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NOTE ON THE COMPOSITION OF INFINITESIMAL ROTATIONS.

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THE following is a solution of a question proposed by me in the last Smith's Prize Examination:

"Show that infinitesimal rotations impressed upon a solid body may be compounded together according to the rules for the composition of forces."

DEFINITION. The "six coordinates" of a line passing through the point (x_0, y_0, z_0), and inclined at angles (α, β, γ), to the axes, are

$$\begin{aligned} a &= \cos\alpha, \quad f = y_0\cos\gamma - z_0\cos\beta, \\ b &= \cos\beta, \quad g = z_0\cos\alpha - x_0\cos\gamma, \\ c &= \cos\gamma, \quad h = x_0\cos\beta - y_0\cos\alpha. \end{aligned}$$

I use, throughout, the term rotation to denote an infinitesimal rotation; this being so,

LEMMA 1. A rotation ω round the line (a, b, c, f, g, h), produces in the point (x, y, z), rigidly connected with the line, the displacements

$$\begin{aligned} \delta x &= \omega(\quad . \quad cy - bz + f), \\ \delta y &= \omega(-cx \quad . \quad + az + g), \\ \delta z &= \omega(\quad bx - ay \quad . \quad + h). \end{aligned}$$

LEMMA 2. Considering in a solid body the point (x, y, z), situate in the line (a, b, c, f, g, h), then for any infinitesimal motion of the solid body, the displacement of the point in the direction of the line is

$$= al + bm + cn + fp + gq + hr,$$

where l, m, n, p, q, r are constants depending on the infinitesimal motion of the solid body.

Hence, *first*, for a system of rotations

$$\begin{array}{ll} \omega_1 \text{ about the line } & (a_1, b_1, c_1, f_1, g_1, h_1), \\ \omega_2 \quad ,, \quad ,, & (a_2, b_2, c_2, f_2, g_2, h_2), \\ \&\text{c.} & \end{array}$$

the displacements of the point (x, y, z), are

$$\begin{aligned} \delta x &= \quad . \quad y\Sigma c\omega - z\Sigma b\omega + \Sigma f\omega, \\ \delta y &= -x\Sigma c\omega \quad . \quad + z\Sigma a\omega + \Sigma g\omega, \\ \delta z &= \quad x\Sigma b\omega + y\Sigma a\omega \quad . \quad + \Sigma h\omega ; \end{aligned}$$

and when the rotations are in equilibrium, the displacements $(\delta x, \delta y, \delta z)$ of any point (x, y, z) whatever must each of them vanish; that is, we must have

$$\Sigma\omega a = 0, \quad \Sigma\omega b = 0, \quad \Sigma\omega c = 0, \quad \Sigma\omega f = 0, \quad \Sigma\omega g = 0, \quad \Sigma\omega h = 0,$$

which are therefore the conditions for the equilibrium of the rotations ω_1, ω_2, &c.

Secondly, for a system of forces

$$\begin{array}{ll} P_1 \text{ along the line } & (a_1, b_1, c_1, f_1, g_1, h_1), \\ P_2 \quad ,, \quad ,, & (a_2, b_2, c_2, f_2, g_2, h_2), \\ \&\text{c.} & \end{array}$$

the condition of equilibrium as given by the principle of virtual velocities is

$$\Sigma P\,(al + bm + cn + fp + gq + hr) = 0 ;$$

or, what is the same thing, we must have

$$\Sigma Pa = 0, \quad \Sigma Pb = 0, \quad \Sigma Pc = 0, \quad \Sigma Pf = 0, \quad \Sigma Pg = 0, \quad \Sigma Ph = 0,$$

which are therefore the conditions for the equilibrium of the forces P_1, P_2, &c.

Comparing the two results we see that the conditions for the equilibrium of the rotations ω_1, ω_2, &c. are the same as those for the equilibrium of the forces P_1, P_2, &c.; and since, for rotations and forces respectively, we pass at once from the theory of equilibrium to that of composition; the rules of composition are the same in each case.

Demonstration of Lemma 1.

Assuming for a moment that the axis of rotation passes through the origin, then for the point P, coordinates (x, y, z), the square of the perpendicular distance from the axis is

$$\begin{aligned} = &\;(\quad . \quad - y\cos\gamma + z\cos\beta)^2 \\ + &\;(\quad x\cos\gamma \quad . \quad - z\cos\alpha)^2 \\ + &\;(- x\cos\beta + y\cos\alpha \quad . \quad)^2, \end{aligned}$$

and the expressions which enter into this formula denote as follows; viz. if through the point P, at right angles to the plane through P and the axis of rotation, we draw a line PQ, = perpendicular distance of P from the axis of rotation, then the coordinates of Q referred to P as origin are

$$\begin{array}{ccc} . & -y\cos\gamma + z\cos\beta, & \\ x\cos\gamma & . & -z\cos\alpha, \\ -x\cos\beta + y\cos\alpha & . & , \end{array}$$

respectively. Hence the foregoing quantities each multiplied by ω are the displacements of the point P in the directions of the axes, produced by the rotation ω. Suppose that the axis of rotation (instead of passing through the origin) passes through the point (x_0, y_0, z_0); the only difference is that we must in the formulæ write $(x-x_0, y-y_0, z-z_0)$ in place of (x, y, z): and attending to the significations of the six coordinates (a, b, c, f, g, h) it thus appears that the displacements produced by the rotation are equal to ω into the expressions

$$\begin{array}{cccc} . & -cy + bz & +f, \\ cx & . & -az & +g, \\ -bx + ay & . & & +h, \end{array}$$

respectively.

Demonstration of Lemma 2.

For any infinitesimal motion whatever of a solid body, the displacements of the point (x, y, z) in the directions of the axes are

$$\begin{aligned} \delta x &= l \quad . \quad - ry + qz, \\ \delta y &= m + rx \quad . \quad - pz, \\ \delta z &= n - qx + py \quad . \quad , \end{aligned}$$

and hence the displacement in the direction of the line (α, β, γ), is

$$\delta x\cos\alpha + \delta y\cos\beta + \delta z\cos\gamma,$$

which, attending to the signification of the six coordinates (a, b, c, f, g, h), is

$$= al + bm + cn + fp + gq + hr,$$

which is the required expression.

It is proper to remark that the last-mentioned expressions of $(\delta x, \delta y, \delta z)$ are in fact the displacements produced by a translation and a rotation. If we *assume* that every infinitesimal motion of a solid body can be resolved into a translation and a rotation, then, since a translation can be produced by two rotations, every infinitesimal motion of a solid body can be resolved into rotations alone, and the foregoing expressions for the displacements produced by a rotation, combining any number of them and writing $(\Sigma\omega a, \Sigma\omega b, \Sigma\omega c, \Sigma\omega f, \Sigma\omega g, \Sigma\omega h) = (-p, -q, -r, l, m, n)$ respectively, lead to the expressions for the displacements $\delta x, \delta y, \delta z$ produced by the infinitesimal motion of the solid body.