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The basic equations for a grade 2 material viewed as an oriented continuum

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THE field equations for a hyperelastic material of grade 2 are derived from a variational principle for an oriented, hyperelastic (Toupin) material having at each point three linearly independent directors. It is demonstrated that when the directors are constrained to deform with the local deformation of the continuum, the oriented material is equivalent to a material of grade 2; and on this hypothesis the basic equations are obtained from the variational principle. Although the development and form of the results differ from those provided by TOUPIN and others, it is shown that the equations derived here are equivalent to those presented elsewhere, yet the present formulation seems easier to use and interpret in familiar physical terms.

Równania pola dla hipersprężystego materiału stopnia 2 wyprowadzono z zasady wariacyjnej dla zorientowanego hipersprężystego materiału (Toupina), posiadającego w każdym punkcie trzy liniowo niezależne kierunki. Pokazano, że gdy kierunki te przy ograniczeniu więzami odkształcają się wraz z lokalną deformacją ośrodka ciągłego, zorientowany materiał jest równoważny materiałowi stopnia 2. Wychodząc z tej hipotezy równania podstawowe otrzymano z zasady wariacyjnej. Chociaż metoda postępowania i wynik końcowy różnią się od rezultatów otrzymanych przez TOUPINA i innych autorów, wykazano, że wyprowadzone tu równania są równoważne równaniom przedstawionym w innych pracach, a ponadto wydają się być łatwiejsze w zastosowaniu i interpretacji przez znane terminy fizyczne.

Уравнения поля для гиперупругого материала второй степени выведены из вариационного принципа для ориентированного гиперупругого материала (Тупина), обладающего в каждой точке тремя линейно независимыми направлениями. Показано, что эти направления, из-за ограничения связями, деформируются совместно с локальной деформацией сплошной среды; ориентированный материал эквивалентен материалу второй степени. Исходя из этой гипотезы основные уравнения получены из вариационного принципа. Хотя метод поступания и заключительный результат отличаются от результатов полученных ТИПИНЫМ и другими авторами доказано, что выведенные здесь уравнения эквивалентны уравнениям представленным в других работах и кроме этого они кажутся быть более легкими в применении и в интерпретации в известных физических терминах.

1. Introduction

IN 1964, TOUPIN [1] developed a deep and comprehensive general theory of hyperelastic materials in which couple stresses and material directors are a central consideration, his principal objective being to expose the basic concepts and principles of continuum mechanics common to several mathematical models of such materials. We have observed, however, that nowhere in his elegant construction for materials of grade n, and specifically for materials of grade 2, does he relate this theory to the general developments for materials characterized by a set of deformable directors. Therefore, we are led to question the possible connection between the director-oriented and non-simple, grade 2 continua.

We recall that a Cosserat continuum can be characterized by a rigid director triad that is free to rotate relative to the local rigid rotation of the continuum, and a constrained

Cosserat continuum is characterized by the additional restriction that the director triad rotate with the local rigid body rotation of the material. Of course, in general the directors need not be rigid and they need not be restricted to three in number, so other interesting situations are conceivable. In particular, we recall that in classical molecular theories of elasticity of perfect crystals the directors are identified with the lattice vectors, which are assumed to deform with the continuum. In plastic deformation, on the other hand, it seems more appropriate to consider the director motion and the particle motion as independent, and thus regard an imperfect crystal as an oriented material, possibly characterized in part by the Burger's vector as a director measure of imperfection. Motivated by the perfect crystal model, we herein wish to consider the case when the director triad is constrained to deform with the local deformation of the continuum, whatever it may be. We show that a certain hyperelastic oriented continuum having this property reduces to a material of grade 2, and we derive the principal equations that follow from this definition. It turns out that our equations for materials of grade 2 assume a form different from those provided by TOUPIN [1, § 10], who developed this theory along quite a different direction; nevertheless, we show easily that our equations are in fact equivalent to those determined in [1]. Therefore, we feel that our results, which seem easier to use and interpret in physical terms, shed considerable light upon the basis for the very elegant though deceptively simple assumed principle of virtual work used there, and some further remarks about this are provided at the end.

2. Toupin materials

A motion of a body $\mathfrak{P}$ is defined by a mapping

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \quad \mathbf{X} = \boldsymbol{\varkappa}(X), \tag{2.1}$$

wherein $\mathbf{x}$ denotes the place in the configuration χ at time t that is occupied by the particle X whose place was $\mathbf{X}$ in some assigned reference configuration $\boldsymbol{\varkappa}$ initially. In addition, we shall assume that to each particle X three independent vector fields $\mathbf{d}_I$, $I = 1, 2, 3$, called directors, also are assigned. The director motion is similarly characterized by

$$\mathbf{d}_I = \mathbf{d}_I(\mathbf{X}, t), \quad \mathbf{D}_I = \mathbf{D}_I(\mathbf{X}), \tag{2.2}$$

in which $\mathbf{D}_I$ are initially the values of the directors in the reference configuration.

Following TOUPIN [1], we introduce a function

$$L(\mathbf{X}, \tau) = L^*(\mathbf{X}, \mathbf{x}, \mathbf{d}_I\, \dot{\mathbf{x}}, \dot{\mathbf{d}}_I, \mathbf{F}, \mathbf{W}_I, \tau) \tag{2.3}$$

called the action density. Here $\mathbf{F} = \nabla\mathbf{x}$, $\mathbf{W}_I = \nabla\mathbf{d}_I$, $\nabla \equiv \partial/\partial\mathbf{X}$, τ is time and the superposed dot denotes the material time derivative. A material whose action density has the form (2.3) is called a *hyperelastic Toupin material*. The equations of motion and boundary conditions for such materials are obtained from Hamilton's principle in the form of the variational equation

$$\delta \int_{\mathcal{I}}\int_{\mathcal{P}} L(\mathbf{X}, \tau)\, dV d\tau + \int_{\mathcal{I}}\int_{\mathcal{P}} (\mathbf{Z}\cdot\delta\mathbf{x} + \mathbf{G}^I\cdot\delta\mathbf{d}_I)\, dV d\tau + \int_{\mathcal{I}}\int_{\partial\mathcal{P}} (\mathbf{T}_N\cdot\delta\mathbf{x} + \mathbf{H}^I_N\cdot\delta\mathbf{d}_I)\, dA d\tau - \int_{\mathcal{P}} (\mathbf{P}\cdot\delta\mathbf{x} + \mathbf{Q}^I\cdot\delta\mathbf{d}_I)\, dV \Big|_{t_0}^{t} = 0, \tag{2.4}$$

wherein $\mathscr{P} \subset \mathfrak{P}$ with elemental volume dV and elemental surface area dA in $\varkappa$, $\partial\mathscr{P}$ is the boundary of $\mathscr{P}$ with unit normal $\mathbf{N}$ in $\varkappa$, Z and $\mathbf{G}^I$ are certain generalized body forces, $\mathbf{T}_N$ and $\mathbf{H}_N^I$ denote certain generalized surface tractions, $\mathbf{P}$ and $\mathbf{Q}^I$ are certain generalized momenta, and $\mathscr{I} = [t_0, t]$. Neither $\mathscr{P}$ nor $\mathscr{I}$ are to be varied, and the variations $\delta\mathbf{x}$ and $\delta\mathbf{d}_I$ are to vanish at t_0 and t. In the above and all subsequent relations the diagonally repeated indices are to be summed over the range of three as usual. TOUPIN [1] thus derives from (2.3) and (2.4) differential equations of the form

$$\text{Div}\,\boldsymbol{\Sigma}+\mathbf{Z} = \dot{\mathbf{P}}+\mathbf{A}, \tag{2.5a}$$

$$\text{Div}\,\mathbf{H}^I+\mathbf{G}^I = \dot{\mathbf{Q}}^I+\mathbf{B}^I \quad \text{for} \quad \text{all}\, X \in \mathscr{P},\ \tau \in \mathscr{I}, \tag{2.5b}$$

and natural boundary conditions of the form

$$\mathbf{T}_N = \boldsymbol{\Sigma}\mathbf{N}, \quad \mathbf{H}_N^I = \mathbf{H}^I\mathbf{N} \quad \text{for} \quad X \in \partial\mathscr{P},\ \tau \in \mathscr{I}, \tag{2.6}$$

where Div is the divergence operator with respect to $\mathbf{X}$ in $\varkappa$ and

$$\boldsymbol{\Sigma} = -\frac{\partial L^*}{\partial \mathbf{F}}, \quad \mathbf{P} = \frac{\partial L^*}{\partial \dot{\mathbf{x}}}, \quad \mathbf{A} = -\frac{\partial L^*}{\partial \mathbf{x}}, \tag{2.7a}$$

$$\mathbf{H}^I = -\frac{\partial L^*}{\partial \mathbf{W}_I}, \quad \mathbf{Q}^I = \frac{\partial L^*}{\partial \dot{\mathbf{d}}_I}, \quad \mathbf{B}^I = -\frac{\partial L^*}{\partial \mathbf{d}_I}. \tag{2.7b}$$

Equations (2.5)–(2.7) coincide with those derived also by GREEN and RIVLIN [2, Eqs. (2.15), (2.18), (2.23), (3.2)] for a slightly more special form of (2.3). Their development is based upon a rate of work equation, an entropy production inequality, and certain assumed invariance conditions. They show, based on a comparison of the structure of the two theories, that their earlier multipolar theory can be considered as a special case of the director theory in which the $3n$ scalar components of n director fields are identified as the scalar components of n multipolar deformation fields. Of course, the deformation gradients of this theory do not coincide with the multipolar deformations generally, and there is no mention in [2] of circumstances when they do. In particular, the theory of grade 2 materials is not discussed there.

3. Hyperelastic materials of grade 2

A hyperelastic material whose response depends on no more than n material gradients of (2.1) is called a hyperelastic material of grade n. In particular, a *hyperelastic material of grade* 2 is characterized by the action density

$$L(\mathbf{X}, \tau) = L(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{F}, \dot{\mathbf{F}}, \nabla\mathbf{F}, \tau). \tag{3.1}$$

We wish to relate (3.1) to our earlier remarks on Toupin materials.

Let us suppose that in every motion of a hyperelastic Toupin material the directors behave like infinitesimal material line elements at each point of $\mathfrak{P}$, i.e., like the lattice vectors of a perfect crystal as remarked earlier. Then

$$\mathbf{d}_I(\mathbf{X}, \tau) = \mathbf{F}(\mathbf{X}, \tau)\mathbf{D}_I(\mathbf{X}), \tag{3.2}$$

and (2.3) can be rewritten as

$$L(\mathbf{X}, \tau) = L^*(\mathbf{X}, \mathbf{x}, \mathbf{F}\mathbf{D}_I, \dot{\mathbf{x}}, \dot{\mathbf{F}}\mathbf{D}_I, \mathbf{F}, \nabla(\mathbf{F}\mathbf{D}_I), \tau) = L(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{F}, \dot{\mathbf{F}}, \nabla\mathbf{F}, \tau),$$

which has the form (3.1). We thus discover that a hyperelastic Toupin material in which the directors are constrained to stretch and rotate in harmony with the local stretch and local rigid body rotation of material line elements of the continuum is a grade 2 material. In view of this fact we perceive, based upon (2.4), that the variational principle for hyperelastic materials of grade 2 has the form

$$(3.3)\quad \delta \int_{\mathcal{I}}\int_{\mathcal{P}} L(\mathbf{X}, \tau)\,dV d\tau + \int_{\mathcal{I}}\int_{\mathcal{P}} (\mathbf{Z}\cdot\delta\mathbf{x}+\mathbf{G}\cdot\delta\mathbf{F})\,dV d\tau + \int_{\mathcal{I}}\int_{\partial\mathcal{P}} (T_N\cdot\delta\mathbf{x}+\mathbf{H}_N\cdot\delta\mathbf{F})\,dA d\tau - \int_{\mathcal{P}} (\mathbf{P}\cdot\delta\mathbf{x}+\mathbf{Q}\cdot\delta\mathbf{F})\,dV\Big|_{t_0}^{t} = 0,$$

wherein the action is given by (3.1) and

$$(3.4)\qquad \mathbf{G} \equiv \mathbf{G}^I\otimes\mathbf{D}_I, \quad \mathbf{H}_N \equiv \mathbf{H}_N^I\otimes\mathbf{D}_I, \quad \mathbf{Q} \equiv \mathbf{Q}^I\otimes\mathbf{D}_I$$

are certain second-rank, two point tensor fields of the general form $\mathbf{S} = S_i^\alpha \mathbf{e}^i\otimes\mathbf{E}_\alpha$. These are called the *hyper-body force*, the *hyper-traction*, and the *hyper-momentum*, respectively. Of course, in view of (3.4) it is no longer necessary to refer to the director fields at all; on the other hand, equations (2.5)–(2.7) require further modification due to the constraint (3.2). We shall derive the differential equations and boundary conditions from (3.3).

For the action density (3.1) we find with the help of the divergence theorem and following some lengthy calculations that our variational principle takes the form

$$(3.5)\quad \int_{\mathcal{I}}\int_{\mathcal{P}} \{\mathrm{Div}\,\mathbf{T}^*+\mathbf{Z}-\dot{\mathbf{P}}^*-\mathbf{A}\}\cdot\delta\mathbf{x}\,dV d\tau + \int_{\mathcal{I}}\int_{\partial\mathcal{P}} \{\mathbf{T}_N-\mathbf{T}^*\mathbf{N}\}\cdot\delta\mathbf{x}\,dA d\tau + \int_{\mathcal{I}}\int_{\partial\mathcal{P}} \mathbf{\Phi}\cdot\delta\mathbf{F}\,dA d\tau + \int_{\mathcal{P}} \{\Delta\mathbf{P}-\mathrm{Div}\,\Delta\mathbf{Q}\}\cdot\delta\mathbf{x}\,dV\Big|_{t_0}^{t} + \int_{\partial\mathcal{P}} \Delta\mathbf{Q}\mathbf{N}\cdot\delta\mathbf{x}\,dA\Big|_{t_0}^{t} = 0,$$

wherein, for convenience, we have written

$$(3.6a)\qquad \mathbf{T}^* \equiv \mathbf{T}-\mathbf{G}+\dot{\mathbf{Q}}^*, \quad \Delta\mathbf{P} \equiv \mathbf{P}^*-\mathbf{P}, \quad \Delta\mathbf{Q} \equiv \mathbf{Q}^*-\mathbf{Q},$$

and more permanently, by definition,

$$(3.6b)\qquad \mathbf{T} = -\left(\frac{\partial L}{\partial \mathbf{F}}+\mathrm{Div}\,\mathbf{H}\right), \quad \mathbf{H} = -\frac{\partial L}{\partial \nabla\mathbf{F}},$$

$$(3.6c)\qquad \mathbf{P}^* = \frac{\partial L}{\partial \dot{\mathbf{x}}}, \quad \mathbf{Q}^* = \frac{\partial L}{\partial \dot{\mathbf{F}}}, \quad \mathbf{A} = -\frac{\partial L}{\partial \mathbf{x}}, \quad \mathbf{\Phi} = \mathbf{H}_N-\mathbf{H}\mathbf{N}.$$

To finish the analysis we must recall that on $\partial\mathcal{P}$ the variations $\delta\mathbf{x}$ and $\delta\mathbf{F}$ are not independent, rather

$$(3.7)\qquad \delta\mathbf{F} \equiv \mathrm{Grad}\,\delta\mathbf{x} = \mathbf{D}\delta\mathbf{x}+D(\delta\mathbf{x})\otimes\mathbf{N},$$

where D denotes the normal derivative and $\mathbf{D}$ is the surface gradient operator [cf. 3, 4]. Since we may write (3.5) in terms of the independent variations $\delta\mathbf{x}$ and $D\delta\mathbf{x}$, we can introduce a general two point tensor field $\mathbf{U} = U_i^\alpha\mathbf{e}^i\otimes\mathbf{E}_\alpha$, and with (3.7) we find

$$\mathbf{U}\cdot\delta\mathbf{F} = U_i^\alpha\,\delta x^i{}_{.\alpha} = U_i^\alpha(D_\alpha\delta x^i+N_\alpha D\delta x^i) = D_\alpha(U_i^\alpha\,\delta x^i)-(D_\alpha U_i^\alpha)\,\delta x^i+(U_i^\alpha N_\alpha)D\delta x^i,$$

which serves also to assist the reader with our notation. We may write

$$\mathbf{U}\cdot\delta\mathbf{F} = \mathbf{D}\cdot(\mathbf{U}^T\delta\mathbf{x}) - (\mathbf{D}\cdot\mathbf{U})\cdot\delta\mathbf{x} + (\mathbf{U}\mathbf{N})\cdot D\delta\mathbf{x}, \tag{3.8}$$

in which the transpose is noted as usual. In addition, $\mathbf{N}$ being a unit normal vector,

$$\frac{1}{2}\mathbf{D}(\mathbf{N}\cdot\mathbf{N}) = -\mathbf{B}\mathbf{N} = \mathbf{0}, \quad \text{where} \quad \mathbf{B} \equiv -\mathbf{D}\mathbf{N} \tag{3.9}$$

is the symmetric second fundamental form for $\partial\mathscr{P}$ in $\varkappa$; thus

$$D_\alpha(U_i^\alpha\delta x^i) = D_\alpha(N^\beta U_i^\alpha \delta x^i)N_\beta. \tag{3.10}$$

We recall too that $\operatorname{tr}\mathbf{B} = -\mathbf{D}\cdot\mathbf{N} = -D_\alpha N^\alpha$. Further, for any 2-tensor $\mathbf{V}$, Toupin's integral identity [3, 5] is given by(1)

$$\int\limits_{\partial\mathscr{P}} (\mathbf{N}\otimes\mathbf{D})\cdot\mathbf{V}^T dA = \int\limits_{\mathscr{P}} \mathbf{V}\cdot(\mathbf{B}-\mathbf{N}\otimes\mathbf{N}\operatorname{tr}\mathbf{B})dA + \int\limits_{\mathscr{C}} [\mathbf{V}\cdot(\mathbf{M}\otimes\mathbf{N})]dl, \tag{3.11}$$

wherein $(\mathbf{N}\otimes\mathbf{D})\cdot\mathbf{V}^T = N_\alpha D_\beta V^{\beta\alpha}$ and $\mathbf{M} = \mathbf{S}\times\mathbf{N}$ with $\mathbf{S}$ denoting the usual continuous, unit tangent vector to the boundary edges $\mathscr{C}$ in $\varkappa$. Also, $[K]$ denotes the jump in the entity K as a given point on an edge is approached from each side. Of course if $\partial\mathscr{P}$ has no edges $\mathscr{C}$, and K is smooth throughout $\mathscr{P}$, the line integral will vanish. Thus, with (3.7)–(3.10) and application of Toupin's integral identity (3.11) with $\mathbf{V} = \mathbf{N}\otimes\mathbf{U}^T\delta\mathbf{x}$, we find

$$\int\limits_{\partial\mathscr{P}} \mathbf{U}\cdot\delta\mathbf{F}dA = \int\limits_{\partial\mathscr{P}} (\mathbf{U}\mathbf{N})\cdot D\delta\mathbf{x}dA - \int\limits_{\partial\mathscr{P}} \{\mathbf{D}\cdot\mathbf{U} + \mathbf{U}\mathbf{N}\operatorname{tr}\mathbf{B}\}\cdot\delta\mathbf{x}dA + \int\limits_{\mathscr{C}} [\mathbf{U}\mathbf{M}]\cdot\delta\mathbf{x}\,dl. \tag{3.12}$$

We now identify $\mathbf{U} = \mathbf{\Phi}$ in (3.5) and with the aid of (3.12) we determine that our variational equation (3.5) has the final form

$$\begin{aligned}\int\limits_{\mathscr{I}}\int\limits_{\mathscr{P}} \{\operatorname{Div}\mathbf{T}^* + \mathbf{Z} - \dot{\mathbf{P}}^* - \mathbf{A}\}\cdot\delta\mathbf{x}dVd\tau + \int\limits_{\mathscr{I}}\int\limits_{\partial\mathscr{P}} \{\mathbf{T}_N - \mathbf{T}^*\mathbf{N} - \mathbf{D}\cdot\mathbf{\Phi} \\ - \mathbf{\Phi}\mathbf{N}\operatorname{tr}\mathbf{B}\}\cdot\delta\mathbf{x}dAd\tau + \int\limits_{\mathscr{I}}\int\limits_{\partial\mathscr{P}} (\mathbf{\Phi}\mathbf{N})\cdot\mathbf{D}\delta\mathbf{x}dA\,d\tau + \int\limits_{\mathscr{I}}\int\limits_{\mathscr{C}} [\mathbf{\Phi}\mathbf{M}]\cdot\delta\mathbf{x}\,dl\,d\tau \\ + \int\limits_{\mathscr{P}} \{\Delta\mathbf{P} - \operatorname{Div}\Delta\mathbf{Q}\}\cdot\delta\mathbf{x}dV\Big|_{t_0}^{t} + \int\limits_{\partial\mathscr{P}} \Delta\mathbf{Q}\mathbf{N}\cdot\delta\mathbf{x}dA\Big|_{t_0}^{t} = 0.\end{aligned} \tag{3.13}$$

This variational equation is to hold for all arbitrary variations $\delta\mathbf{x}$ and $D\delta\mathbf{x}$. By routine arguments we obtain for all times $\tau\in\mathscr{I}$ the following differential equations and boundary conditions for hyperelastic materials of grade 2:

$$\operatorname{Div}\mathbf{T}^* + \mathbf{Z} = \dot{\mathbf{P}} + \mathbf{A} \quad \text{for all } X\in\mathscr{P}, \tag{3.14a}$$

$$\mathbf{T}_N = \mathbf{T}^*\mathbf{N} + \mathbf{D}\cdot\mathbf{\Phi}, \quad \mathbf{\Phi}\mathbf{N} = \mathbf{0} \quad \text{for all } X\in\mathscr{P}, \tag{3.14b}$$

$$[\mathbf{\Phi}\mathbf{M}] = \mathbf{0} \quad \text{for all } X\in\mathscr{C}, \tag{3.14c}$$

$$\mathbf{P}^* = \mathbf{P}, \quad \mathbf{Q}^* = \mathbf{Q} \quad \text{for all } X\in\mathscr{P}, \tag{3.14d}$$

wherein we recall (3.6a)–(3.6c).

(1) A general derivation of a class of integrals of the Toupin type is provided in [6].

4. Reduction to Toupin's theory of grade 2 materials

The foregoing theory is essentially equivalent to the theory developed along different lines by TOUPIN [1, § 10]. To see this it is necessary to assume that $\mathbf{A} = \mathbf{0}$ in (3.6c) and to restrict attention to the static case for smooth bodies, so that the terms in (3.14c) and (3.14d) are absent from all equations. Then, in particular, it can be shown that (3.3) can be rewritten as

$$(4.1) \qquad \delta\tilde{E}(\mathscr{P}) = \int_{\mathscr{P}} \tilde{\mathbf{F}} \cdot \delta\mathbf{x}\, dV + \int_{\partial\mathscr{P}} \tilde{\mathbf{T}} \cdot \delta\mathbf{x}\, dA + \int_{\partial\mathscr{P}} \tilde{\mathbf{H}} \cdot D\delta\mathbf{x}\, dA,$$

where, for correspondence with [1], we define

$$(4.2) \qquad \tilde{E}(\mathscr{P}) = \int_{\mathscr{P}} \tilde{W} dV \quad \text{with} \quad \tilde{W} = -L(\mathbf{X}, \mathbf{F}, \nabla\mathbf{F})$$

for the total strain energy, and

$$(4.3) \qquad \tilde{\mathbf{F}} = \mathbf{Z} - \operatorname{Div}\mathbf{G}, \quad \tilde{\mathbf{T}} = \mathbf{T}_N - \mathbf{D} \cdot \mathbf{H}_N - \tilde{\mathbf{H}} \operatorname{tr}\mathbf{B} + \mathbf{G}\mathbf{N}, \quad \tilde{\mathbf{H}} = \mathbf{H}_N \mathbf{N}.$$

Equation (4.1) coincides precisely with Toupin's assumed principle of virtual work in [1, Eq. (10.6)], and if (3.13) be cast in these terms, we could obtain the appropriate equilibrium equations and boundary conditions given in [1]. In fact, use of (4.3) in (3.14a) and (3.14b) and recollection of (3.6a)–(3.6c) reveals readily that the differential equations and boundary conditions (3.14) can be written

$$(4.4a) \qquad \operatorname{Div}\mathbf{T} + \tilde{\mathbf{F}} = \mathbf{0} \text{ in } \mathscr{P},$$

$$(4.4b) \qquad \tilde{\mathbf{T}} = \mathbf{T}\mathbf{N} - \mathbf{D} \cdot (\mathbf{H}\mathbf{N}) - \mathbf{H} \cdot (\mathbf{N} \otimes \mathbf{N}) \operatorname{tr}\mathbf{B}, \quad (\mathbf{H}_N - \mathbf{H}\mathbf{N})\mathbf{N} = \mathbf{0} \quad \text{on } \partial\mathscr{P},$$

where now the constitutive equations (3.6b) are provided by

$$(4.4c) \qquad \mathbf{T} = \frac{\partial \tilde{W}}{\partial \mathbf{F}} - \operatorname{Div}\mathbf{H}, \quad \mathbf{H} = \frac{\partial \tilde{W}}{\partial \nabla\mathbf{F}}.$$

These are Toupin's equations for hyperelastic materials of grade 2.

5. Euclidean invariance and balance laws

Let us return to the theory of § 3 and now require further, as in [1], that the action density (3.1) be invariant under the group of Euclidean transformations relating two motions to the same reference configuration:

$$(5.1) \qquad \mathbf{x}^*(\mathbf{X}, t^*) = \mathbf{R}\mathbf{x}(\mathbf{X}, t) + \mathbf{a}, \quad t^* = t + c, \qquad (5.1)$$

in which $\mathbf{R}$ is a constant orthogonal tensor, $\mathbf{a}$ is a constant vector and c is a constant scalar. In view of (3.2) this transformation induces a transformation on the directors, namely

$$(5.2) \qquad \mathbf{d}_I^*(\mathbf{X}, \tau^*) = \mathbf{R}\mathbf{d}_I(\mathbf{X}, \tau). \qquad (5.2)$$

Without (3.2) it becomes necessary to include (5.2) in (5.1), as was done by TOUPIN [1]. It suffices to consider only infinitesimal transformations. In this case our invariance requirement translates as

$$\text{(5.3)} \qquad \text{(i) } \delta L = 0 \text{ when } \delta \mathbf{x} = \mathbf{a}, \quad \text{(ii) } L(\mathbf{X}, t) = L(\mathbf{X}, t+c), \quad \text{(iii) } \delta L = 0 \text{ when } \delta \mathbf{x} = \Omega \mathbf{x},$$

where $\mathbf{\Omega} = \mathbf{R} - \mathbf{1}$ is the skew symmetric, infinitesimal rigid rotation tensor. It is clear that (i) and (iii) are equivalent to L being invariant under all infinitesimal rigid variations $\delta \mathbf{x} = \mathbf{\Omega x} + \mathbf{a}$. The conditions (5.3) imply, respectively,

$$\text{(5.4a)} \qquad \text{(i) } \frac{\partial L}{\partial \mathbf{x}} = \mathbf{0}, \quad \text{(ii) } \frac{\partial L}{\partial t} = 0, \quad \text{(iii) } \mathbf{K} = \mathbf{K}^T,$$

where

$$\text{(5.4b)} \qquad K_{ij} = x_i \frac{\partial L}{\partial x^j} + \dot{x}_i \frac{\partial L}{\partial \dot{x}^j} + x_{i,\alpha} \frac{\partial L}{\partial x^j{}_{,\alpha}} + \dot{x}_{i,\alpha} \frac{\partial L}{\partial \dot{x}^j{}_{,\alpha}} + x_{i,\alpha\beta} \frac{\partial L}{\partial x^j{}_{,\alpha\beta}}.$$

These constitute the grade 2 counterparts of Toupin's more general conditions on Toupin materials [cf. 1, § 6].

Bearing in mind (5.3) and (5.4), we return to (3.3) and consider the following particular variations: (i) $\delta \mathbf{x} = \mathbf{a}$, (ii) $\delta \mathbf{x} = \Omega \mathbf{x}$, (iii) $\delta \mathbf{x} = c\mathbf{x}$. Taking these in their turn, we find that (3.3) holds if, and only if,

$$\text{(5.5a)} \qquad \text{(i) } \frac{d}{dt} \int_{\mathscr{P}} \mathbf{P} dV = \int_{\partial \mathscr{P}} \mathbf{T}_N dA + \int_{\mathscr{P}} \mathbf{Z} dV,$$

$$\text{(5.5b)} \qquad \text{(ii) } \frac{d}{dt} \int_{\mathscr{P}} \mathfrak{H} dV = \int_{\partial \mathscr{P}} (\mathbf{x} \otimes \mathbf{T}_N - \mathbf{H}_N \mathbf{F}^T)_A dA + \int_{\mathscr{P}} (\mathbf{x} \otimes \mathbf{Z} - \mathbf{G}\mathbf{F}^T)_A dV,$$

$$\text{(5.5c)} \qquad \text{(iii) } \frac{d}{dt} \int_{\mathscr{P}} E dV = \int_{\partial \mathscr{P}} (\mathbf{T}_N \cdot \dot{\mathbf{x}} + \mathbf{H}_N \cdot \dot{\mathbf{F}}) dA + \int_{\mathscr{P}} (\mathbf{Z} \cdot \dot{\mathbf{x}} + \mathbf{G} \cdot \dot{\mathbf{F}}) dV,$$

each of which must hold separately for all motions of $\mathscr{P}$. Here the subscript A denotes the antisymmetric part of the tensor shown, and

$$\text{(5.6a)} \qquad E \equiv \mathbf{P} \cdot \dot{\mathbf{x}} + \mathbf{Q} \cdot \dot{\mathbf{F}} - L(\mathbf{X}, \dot{\mathbf{x}}, \mathbf{F}, \dot{\mathbf{F}}, \nabla \mathbf{F}),$$

$$\text{(5.6b)} \qquad \mathfrak{H} \equiv (\mathbf{x} \otimes \mathbf{P} - \mathbf{Q}\mathbf{F}^T)_A.$$

Thus, identifying $\mathbf{P}$ as the linear momentum, $\mathfrak{H}$ as the moment of momentum and E as the energy, our equations are expressed in the familiar form of integral balance laws. With the interpretation (3.4) and the invariance (5.4) it can be shown subject to the constraint (3.2) that the integral balance laws for Toupin materials reduce to our (5.5).

It is also interesting to see how (5.5a) can be obtained by integration of (3.14a), as ought to be the case. We note that

$$\int_{\mathscr{P}} \operatorname{Div} \mathbf{T}^* dV = \int_{\partial \mathscr{P}} \mathbf{T}^* \mathbf{N} dA = \int_{\partial \mathscr{P}} \mathbf{T}_N dA - \int_{\partial \mathscr{P}} \mathbf{D} \cdot \mathbf{\Phi} dA,$$

where we have introduced $(3.14b)_1$. However, we can show with the help of (3.9) and (3.12) that

$$\int_{\partial\mathscr{P}} \mathbf{D}\cdot\mathbf{\Phi}\,dA = \int_{\mathscr{C}} [\mathbf{\Phi}\mathbf{M}]\,dl - \int_{\partial\mathscr{P}} (\mathrm{tr}\,\mathbf{B})\mathbf{\Phi}\mathbf{N}\,dA.$$

Use of $(3.14b)_2$ and (3.14c) shows at once that this integral vanishes. Thus, (5.5a) follows from (3.14), as asserted, when (5.4a(i) holds. Equations (5.5b)–(5.5c) can be similarly derived using (5.6) and (5.4a) together with (3.6) and (3.14).

6. Closure

We have shown that a hyperelastic Toupin material whose directors are constrained to follow the local deformation of the continuum at each point is a hyperelastic material of grade 2. Though both the director theory sketched in § 1 and the theory of grade 2 materials described in § 3 were developed in [1], and though Toupin may have perceived the connection we have demonstrated here by virtue of this elegant *assumed* form (4.1), herein derived from a more general variational principle, the relationships and structure we furnish are none the less altogether new. We see clearly that to use the conventional identification of body force and surface traction for $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{T}}$ in the first two integrals of (4.1) is potentially deceiving. In fact, if this familiar interpretation is carried into (4.1), one must assume unnecessarily that body couples are neglected; moreover, we note too that nowhere in [1, § 10] are these even mentioned. But let us see where this line of thought may take us. Thus let us suppose that $\tilde{\mathbf{F}} = \mathbf{Z}$, $\tilde{\mathbf{T}} = \mathbf{T}_N$, $\mathbf{G} = \mathbf{0}$ and $\tilde{\mathbf{H}} = \mathbf{H}_N$. Then, it is clear from our definition of the tilda quantities in (4.3) that this is possible, if and only if, $\mathbf{D}\cdot\mathbf{H}_N + \mathbf{H}_N\mathbf{N}\,\mathrm{tr}\,\mathbf{B} = \mathbf{0}$, i.e., when and only when $\int_{\partial\mathscr{P}} \mathbf{H}_N\cdot(\mathbf{D}\delta\mathbf{x})\,dA = 0$ for all variations $\delta\mathbf{x}$. None the less, in this case we shall have exactly the same form for our variational equation, and for our equations of motion and boundary conditions as provided by TOUPIN [1, Eqs. (10.6)–(10.11)], i.e., (4.1)–(4.4) above! Of course, because of the special restriction on the hypertractions necessary for this reduction, it is evident that the classical interpretation is not to be assigned to $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{T}}$ as described. Our equations (3.6), (3.14) and (5.5) admit the conventional identification implied clearly in (5.5).

Finally, all the foregoing developments become somewhat troublesome in applications where spatial variables rather than material variables may prove more useful. However, all of the results derived here can readily be converted by using (3.2) and (3.4) in the transformation relations defined in [1, § 7].

Some applications of our equations in the proof of several theorems on materials of grade 2 are considered in a another paper [7].

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