discrete systems, i.e. polycrystalline solids and fibrous networks have been established by means of X-ray diffraction techniques [22], holographic interferometry and electron-microscopy.

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