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ON THE CONICS WHICH TOUCH THREE GIVEN LINES AND PASS THROUGH A GIVEN POINT.

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CONSIDER the triangles which touch three given lines; the three lines form a triangle, and the lines joining the angles of the triangle with the points of contact of the opposite sides respectively meet in a point S: conversely given the three lines and the point S, then joining this point with the angles of the triangle the joining lines meet the opposite sides respectively in three points which are the points of contact with the three given lines respectively of a conic; such conic is determinate and unique. Suppose now that the conic passes through a given point; the point S is no longer arbitrary, but it must lie on a certain curve; and this curve being known, then taking upon it any point whatever for the point S, and constructing as before the conic which corresponds to such point, the conic in question will pass through the given point, and will thus be a conic touching the three given lines and passing through the given point. And the series of such conics corresponds of course to the series of points on the curve.

I proceed to find the curve which is the locus of the point S.

We may take $x=0$, $y=0$, $z=0$ for the equations of the given lines, and $x:y:z=1:1:1$ for the coordinates of the given point. The equation of a conic touching the three given lines is

$$a\sqrt{(x)}+b\sqrt{(y)}+c\sqrt{(z)}=0,$$

and the coordinates of the corresponding point S are as $\frac{1}{a^2}:\frac{1}{b^2}:\frac{1}{c^2}$, that is, taking (x, y, z) for the coordinates of the point in question, we have

$$a:b:c=\frac{1}{\sqrt{(x)}}:\frac{1}{\sqrt{(y)}}:\frac{1}{\sqrt{(z)}},$$

the condition in order that the conic may pass through the given point is $a+b+c=0$, and we thus find for the curve, which is the locus of the point S, the equation

$$\frac{1}{\sqrt{(x)}}+\frac{1}{\sqrt{(y)}}+\frac{1}{\sqrt{(z)}}=0,$$

or, what is the same thing,

$$\sqrt{(yz)}+\sqrt{(zx)}+\sqrt{(xy)}=0,$$

the rationalised form of which is

$$y^2z^2+z^2x^2+x^2y^2-2xyz\,(x+y+z)=0.$$

This is a quartic curve with three cusps, viz. each angle of the triangle is a cusp; and by considering for example the cusp ($y=0$, $z=0$) and writing the equation under the form

$$x^2\,(y-z)^2-2x\,(yz^2+y^2z)+y^2z^2=0,$$

we see that the tangent at the cusp in question is the line $y-z=0$; that is, the tangents at the three cusps are the lines joining these points respectively with the given point (1, 1, 1). Each cuspidal tangent meets the curve in the cusp counting as three points and in a fourth point of intersection, the coordinates whereof in the case of the tangent $y-z=0$, are at once found to be $x:y:z=1:4:4$, or say this is the point (1, 4, 4); the point on the tangent $z-x=0$ is of course (4, 1, 4), and that on the tangent $x-y=0$ is (4, 4, 1). To find the tangents at these points respectively, I remark that the general equation of the tangent is

$$(X\delta_x+Y\delta_y+Z\delta_z)\left\{\frac{1}{\sqrt{(x)}}+\frac{1}{\sqrt{(y)}}+\frac{1}{\sqrt{(z)}}\right\}=0,$$

that is

$$\frac{X}{x^{\frac{3}{2}}}+\frac{Y}{y^{\frac{3}{2}}}+\frac{Z}{z^{\frac{3}{2}}}=0,$$

or for the point (1, 4, 4) the equation of the tangent is $8X+Y+Z=0$, or say $8x+y+z=0$; that is, the tangent passes through the point $x=0$, $x+y+z=0$, being the point of intersection of the line $x=0$ with the line $x+y+z=0$, which is the harmonic of the given point (1, 1, 1) in regard to the triangle; the tangents at the points (1, 4, 4), (4, 1, 4), (4, 4, 1) respectively pass through the points of intersection of the harmonic line $x+y+z=0$ with the three given lines respectively.

In the case where the given point lies within the triangle, the curve the locus of S lies wholly within the triangle, and is of the form shown in fig. 3 in the plate opposite; it is clear that in this case the conics of the system are all of them ellipses; there are however three limiting forms, viz. the line joining the given point with any angle of the triangle, such line being regarded as a twofold line or pair of coincident lines, is a conic of the system. The discussion of the two cases in which the given point lies outside the triangle, viz. in the infinite space bounded by two sides produced, or in the infinite space bounded by a side and two sides produced, may be effected without much difficulty.