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On a certain general formulation of the fracture criterion in solids

V. N. NIKOLAEVSKII (MOSKOW)

THE solid body containing pores and cracks and deforming in a process of loading with or without fracture is considered. The criterion of stationary fracture is formulated as a condition of a dependence of the energy flux (increase of entropy) in a controlled region of a body solely on the crack propagation velocity and on the rheological properties of a material. From this condition the known criteria of fracture for brittle, plastic and viscous bodies are obtained as the particular cases.

Rozważono ośrodek stały, zawierający pory i szczeliny odkształcające się w procesie obciążenia z udziałem lub bez udziału pękania. Kryterium stacjonarnego pękania sformułowano jako warunek zależności strumienia energii (wzrostu entropii) w kontrolnym obszarze ciała wyłącznie od prędkości propagacji szczeliny i od reologicznych właściwości materiału. Jako przypadki szczególne otrzymuje się z tego warunku znane kryteria pękania dla ciał kruchych, plastycznych i lepkich.

Рассмотрена твердая среда содержащая поры и трещины, деформирующиеся в процессе нагружения при наличии или без участия разрушения. Критерий стационарного разрушения сформулирован как условие зависимости потока энергии (роста энтропии) в контрольной области тела исключительно от скорости распространения трещины и от реологических свойств материала. Как частные случаи из этого условия получаются известные критерии разрушения для хрупких, пластических и вязких тел.

EQUATIONS of balance are constructed for a material containing voids and cracks. The cases of void (crack) growth without or with fracture are considered separately. In the latter case the normal velocities of propagation of a free surface and that of a particle at the point of fracture differ by the crack propagation velocity. Corresponding expressions for the flows of mass, momentum, energy and entropy at the fracture point are written; to that end, the singular point is surrounded by a suitable control surface. The stationary fracture criterion is formulated as a criterion of autonomy — the energy release (increase of entropy) in the control volume depending solely on the crack velocity and on the rheological properties of the body. Brittle (Griffith type) fracture corresponds to the energy outflow through the singular point at the expense of the internal energy variation of the particles located inside and on the surface of the volume. In the cases of plastic or viscous fracture, leading role is played by the energy dissipation within the control volume. In the plastic case the dissipation function is a first-order homogeneous function, and in the viscous case — a second-order function of the strain rates. Therefore, in the case of perfect plasticity, the energy outflow is proportional to the first power of the crack propagation velocity (Irvin-Orowan criterion), while in the viscous case — to its second power (Kachanov's criterion).

1. Balance equations

It is assumed that at every point of the continuous medium considered, the following equations of conservation of mass, momentum, complete energy and entropy are satisfied:

$$
\begin{gathered}
\frac{\partial \varrho}{\partial t}+\frac{\partial \varrho v_j}{\partial x_j}=0, \quad \frac{\partial \varrho v_i}{\partial t}+\frac{\partial \varrho v_i v_j}{\partial x_j}=\frac{\partial t_{ij}}{\partial x_j}+\varrho \mathscr{F}_i, \\
\frac{\partial}{\partial t} \varrho\left(\varepsilon+\frac{v_i^2}{2}\right)+\frac{\partial}{\partial x_j} \varrho\left(\varepsilon+\frac{v_i^2}{2}\right) v_j=\frac{\partial t_{ij} v_i}{\partial x_j}+\frac{\partial q_j}{\partial x_j}+\varrho Q+\varrho \mathscr{F}_i v_i, \\
\frac{\partial}{\partial t} \varrho \,\text{э}+\frac{\partial}{\partial x_j} \varrho \,\text{э}\, v_j=\frac{1}{T} t_{ij} \dot{e}_{ij}^p+\frac{1}{T} \varrho Q+\frac{1}{T} \frac{\partial q_i}{\partial x_j}+\frac{\dot{h}}{T} .
\end{gathered} \tag{1.1}
$$

Here ϱ — density of the continuous material, v_i — velocity of displacement, t_{ij} — stress, $\mathscr{F}_i$ — body force, ε — specific internal energy of the continuous material, q_j — heat flux, Q — body heat source, э — specific entropy, T — temperature, h — power of irreversible work, $\dot{e}_{ij}=(\partial v_i/\partial x_j+\partial v_j/\partial x_i)/2$ — velocity of total deformation, $e_{ij}=e_{ij}^e+ +e_{ij}^p$ — total deformation consisting of the elastic e_{ij}^e and irreversible e_{ij}^p deformation.

Let us integrate the system (1.1) over the volume $V=(1-m)V+mV$ bounded by a fixed surface S with a true area $S(1-n)$ and divided into the matrix and the voids of the surface S_n. Here m — porosity, n — "transparency" of the medium; we assume, as usual, that $n=m$. Then

$$
\begin{gathered}
\int_{(1-m)V} \frac{\partial}{\partial t} \varrho dV+\int_{(1-m)S} \varrho v_n dS_n+M=0, \\
\int_{(1-m)V} \frac{\partial}{\partial t} \varrho v_i dV+\int_{(1-m)S} \varrho v_i v_n dS_n+W=\int_{(1-m)V} \varrho \mathscr{F}_i dV+\int_{(1-m)S} t_{in} dS_n, \\
\int_{(1-m)V} \frac{\partial}{\partial t} \varrho\left(\varepsilon+\frac{v_i^2}{2}\right) dV+\int_{(1-m)S} \varrho\left(\varepsilon+\frac{v_i^2}{2}\right) v_n dS_n+E=\int_{(1-m)V} \varrho \mathscr{F}_i v_i dV \\
+\int_{(1-m)S} t_{in} v_i dS_n+\int_{(1-m)V} \varrho Q dV+\int_{(1-m)S} q_n dS_n, \\
\int_{(1-m)V} \frac{\partial}{\partial t} \varrho \,\text{э}\, dV+\int_{(1-m)V} \varrho \,\text{э}\, v_n dS_n+\text{Э}=\int_{(1-m)V} \frac{t_{ij} \dot{e}_{ij}^p}{T} dV+\int_{(1-m)V} \frac{Q}{T} dV \\
+\int_{(1-m)S} \frac{q_n}{T} dS_n+\int_{(1-m)V} \frac{\dot{h}}{T} dV-\int_{(1-m)V} \frac{q_n}{T} \frac{\partial T}{\partial x_n} dV .
\end{gathered} \tag{1.2}
$$

Here the following identity has been used which holds true for a volume $V(t)$ bounded by surface S propagating at the velocity W_n along the normal n, and for an arbitrary function ψ:

$$
\int_{V(t)} \frac{\partial \psi}{\partial t} dV=\frac{\partial}{\partial t} \int_{V(t)} \psi dV-\int_S \psi w_n dS_n ,
$$

and M, W, E, Э denote the flows of mass, momentum, energy and entropy at the surface S_m of the voids (cracks):

$$M = \int_{S_m} \varrho(v_n - w_n)dS_n, \quad W = \int_{S_m} \{\varrho v_i(v_n - w_n) - t_{in}\} dS_n,$$

$$E = \int_{S_m} \left\{ \varrho\left(\varepsilon + \frac{v_i^2}{2}\right)(v_n - w_n) - t_{in}v_i - q_n \right\} dS_n, \tag{1.3}$$

$$\ni = \int_{S_m} \left\{ \varrho \ni (v_n - w_n) - \frac{q_n}{T} \right\} dS_n + \ni_m.$$

Here $\ni_m$ — certain additional entropy source on the surface S_m.

It is seen that the outflow of mass, part of the impulse, energy and entropy is determined by the difference of velocities $v_n - w_n$ of the particles, v_n, and of the points at the surface of voids, w_n. This difference is non-zero in the case of mass sinks at the void surfaces (e.g. transition to the gaseous state). If the voids are deformed in such a manner that their surfaces consist of the same material particles, then for these particles we obtain $v_n = w_n$. This kind of deformation (including irreversible phenomena) is not accompanied by fracture of the material of the matrix. If, however, at a certain point α of the surface S_m of the voids $v_n - w_n \neq 0$, then this is the point at which fracture occurs; the velocities of motion of the void boundary and of the particle located at the given instant on that boundary are different. At a following instant another particle appears on the boundary — a particle which originally belonged to the voluminal phase of the matrix material. If the fracture point is isolated, a crack will be propagated from the void surface into the volume, the difference $\dot{l} = v_n - w_n$ representing the normal component of the crack growth velocity (motion of the crack tip) measured with respect to the material of the medium.

Due to the stress concentration phenomena, the stresses at isolated points may exhibit a singular character and so the work done by them on the corresponding displacements is, in general, different from zero. If such points are excluded from the continuum, they should be replaced by concentrated forces (and concentrated couples, but then the Eqs. (1.2) must be completed by the condition of balance of moment of momentum). If the isolated points are not excluded from the continuum, then those forces (and couples) should be considered as internal forces.

At the portions of surfaces of the voids (cracks) which are free of loading the normal stress components t_{nn} vanish. If the opposite crack boundaries touch each other, then the stresses play the role of internal forces, their values at the surfaces in contact being equal but their directions — opposite. If the velocities of the conterminous crack boundaries are different, then the work done by the boundary stresses on the velocity jump $t_{\tau n}[v_\tau]$ will be different from zero and will contribute to the integrand for E. In the cases of contact of rigid surfaces, the tangential displacement jump and the contact forces are usually connected by means of the rheological boundary Coulomb law; the work $t_{\tau n}[v_\tau]$ is dissipated and plays the role of an additional heat source of intensity equal to the area of the hysteresis loops characteristic for the dry friction law.

The existence of such surface heat sources corresponds to the action of additional entropy sources $\ni_m$ located on the internal surfaces (discontinuities). In that case

$$\ni_m = \int_{S_m} \frac{1}{T} t_{\tau n}[v_\tau] dS_n.$$

2. Control volume and the criterion of fracture

The fracture process occurring at a point is actually determined by the state of stress and strain in its neighbourhood which will be called the concentration zone. The zone is small in comparison with the dimension of the entire body. Let us write the balance equations for the control volume V_α bounded by the surface S_α containing the fracture point α and the concentration zone. Then the integrals (1.3) are reduced to $\sum_\alpha S_\alpha$ — the sum of surfaces of the control volumes V_α containing the energy sinks which, on the other hand, may be estimated by means of the dissipative processes occurring within the concentration zone.

Following GIBBS [1] let us introduce the surface phase and assume the particles belonging to that phase to be characterized by the internal energy ε_m different from the internal energy ε_0 of particles located inside the body. Difference of these two energies $\varepsilon_0-\varepsilon_m$ obtained in passing with the particle to the surface corresponds, in the integral sense, to the change of the energy of cohesion (an additive constant in the elastic potential of perfectly elastic bodies) according to the approach by L. I. SEDOV [2]. The energy change corresponds also to the entropy variation provided the changes of particle deformation due to the phase transition are relatively small,

$$\text{Э}_0-\text{Э}_m = \frac{1}{T}(\varepsilon_0-\varepsilon_m).$$

Let us consider a two-dimensional case; the control volume (now a control area S_α) for a two-dimensional crack will be selected in the form of a rectangle with the contour $L_\alpha = ABCD$ $(AB = 2b \gg BC = 2a)$ moving in parallel at the velocity $v_n-w_n = \dot{l}$

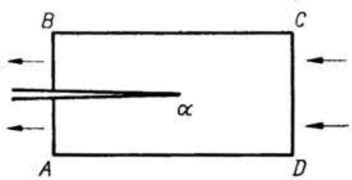

FIG. 1.

of propagation of the fracture point. The particles with specific energy ε_0 flow into the control volume S_α through the side CD, while the particles with energy ε_m flow out of that region (across the side BA). For the sake of simplicity the secondary flow of energy across the sides $BC+AD$ will be assumed to consist of particles with energy ε_0. Expression for the dissipation $\mathscr{D}_\alpha$ of energy in the volume S_α takes now the form

$$(2.1)\qquad \mathscr{D}_\alpha = \int_{S_\alpha} t_{ij}\dot{e}^p_{ij}\,dS = \Big(\int_{AB}\varrho\varepsilon_m\,dS - \int_{CD}\varrho\varepsilon_0\,dS\Big)\dot{l}$$
$$+\int_{BC+AD}\varrho\varepsilon_0(v_n-w_n)\,dL_n + \int_{L_\alpha}\varrho\frac{v_i^2}{2}(v_n-w_n)\,dL_n - \int_{L_\alpha}t_{in}v_i\,dL - \int_{L_\alpha}q_n\,dL_n.$$

However, in the equations of balance (1.2) and in expressions (1.3) only the internal energy of voluminal phase $\varepsilon \equiv \varepsilon_0$ is used, and in order to apply the equations of balance correctly we should introduce the following expression for the energy outflow E_α:

$$(2.2)\qquad E_\alpha = \mathcal{D}_\alpha + \dot{l}_\alpha \int\limits_{AB} \varrho(\varepsilon_0 - \varepsilon_m)\,dL$$
$$= \int\limits_{L_\alpha} \varrho\left(\varepsilon_0 + \frac{v_i^2}{2}\right)(v_n - w_n)\,dL_n - \int\limits_{L_\alpha} t_{in} v_i\,dL_n - \varrho_{L_\alpha} q_n\,dL_n.$$

The left-hand side expressions in (2.2) characterize the fracture process locally (inside the control volume — within the concentration zone and at the fracture point α), while the right-hand expressions represent the kinetic and potential energy losses as also the work done by the stresses in the medium. The corresponding entropy source may be written in the form

$$(2.3)\qquad \Theta_\alpha = \frac{1}{T}\mathcal{D}_\alpha + \frac{1}{T}\dot{l}_\alpha \int\limits_{AB} \varrho(\varepsilon_0 - \varepsilon_m)\,dL$$
$$= \frac{1}{T}\int\limits_{L_\alpha} \varrho\left(\varepsilon_0 + \frac{v_i^2}{2}\right)(v_n - w_n)\,dL - \frac{1}{T}\int\limits_{L_\alpha} t_{in} v_i\,dL - \frac{1}{T}\int\limits_{L_\alpha} q_n\,dL.$$

The fracture criterion of the body at a point α is now formulated as the condition of balance between the energy outflow E_α at that point [cf. the right-hand side of Eq. (2.3)] and the entropy source Θ_α corresponding to the autonomy of the process in the control volume [integral relations (2.3) of that process with the deformation of the body outside V_α].

Consequently, the source Θ_α will be assumed to be determined by the rheological properties of the body, by the crack propagation velocity $\dot{l}_\alpha$ and by the characteristic dimensions of the concentration zone (area S_α). The autonomy introduced here is possible provided the crack length l_α is much greater than $\sqrt{S_\alpha}$.

Let us start with the case of a perfectly elastic body in which the dissipation $\mathcal{D}_\alpha = 0$. If the limit

$$(2.4)\qquad \lim_{a\to 0} 2\varrho(\varepsilon_0 - \varepsilon_m)a = 2\gamma, \qquad [\gamma] = \text{dyne/cm}$$

exists, then γ may be interpreted as the energy used for creating a unit free surface, and with $\mathcal{D}_\alpha = 0$, $a \to 0$ condition (2.3) corresponds to the criterion of brittle fracture due to GRIFFITH [3] in the form used in [4, 5]. The coefficient γ may be variable due to the medium non-homogeneity and thermal or chemical effects.

It is known [2] that in dissipative media the work lost in the fracture process is spent mainly on proper dissipation $\mathcal{D}_\alpha$ rather than the surface energy γ. In a plastic body the dissipative function is homogeneous in the strain rates, i.e., it is proportional to the term $(\dot{e}^p_{ij}\dot{e}^p_{ij})^{1/2}$. By estimating this term

$$(\dot{e}^p_{ij}\dot{e}^p_{ij})^{1/2} \sim \dot{l}_\alpha S_\alpha^{-1/2},$$

we obtain the final expression for the dissipation $\mathcal{D}_\alpha$:

$$(2.5)\qquad \mathcal{D}_\alpha = kS_\alpha(\dot{e}^p_{ij}\dot{e}^p_{ij})^{1/2} = \gamma_*\dot{l}_\alpha, \qquad \gamma_* = k\sqrt{S_\alpha}, \qquad [\gamma_*] = \text{dyne/cm}$$

and if the plastic zone at the crack is stationary ($\gamma_* = \text{const}$), the fracture condition (2.3) takes the form of the Irvin-Orowan quasi-brittle fracture criterion [6, 7].

In viscous bodies, however, the dissipation function is a second order homogeneous function of the strain rates, i.e., it is proportional to the product $\dot{e}^p_{ij}\dot{e}^p_{ij}$. The corresponding estimate

$$\dot{e}^p_{ij}\dot{e}^p_{ij} \sim \dot{l}^2_\alpha S^{-1}_\alpha$$

enables us to write the formula for dissipation $\mathcal{D}_\alpha$,

$$\mathcal{D}_\alpha = \eta S_\alpha \dot{e}^p_{ij}\dot{e}^p_{ij} = \eta \dot{l}^2_\alpha, \quad [\eta] = \mathrm{g(cm\,s)^{-1}}, \tag{2.6}$$

i.e., the dimension of η coincides with the dimension of dynamic viscosity coefficient. In that case the fracture condition (2.3) takes the form of the dissipative fracture criterion given by KACHANOV [8, 9] (which is alternative to the condition of the type (2.5) generalized [5, 10, 11] to the case of viscous bodies). L. M. KACHANOV calls η the "strength coefficient".

It is seen that the irreversible work of deformation in the neighbourhood of the crack tip cannot be always reduced to the concept of effective surface energy. The assumption [10] should also be mentioned stating that "the condition of local fracture corresponds to the entropy density attaining a critical value". In the present paper the criterion of grack growth (local fracture) is shown to require the time rate of entropy to assume a critical value.

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