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Global thermodynamic field equations of balance for anelastic bodies

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THE mechanical model of anelastic response is generalized to allow for thermodynamical influences and leads to restrictions on the constitutive functions and flow rate which are implied by the Clausius-Duhem inequality. From the concept of symmetry isomorphism the anelastic global structure and the associated geometric structures are generated and used to derive the global thermodynamic field equations of balance.

Mechaniczny model niesprężystej reakcji ośrodka uogólniony tak, że uwzględnione są efekty termodynamiczne, prowadzi do ograniczeń na funkcje konstytutywne i prędkość płynięcia, wynikających z nierówności Clausiusa-Duhema. Wykorzystując koncepcję symetrii izomorfizmu utworzono globalną strukturę niesprężystą oraz towarzyszące odpowiednie struktury geometryczne, które zastosowano do wyprowadzenia globalnych termodynamicznych równań równowagi.

Механическая модель неупругой реакции среды, обобщенная так, что учтены термодинамические эффекты, ведет к ограничениям для определяющих функций и для скорости течения, следующим из неравенства Клаузиуса-Дюгема. Используя концепцию симметрии изоморфизма образованы глобальная неупругая структура и сопутствующие соответствующие геометрические структуры, которые применены для вывода глобальных термодинамических уравнений равновесия.

Introduction

IN a previous paper [1] the authors constructed a theory of anelastic response for a purely mechanical model; in the present work we generalize that model to a thermomechanical one. The restrictions on the constitutive functions and on the anelastic flow rate, which follow from a standard application of the Clausius-Duhem inequality, are derived in §1. Use of the concept of symmetry isomorphism, first considered by WANG in [4], allows us then to set forth our global constitutive assumptions in §2; here, as in [1], the anelastic global structure of B is generated from the elastic global structure of B via the field of anelastic transformation functions and the concept of a symmetry connection on an elastic body is introduced. Using the global balance equations of momentum and energy for a thermoelastic body with uniform symmetry (e.g. [4]) we derive, in §3, the global thermodynamic field equations of balance which apply to materials exhibiting anelastic response.

1. Local constitutive assumptions and symmetry groups

Let $\mathscr{B}$ be a body manifold and p a body point in $\mathscr{B}$. Then as in [1] p is called an *anelastic point* if it is a quasi-elastic point such that the instantaneous anelastic response function $\mathbf{G}^t$ is given by transforming a fixed elastic response function $\mathbf{G}$ according to the rule

$$\mathbf{G}^t(\mathbf{v}) = \mathbf{G}(\mathbf{v} \circ \boldsymbol{\alpha}(t)), \tag{1.1}$$

where $\mathbf{v}$ is an arbitrary local configuration of p, and where $\boldsymbol{\alpha}(t)$ is an isochoric automorphism of $\mathscr{B}_p$ called the anelastic transformation. It is assumed that $\boldsymbol{\alpha}(t) = id_{\mathscr{B}_p}$ in any rigid or rest process of p and that $\boldsymbol{\alpha}(t) \to id_{\mathscr{B}_p}$ as $t \to \infty$ in any process of p in general. The governing equation of the anelastic transformation $\boldsymbol{\alpha}(t)$ is called the flow rule. The model defined by (1.1) is a purely mechanical one not involving the use of thermodynamical state variables and state functions.

To generalize the mechanical model (1.1) to a thermomechanical one, we introduce first the usual list of constitutive relations for a thermoelastic point:

$$(\mathbf{T}, \mathbf{q}, \varepsilon, \eta) = (\mathbf{G}, \mathbf{l}, e, h)(\mathbf{v}, \theta, \mathbf{g}), \tag{1.2}$$

where $\mathbf{T}$, $\mathbf{q}$, ε, η, θ, and $\mathbf{g}$ are the stress tensor, the heat flux vector, the internal energy, the entropy, the temperature, and the temperature gradient, respectively. Now we regard $(\mathbf{G}, \mathbf{l}, e, h)$ to be the elastic response functions which hold for an anelastic point p in any rest process with constant $(\mathbf{v}, \theta, \mathbf{g})$. When the process is not a rest one, we require that the instantaneous anelastic response functions $(\mathbf{G}^t, \mathbf{l}^t, e^t, h^t)$ be related to $(\mathbf{G}, \mathbf{l}, e, h)$ by

$$(\mathbf{G}^t, \mathbf{l}^t, e^t, h^t)(\mathbf{v}, \theta, \mathbf{g}) = (\mathbf{G}, \mathbf{l}, e, h)(\mathbf{v} \circ \boldsymbol{\alpha}(t), \theta, \mathbf{g}), \tag{1.3}$$

where $\boldsymbol{\alpha}(t)$ is governed by the flow rule associated with the process.

By using the original argument of COLEMAN and NOLL [2], as applied by WANG and BOWEN [3] to quasi-elastic materials, we can determine the thermodynamic restrictions on the constitutive relations (1.2) and (1.3) which follow from the Clausius-Duhem inequailty. As usual, we choose first a local reference configuration $\boldsymbol{\mu}$ for p and rewrite (1.2) and (1.3) in the form

$$(\mathbf{T}, \mathbf{q}, \varepsilon, \eta) = (\mathbf{G}_\mu, \mathbf{l}_\mu, e_\mu, h_\mu)(\mathbf{F}, \theta, \mathbf{g}), \tag{1.4}$$

$$(\mathbf{G}^t_\mu, \mathbf{l}^t_\mu, e^t_\mu, h^t_\mu)(\mathbf{F}, \theta, \mathbf{g}) = (\mathbf{G}_\mu, \mathbf{l}_\mu, e_\mu, h_\mu)(\mathbf{F}\mathbf{A}_\mu(t), \theta, \mathbf{g}), \tag{1.5}$$

where $\mathbf{F}$ denotes the deformation gradient relative to $\boldsymbol{\mu}$, i.e., $\mathbf{F} = \mathbf{v} \circ \boldsymbol{\mu}^{-1}$, and where $\mathbf{A}_\mu(t) = \boldsymbol{\mu} \circ \boldsymbol{\alpha}(t) \circ \boldsymbol{\mu}^{-1}$. Since $\boldsymbol{\mu}$ is held fixed in the analysis for the sake of brevity, we shall supress it from the notation here.

Our starting point is the reduced entropy inequality:

$$\varrho(\dot{\psi} + \eta\dot{\theta}) - \operatorname{tr}(\mathbf{T} \operatorname{grad} \mathbf{v}) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leqslant 0, \tag{1.6}$$

where ϱ is the density, $\mathbf{v}$ is the velocity, and ψ is the free energy function defined by $\psi = \varepsilon - \eta\theta$. From (1.4) and (1.5) we then have

$$\psi = f(\mathbf{F}, \theta, \mathbf{g}) \tag{1.7}$$

in any rest process and

$$\psi = f^t(\mathbf{F}, \theta, \mathbf{g}) = f(\mathbf{F}\mathbf{A}(t), \theta, \mathbf{g}) \tag{1.8}$$

at any instant t in a general process. By (1.8) the gradients of f^t and f are related by

$$\frac{\partial f^t}{\partial F^i_j} = \frac{\partial f}{\partial F^i_k} A^j_k(t), \quad \frac{\partial f^t}{\partial \theta} = \frac{\partial f}{\partial \theta}, \quad \frac{\partial f^t}{\partial g_i} = \frac{\partial f}{\partial g_i}, \tag{1.9}$$

where the arguments of f^t and f are as shown in (1.8). By use of the chain rule we see that the total time derivative $\dot{\psi}$ is given by

$$\dot{\psi} = \frac{\partial f}{\partial F^a_b}(\dot{F}^a_c A^c_b + F^a_c \dot{A}^c_b) + \frac{\partial f}{\partial \theta}\dot{\theta} + \frac{\partial f}{\partial g_a}\dot{g}_a \tag{1.10}$$
$$= \frac{\partial f^t}{\partial F^a_b}\dot{F}^a_b + \frac{\partial f^t}{\partial F^a_b} F^a_c \dot{A}^c_d A^{-1d}_b + \frac{\partial f^t}{\partial \theta}\dot{\theta} + \frac{\partial f^t}{\partial g_a}\dot{g}_a,$$

where (1.9) has been used. Substituting (1.10) into (1.6), we obtain the following results:

(i) f^t and f are independent of $\mathbf{g}$ so that

$$\psi = f^t(\mathbf{F}, \theta) = f(\mathbf{F}\mathbf{A}(t), \theta). \tag{1.11}$$

(ii) h^t and h are independent of $\mathbf{g}$ and are related to f^t and f by

$$\eta = h^t(\mathbf{F}, \theta) = h(\mathbf{F}\mathbf{A}(t), \theta) = -\frac{\partial f^t(\mathbf{F}, \theta)}{\partial \theta} = -\frac{\partial f(\mathbf{F}\mathbf{A}(t), \theta)}{\partial \theta}. \tag{1.12}$$

(iii) $\mathbf{G}^t$ and $\mathbf{G}$ are independent of $\mathbf{g}$ and are related to f^t and f by

$$T^a_b = G^{ta}_b(\mathbf{F}, \theta) = G^a_b(\mathbf{F}\mathbf{A}(t), \theta) = \varrho F^a_c \frac{\partial f^t(\mathbf{F}, \theta)}{\partial F^b_c} = \varrho F^a_c A^c_d(t) \frac{\partial f(\mathbf{F}\mathbf{A}(t), \theta)}{\partial F^b_d}.$$

(iv) $\mathbf{l}^t$ and $\mathbf{l}$ satisfy the inequality

$$l^{tj}(\mathbf{F}, \theta, \mathbf{g}) g_j = l^j(\mathbf{F}\mathbf{A}(t), \theta, \mathbf{g}) g_j \leqslant 0. \tag{1.14}$$

(v) $\dot{\mathbf{A}}(t)$ satisfies the inequality

$$F^{-1a}_b G^{tb}_c(\mathbf{F}, \theta) F^c_d \dot{A}^d_e(t) A^{-1e}_a \leqslant 0. \tag{1.15}$$

We note here that the result in (iv) is somewhat stronger than the restriction on the heat flux of a quasi-elastic point in general, cf. [3, (3.14)]. To prove (iv) we consider (1.6) in any rest process with constant $(\mathbf{F}, \theta, \mathbf{g})$ and we obtain the following inequality for the elastic response function $\mathbf{l}(\mathbf{F}, \theta, \mathbf{g})$:

$$l^j(\mathbf{F}, \theta, \mathbf{g}) g_j \leqslant 0. \tag{1.16}$$

Then (iv) follows from (1.6) and (1.5).

The preceding results, i.e. (i)–(iv), show that the elastic response functions $\mathbf{G}$, $\mathbf{l}$, e, h, and f of an anelastic point satisfy exactly the same thermodynamic restrictions as those of a thermoelastic point. Next, the anelastic response functions $\mathbf{G}^t$, $\mathbf{l}^t$, e^t, h^t, and f^t are obtained from the elastic response functions $\mathbf{G}$, $\mathbf{l}$, e, h, and f by (1.3) or (1.5) through the anelastic transformations $\boldsymbol{\alpha}(t)$ or $\mathbf{A}(t)$, respectively. Finally, the flow rate $\dot{\mathbf{A}}(t)$ must satisfy the inequality (1.15). The constitutive relations (1.4) and (1.5) together with the flow rule and the thermodynamic restrictions (1.11)–(1.15) are now admitted as the local constitutive assumptions of an anelastic point.

Since the elastic response functions **G**, **l**, e, h, and f obey the same restrictions as those of a thermoelastic point, we define the elastic symmetry groups g and g^* in the same way as in [4, Sect. III]:

$$\boldsymbol{\xi} \in g \Leftrightarrow \boldsymbol{\xi} \in \mathscr{SL}(\mathscr{B}_p): \begin{cases} (\mathbf{G}, \mathbf{l})(\boldsymbol{\nu} \circ \boldsymbol{\xi}, \theta, \mathbf{g}) = (\mathbf{G}, \mathbf{l})(\boldsymbol{\nu}, \theta, \mathbf{g}) \\ \forall(\boldsymbol{\nu}, \theta), \end{cases} \tag{1.17}$$

and

$$\boldsymbol{\xi} \in g^* \Leftrightarrow \boldsymbol{\xi} \in \mathscr{SL}(\mathscr{B}_p): \begin{cases} (e, h, f)(\boldsymbol{\nu} \circ \boldsymbol{\xi}, \theta) = (e, h, f)(\boldsymbol{\nu}, \theta) \\ \forall(\boldsymbol{\nu}, \theta). \end{cases} \tag{1.18}$$

We note here that in writing down (1.17) we have assumed, for the sake of simplicity (as explained in [4 Sect. III]) that the symmetry group of **G** is also that of **l**. As shown in [5], g^* is generally a subgroup of g. However, $g^* = g$ when p is a solid point (i.e., g is compact) or when p is a fluid point $\big($i.e. $g = \mathscr{SL}(\mathscr{B}_p)\big)$; whenever p is a fluid crystal point g^* may or may not coincide with g. As explained in [1], the elastic symmetry groups g and g^* give rise to the anelastic symmetry groups g^t and g^{t*} via

$$g^t = \boldsymbol{\alpha}(t) \circ g \circ \boldsymbol{\alpha}(t)^{-1}, \quad g^{t*} = \boldsymbol{\alpha}(t) \circ g^* \circ \boldsymbol{\alpha}(t)^{-1} \tag{1.19}$$

which then satisfy the conditions

$$\boldsymbol{\xi} \in g^t \Leftrightarrow \boldsymbol{\xi} \in \mathscr{SL}(\mathscr{B}_p): \begin{cases} (\mathbf{G}^t, \mathbf{l}^t)(\boldsymbol{\nu} \circ \boldsymbol{\xi}, \theta, \mathbf{g}) = (\mathbf{G}^t, \mathbf{l}^t)(\boldsymbol{\nu}, \theta, \mathbf{g}) \\ \forall(\boldsymbol{\nu}, \theta, \mathbf{g}), \end{cases} \tag{1.20}$$

and

$$\boldsymbol{\xi} \in g^{t*} \Leftrightarrow \boldsymbol{\xi} \in \mathscr{SL}(\mathscr{B}_p): \begin{cases} (e^t, h^t, f^t)(\boldsymbol{\nu} \circ \boldsymbol{\xi}, \theta) = (e^t, h^t, f^t)(\boldsymbol{\nu}, \theta) \\ \forall(\boldsymbol{\nu}, \theta). \end{cases} \tag{1.21}$$

Relative to a local reference configuration $\boldsymbol{\mu}$ of p, the groups g, g^*, g^t, and g^{t*} are represented by the groups $\mathscr{G}$, $\mathscr{G}^*$, $\mathscr{G}^t$, and $\mathscr{G}^{t*}$, respectively, with

$$(\mathscr{G}, \mathscr{G}^*, \mathscr{G}^t, \mathscr{G}^{t*}) = \boldsymbol{\mu} \circ (g, g^*, g, g^{t*}) \circ \boldsymbol{\mu}^{-1}, \tag{1.22}$$

where

$$\mathbf{K} \in \mathscr{G} \Leftrightarrow \mathbf{K} \in \mathscr{SL}(3): \begin{cases} (\mathbf{G}, \mathbf{l})(\mathbf{FK}, \theta, \mathbf{g}) = (\mathbf{G}, \mathbf{l})(\mathbf{F}, \theta, \mathbf{g}) \\ \forall(\mathbf{F}, \theta, \mathbf{g}), \end{cases} \tag{1.23}$$

$$\mathbf{K} \in \mathscr{G}^* \Leftrightarrow \mathbf{K} \in \mathscr{LS}(3): \begin{cases} (e, h, f)(\mathbf{FK}, \theta) = (e, h, f)(\mathbf{F}, \theta) \\ \forall(\mathbf{F}, \theta), \end{cases} \tag{1.24}$$

and similar statements apply for $\mathscr{G}^t$ and $\mathscr{G}^{t*}$. In terms of the response function f the group $\mathscr{G}$ can be characterized by Truesdell's condition [5]:

$$\mathbf{K} \in \mathscr{G}^* \Leftrightarrow \mathbf{K} \in \mathscr{SL}(3): \begin{cases} f(\mathbf{FK}, \theta) = f(\mathbf{F}, \theta) + f(\mathbf{K}, \theta) - f(\mathbf{1}, \theta) \\ \forall(\mathbf{F}, \theta), \end{cases} \tag{1.25}$$

and a similar condition holds for $\mathscr{G}^{t*}$ in terms of f^t. For a fluid crystal point the exact relationship between $\mathscr{G}$ and $\mathscr{G}^*$ is rather complex: for certain special cases of $\mathscr{G}$, $\mathscr{G}^*$ must coincide with $\mathscr{G}$, cf. [6], but for certain other cases of $\mathscr{G}$, $\mathscr{G}^*$ must be different from $\mathscr{G}$, cf. [7]. The problem is solved in general in [8].

2. Global constitutive assumptions and symmetry connections

In the preceding section we have considered the constitutive relations and symmetry groups of an arbitrary anelastic point $p \in \mathscr{B}$. We now assume that all body points of $\mathscr{B}$ are anelastic and that, moreover, $\mathscr{B}$ satisfies the following global constitutive assumptions:

(a) The elastic response functions of the body points are distributed on $\mathscr{B}$ in the same way as those of a thermoelastic body with uniform symmetry as defined in reference [4, Sect. III]. In other words, in all rest processes the global structure of $\mathscr{B}$ is exactly the same as that formulated in [4].

(b) In a general process of $\mathscr{B}$ the elastic global structure evolves smoothly into the anelastic global structure through the anelastic transformation $\boldsymbol{\alpha}$ as explained in [1].

We shall now summarize some of the basic concepts introduced in [1] and [4]; first, we explain the elastic global structure of $\mathscr{B}$. This structure requires that for any two points p and q in $\mathscr{B}$ there exists an isomorphism $\mathbf{A}(p, q): \mathscr{B}_p \to \mathscr{B}_q$, where $\mathscr{B}_p$ and $\mathscr{B}_q$ are, respectively, the tangent spaces to $\mathscr{B}$ at p and q such that the symmetry groups g_p and g_q are related by

$$g_q = \mathbf{A}(p, q) \circ g_p \circ \mathbf{A}(p, q)^{-1}. \tag{2.1}$$

Any such isomorphism $\mathbf{A}(p, q)$ is then called a symmetry isomorphism. In terms of the relative symmetry groups the condition expressed by (2.1) means that there exist local reference configurations $\boldsymbol{\mu}_p$ and $\boldsymbol{\mu}_q$, which are related via $\mathbf{A}(p, q) = \boldsymbol{\mu}_q \circ \mathbf{u}_p^{-1}$ such that $\mathscr{G}_{\boldsymbol{\mu}_q}$ coincides with $\mathscr{G}_{\boldsymbol{\mu}_q}$. Thus the relative symmetry groups of the body points of $\mathscr{B}$ are all of the same type in the sense that the conjugate classes of the relative symmetry groups of $\mathscr{B}$ coincide with one another.

R e m a r k. As has already been explained in [4], symmetry isomorphisms are to be distinguished from the material isomorphisms which have been employed, for example, in [1] and [4]. An isomorphism $\mathbf{I}(p, q): \mathscr{B}_p \to \mathscr{B}_q$ is called a material isomorphism of p with q if

$$\mathbf{G}(\boldsymbol{\nu}_q, \theta) = \mathbf{G}(\boldsymbol{\nu}_q \circ \mathbf{I}(p, q), \theta) \ \forall (\boldsymbol{\nu}_q, \theta) \tag{2.2}$$

where $\boldsymbol{\nu}_q$ is a local configuration of q. When such an isomorphism $\mathbf{I}(p, q)$ exists, p and q are called materially isomorphic, and it follows immediately that (2.1) holds with $\mathbf{A}(p, q)$ replaced by $\mathbf{I}(p, q)$. Every material isomorphism is, therefore, a symmetry isomorphism but the converse is, in general, not valid.

The smoothness of the distribution of the response functions requires the existence of an elastic atlas $\mathfrak{A}$ on $\mathscr{B}$. Specifically, an elastic atlas $\mathfrak{A}$ is a collection

$$\mathfrak{A} = \{(\mathscr{U}_\gamma, \boldsymbol{\mu}_\gamma), \gamma \in I\}, \tag{2.3}$$

where $\mathscr{U}_\gamma \subset \mathscr{B}$, $\boldsymbol{\mu}_\gamma(\cdot)$ is a smooth field of local reference configurations on $\mathscr{U}_\gamma$, I is an index set and the following conditions are satisfied:

(1) There exists a smooth distribution of relative response functions $(\mathbf{G}_{\mathfrak{A}}, \mathbf{l}_{\mathfrak{A}})(\mathbf{F}, \theta, \mathbf{g}, p)$ on $\mathscr{B}$ such that

$$(\mathbf{G}_{\mathfrak{A}}, \mathbf{l}_{\mathfrak{A}})(\mathbf{F}, \theta, \mathbf{g}, p) = (\mathbf{G}_{\boldsymbol{\mu}_\gamma}, \mathbf{l}_{\boldsymbol{\mu}_\gamma})(\mathbf{F}, \theta, \mathbf{g}, p) \ \forall p \in \mathscr{U}_\gamma, \gamma \in I. \tag{2.4}$$

(2) The distribution $(\mathbf{G}_{\mathfrak{A}}, \mathbf{l}_{\mathfrak{A}})$ has a uniform relative symmetry group $\mathscr{G}_{\mathfrak{A}}$, *viz*,

$$\mathscr{G}_{\mathfrak{A}} = \mathscr{G}_{\mu\gamma}(p) \quad \forall p \in \mathscr{U}_{\gamma}, \gamma \in I. \tag{2.5}$$

Note. The conditions (1) and (2) above imply that the deformation gradient $\mathbf{K}_{\gamma\sigma} = \boldsymbol{\mu}_{\gamma} \circ \boldsymbol{\mu}_{\delta}^{-1}$ from $\boldsymbol{\mu}_{\delta}$ to $\boldsymbol{\mu}_{\gamma}$ is a smooth field on $\mathscr{U}_{\gamma} \cap \mathscr{U}_{\delta}$ with values in $\mathscr{G}_{\mathfrak{A}}$ for all charts $(\mathscr{U}_{\gamma}, \boldsymbol{\mu}_{\gamma})$ and $(\mathscr{U}_{\delta}, \boldsymbol{\mu}_{\delta})$ in $\mathfrak{A}$. Furthermore, the collection $\{\mathbf{K}_{\gamma\delta}; \gamma, \delta \in I\}$ satisfies the following indentities:

$$\begin{gathered} \mathbf{K}_{\gamma\gamma}(p) = \mathbf{1} \quad \forall p \in \mathscr{U}_{\gamma}, \\ \mathbf{K}_{\gamma\delta}(p) = \mathbf{K}_{\delta\gamma}(p)^{-1} \quad \forall p \in \mathscr{U}_{\gamma} \cap \mathscr{U}_{\delta}, \\ \mathbf{K}_{\gamma\delta}(p)\mathbf{K}_{\delta\gamma}(p) = \mathbf{K}_{\gamma\lambda}(p) \quad \forall p \in \mathscr{U}_{\gamma} \cap \mathscr{U}_{\delta} \cap \mathscr{U}_{\lambda}. \end{gathered} \tag{2.6}$$

Hence $\mathfrak{A}$ can be regarded as a bundle atlas with the $\mathbf{K}_{\gamma\delta}$ as the coordinate transformations and $\mathscr{G}_{\mathfrak{A}}$ as the structure group as in the theory of fibre bundles, cf. [9]. The elastic atlas $\mathfrak{A}$ now characterizes completely the elastic global structure in accordance with (a).

From (b) the anelastic global structure of $\mathscr{B}$ at any time t in a general process can be obtained from the elastic glocal structure in the following way: There exists a smooth global anelastic transformation field $\boldsymbol{\alpha}(t, p)$ on $\mathscr{B}$ such that the anelastic atlas $\mathfrak{A}(t)$ is related to the elastic atlas $\mathfrak{A}$ by

$$\mathfrak{A}(t) = \{(\mathscr{U}_{\gamma}, \boldsymbol{\mu}_{\gamma}(t, \cdot)), \gamma \in I\}, \tag{2.7}$$

where

$$\boldsymbol{\mu}_{\gamma}(t, p) = \boldsymbol{\mu}_{\gamma}(p) \circ \boldsymbol{\alpha}(t, p)^{-1} \quad \forall p \in \mathscr{U}_{\gamma}, \gamma \in I. \tag{2.8}$$

Like $\mathfrak{A}$, the anelastic atlas $\mathfrak{A}(t)$ satisfies the following conditions: $(1)^t$. There exists a smooth distribution of relative response functions $(\mathbf{G}^t_{\mathfrak{A}(t)}, \mathbf{l}^t_{\mathfrak{A}(t)})\ (\mathbf{F}, \theta, \mathbf{g}, p)$ on $\mathscr{B}$ such that

$$\begin{cases} (\mathbf{G}^t_{\mathfrak{A}(t)}, \mathbf{l}^t_{\mathfrak{A}(t)})(\mathbf{F}, \theta, \mathbf{g}, p) = (\mathbf{G}^t_{\mu_{\gamma}(t,p)}, \mathbf{l}^t_{\mu_{\gamma}(t,p)})(\mathbf{F}, \theta, \mathbf{g}, p) \\ \forall p \in \mathscr{U}_{\gamma}, \gamma \in I. \end{cases} \tag{2.9}$$

$(2)^t$ The symmetry group of $(\mathbf{G}^t_{\mathfrak{A}(t)}, \mathbf{l}^t_{\mathfrak{A}(t)})$ is uniform on $\mathscr{B}$, i.e.

$$\mathscr{G}^t_{\mathfrak{A}(t)} = \mathscr{G}^t_{\mu_{\gamma}(t,p)} \quad \forall p \in \mathscr{U}_{\gamma}, \gamma \in I. \tag{2.10}$$

Indeed, from (2.8) the relative response functions $(\mathbf{G}^t_{\mathfrak{A}(t)}, \mathbf{l}^t_{\mathfrak{A}(t)})$ are the same as $(\mathbf{G}_{\mathfrak{A}}, \mathbf{l}_{\mathfrak{A}})$ and the relative symmetry group $\mathscr{G}^t_{\mathfrak{A}(t)}$ coincides with $\mathscr{G}_{\mathfrak{A}}$. In the theory of fibre bundles $\mathfrak{A}(t)$ is equivalent to the bundle atlas $\mathfrak{A}$, since its coordinate transformations $\mathbf{K}^t_{\gamma\delta} = \boldsymbol{\mu}_{\gamma}(t, \cdot) \circ \boldsymbol{\mu}_{\delta}(t, \cdot)^{-1}$, $\gamma, \delta \in I$, coincide with those of $\mathfrak{A}$.

Note. As explained in [4] the relative response functions $\mathbf{G}_{\mathfrak{A}}$ and $\mathbf{G}_{\mathfrak{A}(t)}$ would be independent of p were we working with the concept of material isomorphism rather than that of symmetry isomorphism, i.e., the explicit dependence of $\mathbf{G}_{\mathfrak{A}(t)}$, $\mathbf{G}^t_{\mu_{\gamma}}$ (t, p), *etc.*, on p would not appear in (2.9).

Now, just as in [1], we can define an affine connection on the tangent bundle of $\mathscr{B}$ whose induced parallel transports (relative to $\mathfrak{A}$) are contained in $\mathscr{G}_{\mathfrak{A}}$. Since the charts of $\mathfrak{A}$ have a uniform symmetry group, we call such an affine connecttion an elastic symmetry connection. In the theory of fibre bundles, these connections are called structural connections or $\mathscr{G}$-connections, where $\mathscr{G}$ denotes the structure group of the bundle atlas. For the elastic global structure here, the bundle altas is $\mathfrak{A}$ and the structure group is $\mathscr{G}_{\mathfrak{A}}$. It was proved in [9] that a structural connection in general can be characterized in the following way: Choose a coordinate system $\boldsymbol{\varkappa} = (X^A)$ on $\mathscr{B}$. Then the charts $(\mathscr{U}_\gamma, \boldsymbol{\mu}_\gamma)$ of a bundle atlas $\mathfrak{A}$ can be represented by the deformation gradient $\mathbf{F} = \boldsymbol{\varkappa}_* \circ \boldsymbol{\mu}^{-1}$, where $\boldsymbol{\varkappa}_*$ denotes the gradient of $\boldsymbol{\varkappa}$. Now suppose that $\mathscr{H}$ is an affine connection on $\mathscr{I}(\mathscr{B})$, the tangent bundle of $\mathscr{B}$, with connection symbols $\Gamma^A_{BC}(X^D)$ relative to (X^A). Then a necessary and sufficient condition for $\mathscr{H}$ to be a structural connection with respect to $\mathfrak{A}$ is that

$$\left[F_C^{-1A}\left(\frac{\partial F_B^C}{\partial X^D} + \Gamma^C_{ED} F^E_B\right)\right] \in \mathfrak{g}_{\mathfrak{A}}, \quad D = 1, 2, 3, \tag{2.11}$$

where F^A_B denotes the components of $\mathbf{F}$ and $\mathfrak{g}_{\mathfrak{A}}$ denotes the Lie Algebra of the structure group $\mathscr{G}_{\mathfrak{A}}$. It was shown in [1] that a structural connection exists but is generally not unique as the condition expressed by (2.11) does not determine the connection symbols uniquely.

The global anelastic transformation $\boldsymbol{\alpha}$ which maps the atlas $\mathfrak{A}$ of the elastic structure to the atlas $\mathfrak{A}(t)$ of the anelastic structure also maps a structural connection $\mathscr{H}$ relative to $\mathfrak{A}$ to a structural connection $\mathscr{H}^t$ relative to $\mathfrak{A}(t)$. As explained in [1] the parallel transports $\boldsymbol{\rho}(\tau)$ and $\boldsymbol{\rho}^t(\tau)$ of $\mathscr{H}$ and $\mathscr{H}^t$, respectively, along any path λ in $\mathscr{B}$ from $\lambda(0)$ to $\lambda(\tau)$ are related by

$$\boldsymbol{\rho}^t(\tau) = \boldsymbol{\alpha}(t, \lambda(\tau)) \circ \boldsymbol{\rho}(\tau) \circ \boldsymbol{\alpha}(t, \lambda(0))^{-1}. \tag{2.12}$$

As for the connection symbols $\Gamma^A_{BC}(X^D)$ and $\Gamma^A_{BC}(t, X^D)$ relative to (X^A) we have

$$\begin{aligned} \Gamma^A_{BC}(t, X^K) &= \alpha^A_F(t, X^K)\left(\Gamma^F_{DC}(X^K)\alpha_B^{-1D}(t, X^K) + \frac{\partial \alpha_B^{-1F}(t, X^K)}{\partial X^C}\right) \\ &= \alpha_B^{-1D}(t, X^K)\left(\Gamma^F_{DC}(X^K)\alpha^A_F(t, X^K) - \frac{\partial \alpha^A_D(t, X^K)}{\partial X^K}\right), \end{aligned} \tag{2.13}$$

where α^A_B and α_B^{-1A} are the components of $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^{-1}$ with respect to (X^A). The condition expressed by (2.13) implies that the matrices given by (2.11) coincide with the corresponding matrices associated with $\mathscr{H}^t$ and $\mathfrak{A}(t)$, *viz*,

$$\begin{aligned} F_C^{-1A}(t, X^K)&\left(\frac{\partial F^C_B(t, X^K)}{\partial X^D} + \Gamma^C_{ED}(t, X^K) F^E_B(t, X^K)\right) \\ &= F_C^{-1A}(X^K)\left(\frac{\partial F^C_B(X^K)}{\partial X^D} + \Gamma^C_{ED}(X^K) F^E_B(X^K)\right). \end{aligned} \tag{2.14}$$

Since $\mathfrak{g}_{\mathfrak{A}} = \mathfrak{g}_{\mathfrak{A}(t)}$, the identity (2.14) implies, as it should, that $\mathscr{H}^t$ is a structural connection relative to $\mathfrak{A}(t)$. As we shall see, the conditions expressed by (2.11)–(2.14) are important in the derivation of the global field equations of motion for $\mathscr{B}$.

3. Global balance equations

For a thermoelastic body with uniform symmetry the global balance equations of momentum have been derived in [4, Sect. IV]. First, we choose a reference configuration $\boldsymbol{\varkappa} = (X^K)$ and we characterize a motion by the deformation functions $x^i = x^i(t, X^A)$.

Since we shall use a fixed atlas $\mathfrak{A}$ throughout the analysis, for brevity we shall suppress the subscript $\mathfrak{A}$ in what follows. We denote the gradients of the response function G relative to $\mathfrak{A}$ by

$$(3.1)\qquad G^i_{jk}{}^A = \frac{\partial G^i_j(\mathbf{F},\theta,X^K)}{\partial F^k_A},\quad G^i_{j\theta} = \frac{\partial G^i_j(\mathbf{F},\theta,X^K)}{\partial\theta},\quad G^i_{jA} = \frac{\partial G^i_j(\mathbf{F},\theta,X^K)}{\partial X^A}.$$

Then as shown in [4] the global balance equation of momentum takes the form

$$(3.2)\qquad (\tilde{G}^i_{jk}{}^A x^k_{A,B} + \tilde{G}^i_{jB})X^B_i + \tilde{G}^i_{j\theta} g_i + \varrho b_j = \varrho \ddot{x}_j,$$

where x^i_A and X^A_i denote the gradients $\mathbf{x}_*$ and $\mathbf{x}_*^{-1}$ of the deformation $x^i = x^i(t, X^K)$ and the inverse deformation $X^A = X^A(t, x^j)$, i.e.,

$$(3.3)\qquad x^i_A = \frac{\partial x^i(t,X^K)}{\partial X^A},\quad X^A_i = \frac{\partial X^A(t,x^j)}{\partial x^i},$$

and $x^k_{A,B}$ denotes the covariant derivative of x^k_A relative to the connection $\mathscr{H}$, i.e.,

$$(3.4)\qquad x^k_{A,B} = \frac{\partial^2 x^k}{\partial X^A \partial X^B} - \Gamma^C_{AB}\frac{\partial x^k}{\partial X^C},$$

and $\tilde{G}^i_{jk}{}^A$, $\tilde{G}^i_{jB}$, and $\tilde{G}^i_{j\theta}$ are global fields given by the following local formulas:

$$(3.5)\qquad \tilde{G}^i_{jk}{}^A = G^i_{jk}{}^B(\mathbf{x}_*\mathbf{F},\theta,X^K)F^A_B,\quad \tilde{G}^i_{jB} = G^i_{jB}(\mathbf{x}_*\mathbf{F},\theta,X^K),\quad \tilde{G}^i_{j\theta} = G^i_{j\theta}(\mathbf{x}_*\mathbf{F},\theta,X^K),$$

where $\mathbf{F}$, with components F^A_B, denotes the deformation gradient from $\boldsymbol{\mu}$ to $\boldsymbol{\varkappa}$ as shown in (2.11). The formulas in (3.5) are only local because the domain of $\mathbf{F}$ is the subbody $\mathscr{U}$ over which the field $\boldsymbol{\mu}$ is defined, where $(\mathscr{U},\boldsymbol{\mu}) \in \mathfrak{A}$. However, it has been shown in [9] that the values $\tilde{G}^i_{jk}{}^A$, $\tilde{G}^i_{jB}$, and $\tilde{G}^i_{j\theta}$ given by (3.5) are actually independent of the choice of the chart $(\mathscr{U},\boldsymbol{\mu})$ in $\mathfrak{A}$, so that they form global fields on $\mathscr{B}$.

As explained in [10], in order that (3.3) can be applied to an anelastic body $\mathscr{B}$, we must replace the atlas $\mathfrak{A}$ by the atlas $\mathfrak{A}(t)$ and the connection $\mathscr{H}$ by the connection $\mathscr{H}^t$. Thus the balance equations are

$$(3.6)\qquad (\tilde{K}^i_{jk}{}^A x^k_{A|B} + \tilde{K}^i_{jB})X^B_i + \tilde{K}^i_{j\theta} g_i + \varrho b_j = \varrho \ddot{x}_j,$$

where $x^k_{A|B}$ denotes the covariant derivative of x^k_A relative to the connection $\mathscr{H}^t$, and $\tilde{K}^i_{jk}{}^A$, $\tilde{K}^i_{jB}$, and $\tilde{K}^i_{j\theta}$ are given by

$$(3.7)\qquad \tilde{K}^i_{jk}{}^A = G^i_{jk}{}^B(\mathbf{x}_*\boldsymbol{\alpha}\mathbf{F},\theta,X^K)\alpha^A_C F^C_B,\quad \tilde{K}^i_{jB} = G^i_{jB}(\mathbf{x}_*\boldsymbol{\alpha}\mathbf{F},\theta,X^K),$$
$$\tilde{K}^i_{j\theta} = G^i_{j\theta}(\mathbf{x}_*\boldsymbol{\alpha}\mathbf{F},\theta,X^K),$$

where the argument $\mathbf{x}_*\boldsymbol{\alpha}\mathbf{F}$ has components $x^i_A \alpha^A_B F^B_C$, because the deformation gradient $\mathbf{F}(t,\cdot)$ from $\boldsymbol{\mu}(t,\cdot)$ to $\boldsymbol{\varkappa}$ is related to $\mathbf{F}$ by

$$(3.8)\qquad F^A_C(t,X^K) = \alpha^A_B(t,X^K)F^B_C(X^K),$$

cf. [1, Eq. (6.34)]. In [1] we have also proven that the covariant derivatives relative to $\mathscr{H}$ and $\mathscr{H}^t$ are related by

$$(3.9)\qquad \alpha^A_C x^k_{A|B} = (\alpha^A_C x^k_A)_{,B},$$

cf. [1, Eq. (6.41)]. Substituting (3.9) into (3.6), we see that the balance equations can be rewritten as

$$(3.10)\qquad (\tilde{H}^{i}_{jk}{}^{A}(\alpha^{C}_{A}x^{k}_{C})_{,B}+\tilde{K}^{i}_{jB})X^{B}_{i}+\tilde{K}^{i}_{j\theta}g_{i}+\varrho b_{j} = \varrho\ddot{x}_{j},$$

where $\tilde{H}^{i}_{jk}{}^{A}$ is given by the local formula

$$(3.11)\qquad \tilde{H}^{i}_{jk}{}^{A} = G^{i}_{jk}{}^{B}(\mathbf{x}_{*}\boldsymbol{\alpha}\mathbf{F},\theta,X^{K})F^{A}_{B}.$$

We note that (3.10) is exactly the same as (3.2) except that the variable x^{k}_{A} in (3.2) is replaced throughout the equation by the variable $x^{k}_{B}\alpha^{B}_{A}$ in (3.10).

Note. A variant of the balance equations which is based on the response function for the Piola-Kirchhoff stress tensor is given by [4, Eq. (4.28)]; by following the same procedure as used in the derivation of (3.10), we can derive a similar variant of (3.10) here.

Next, we consider the global balance equation of energy. For a thermoelastic body with uniform symmetry, the result is given in [4]. We denote the gradients of the response function l relative to $\mathfrak{A}$ by

$$(3.12)\qquad \begin{aligned} l^{i}{}_{k}{}^{A} &= \frac{\partial l^{i}(\mathbf{F},\theta,\mathbf{g},X^{K})}{\partial F^{k}_{A}}, & l^{i}_{\theta} &= \frac{\partial l^{i}(\mathbf{F},\theta,\mathbf{g},X^{K})}{\partial\theta},\\ l^{ij}_{g} &= \frac{\partial l^{i}(\mathbf{F},\theta,\mathbf{g},X^{K})}{\partial g_{j}}, & l^{i}_{A} &= \frac{\partial l^{i}(\mathbf{F},\theta,\mathbf{g},X^{K})}{\partial X^{A}_{is}}.\end{aligned}$$

Then, as shown in [4, Sect. IV], the global balance equation of energy takes the form

$$(3.13)\qquad (\tilde{l}^{i}{}_{k}{}^{A}x^{k}_{A,B}+\tilde{l}^{i}_{B})X^{B}_{i}+\tilde{l}^{i}_{\theta}g_{i}+\tilde{l}^{ij}_{g}\frac{\partial g_{j}}{\partial x^{i}}+\varrho r = \varrho\theta\eta,$$

where $\tilde{l}^{i}{}_{k}{}^{A}, \tilde{l}^{i}_{B}, \tilde{l}^{i}_{\theta}$, and $\tilde{l}^{ij}_{g}$ are global fields given by the following local formulas:

$$(3.14)\qquad \begin{aligned} \tilde{l}^{i}{}_{k}{}^{A} &= \mathrm{l}^{i}{}_{k}{}^{B}(\mathbf{x}_{*}\mathbf{F},\theta,\mathbf{g},X^{K})F^{A}_{B},\\ \tilde{l}^{ij}_{B} &= l^{i}_{B}(\mathbf{x}_{*}\mathbf{F},\theta,\mathbf{g},X^{K}),\\ \tilde{l}^{i}_{\theta} &= l^{i}_{\theta}(\mathbf{x}_{*}\mathbf{F},\theta,\mathbf{g},X^{K}_{i}),\\ \tilde{l}^{ij}_{g} &= l^{ij}_{g}(\mathbf{x}_{*}F,\theta,\mathbf{g},X^{K}).\end{aligned}$$

To apply the balance equation (3.13) to an anelastic body with uniform symmetry we simply replace $\mathfrak{A}$ by $\mathfrak{A}(t)$ and $\mathscr{H}$ by $\mathscr{H}^{t}$ and the result is

$$(3.15)\qquad (\tilde{m}^{i}{}_{k}{}^{A}_{i}x^{k}_{A|B}+\tilde{m}^{i}_{B})X^{B}_{i_{i}}+\tilde{m}^{i}_{\theta}g_{i}+\tilde{m}^{ij}_{g}\frac{\partial g_{j}}{\partial x^{i}}+\varrho r = \varrho\theta\dot{\eta},$$

where $\tilde{m}^{i}{}_{k}{}^{A}$, $\tilde{m}^{i}_{B}$, $\tilde{m}^{i}_{\theta}$, and $\tilde{m}^{ij}_{g}$ are given by

$$(3.16)\qquad \tilde{m}^{i}{}_{k}{}^{A} = l^{i}{}_{k}{}^{B}(\mathbf{x}_{*}\boldsymbol{\alpha}\mathbf{F},\theta,\mathbf{g},X^{K})\alpha^{A}_{C}F^{C}_{B},\qquad \tilde{m}^{i}_{B} = l^{i}_{B}(\mathbf{x}_{*}\boldsymbol{\alpha}\mathbf{F},\theta,\mathbf{g},X^{K}),$$
$$\tilde{m}^{ij}_{g} = l^{ij}_{g}(\mathbf{x}_{*}\boldsymbol{\alpha}\mathbf{F},\theta,\mathbf{g},X^{K}).$$

Using (3.9), we can rewrite (3.15) as

$$(3.17)\qquad (\tilde{n}^{i}{}_{k}{}^{A}(\alpha^{A}_{B}x^{k}_{B})_{,C}+\tilde{m}^{i}_{C})X^{C}_{i}+\tilde{m}^{i}_{\theta}g_{i}+\tilde{m}^{ij}_{g}\frac{\partial g_{j}}{\partial x^{i}_{i}}+\varrho r = \varrho\theta\dot{\eta},$$

where $\tilde{n}^i{}_k{}^A$ is given by

$$\tilde{n}^i{}_k{}^A = l^i{}_k{}^B(\mathbf{x}_* \boldsymbol{\alpha}\mathbf{F}, \theta, \mathbf{g}, X^K) F^A_B. \tag{3.18}$$

Equation (3.17) is the same as equation (3.13) except that the variable x^i_A of the latter is replaced by the variable $x^i_B \alpha^B_A$ throughout the equation.

Thus far, we have expressed the left-hand side of the balance equation (3.17) in terms of the deformation function and the temperature field. As far as the right-hand side of (3.17) is concerned, we have the following result, cf. [4, Eq. (4.71)]:

$$\dot{\eta} = \tilde{h}^A_j x^i_A \frac{\partial v^j}{\partial x^i} + \tilde{h}_\theta \dot{\theta}, \tag{3.19}$$

where $\tilde{h}^A_j$ and $\tilde{h}_\theta$ are global fields given by the local formulas

$$\tilde{h}^A_j = h^B_j(\mathbf{x}_* \mathbf{F}, \theta, X^K) F^A_B, \qquad \tilde{h}_\theta = h_\theta(\mathbf{x}_* \mathbf{F}, \theta, X^K). \tag{3.20}$$

We remark that h^B_j and h_θ are the gradients of the response function h relative to $\mathfrak{A}$, i.e., locally we have

$$h^B_j = \frac{\partial h(\mathbf{F}, \theta, X^K)}{\partial F^j_B}, \qquad h = \frac{\partial h(\mathbf{F}, \theta, X^K)}{\partial \theta}. \tag{3.21}$$

In order to apply (3.19) to an anelastic body, we replace $\mathfrak{A}$ by $\mathfrak{A}(t)$ as before and obtain

$$\dot{\eta} = \tilde{k}^A_j x^i_B \alpha^A_B \frac{\partial v^j}{\partial x^i} + \tilde{k}_\theta \dot{\theta}, \tag{3.22}$$

where $\tilde{k}^A_j$ and $\tilde{k}_\theta$ are given by

$$\tilde{k}^A_j = h^B_j(\mathbf{x}_* \boldsymbol{\alpha}\mathbf{F}, \theta, X^K) F^A_B, \qquad \tilde{k}_\theta = h_\theta(\mathbf{x}_* \boldsymbol{\alpha}\mathbf{F}, \theta, X^K). \tag{3.23}$$

Thus the global balance equation of energy for an anelastic body with uniform symmetry has the explicit form:

$$(\tilde{\eta}^i{}_k{}^A_i(\alpha^B_A x^k_B)_{,C} + \tilde{m}^i_C) X^C_i + \tilde{m}^i_\theta g_i + \tilde{m}^{ij}_g \frac{\partial g_j}{\partial x^i} + \varrho r = \varrho\theta \left(\tilde{k}^A_j \alpha^B_A x^i_B \frac{\partial v^j}{\partial x^i} + \tilde{k}_\theta \dot{\theta} \right). \tag{3.24}$$

Note. A variant of (3.24), based upon use of the response function for the heat flux vector relative to the reference configuration $\varkappa$, can be derived from the result [4, Eq. (4.58)] for thermoelastic bodies in a manner entirely similar to that which is used in deriving the variant of (3.10) which is based on use of the Piola-Kirchhoff stress tensor.

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