487.

ON THE QUARTIC SURFACES $(*\between U, V, W)^2=0$.

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AMONG the surfaces of the form in question are included the reciprocals of several interesting surfaces of the orders 6, 8, 9, 10, and 12, viz.

Order 6, parabolic ring.
„ 8, elliptic ring.
„ 9, centro-surface of a paraboloid.
„ 10, parallel surface of a paraboloid.
„ „ envelope of planes through the points of an ellipsoid at right angles to the radius vectors from the centre.
„ 12, centro-surface of an ellipsoid.
„ „ parallel surface of an ellipsoid.

I propose to consider these surfaces, not at present in any detail, but merely for the purpose of presenting them in connexion with each other and with the present theory. It will be convenient to use homogeneous equations, but for the metrical interpretation the coordinate W or w may be considered as equal to unity: I have not thought it necessary so to adjust the constants that the equations shall be homogeneous in regard to the constants; this can of course be done without difficulty, and in many cases it would be analytically advantageous to make the change.

I take throughout (X, Y, Z, W) for the coordinates of a point on the quartic surface (so that (U, V, W) in the equation $(*\between U, V, W)^2=0$ are to be considered as quadric functions of (X, Y, Z, W)), reserving (x, y, z, w) for the coordinates of a point on the reciprocal surface of the order 6, 8, 9, 10, or 12. The reciprocation is performed in regard to the imaginary sphere $x^2+y^2+z^2+w^2=0$: the relation between the coordinates (X, Y, Z, W) and (x, y, z, w) is then $Xx+Yy+Zz+Ww=0$, and the equation $(X, Y, Z, W)^4=0$ is the equation in point-coordinates of the quartic surface, or in line-coordinates of the reciprocal surface: and similarly the equation $(x, y, z, w)^n=0$ is the equation in point-coordinates of the reciprocal surface, or in line-coordinates of the quartic surface.

Parabolic ring, or envelope of a sphere of constant radius having its centre on a parabola.

Taking k for the radius of the sphere, and $z=0$, $y^2=4ax$ for the equations of the parabola, then the coordinates of a point on the parabola are $a\theta^2$, $2a\theta$, 0; where θ is a variable parameter. The equation of the sphere therefore is

$$(x-a\theta^2w)^2+(y-2a\theta w)^2+z^2-k^2w^2=0,$$

and the ring is the envelope of this sphere.

The reciprocal of the sphere is

$$k^2(X^2+Y^2+Z^2)-(a\theta^2X+2a\theta Y+W)^2=0\,;$$

writing this in the form

$$a\theta^2X+2a\theta Y+\ W+k\sqrt{(X^2+Y^2+Z^2)}=0,$$

and taking the envelope in regard to θ, we have

$$X\{W+k\sqrt{(X^2+Y^2+Z^2)}\}-aY^2=0,$$

or, what is the same thing,

$$(aY^2-XW)^2-k^2X^2(X^2+Y^2+Z^2)=0,$$

for the equation of the quartic surface. This has the line $X=0$, $Y=0$ for a tacnodal line, but I am not in possession of a theory enabling me thence to infer that the parabolic ring is of the order 6.

To show that it is so, I revert to the equation of the variable sphere

$$(x-a\theta^2w)^2+(y-2a\theta w)^2+z^2-k^2w^2=0,$$

or, what is the same thing,

$$(A, B, C, D, E\between\theta, 1)^4=0,$$

where

$$\begin{aligned}A&=\ 3a^2w^2,\\ B&=\ 0,\\ C&=\ a(2aw^2-xw),\\ D&=-3ayw,\\ E&=\ 3(x^2+y^2+z^2-k^2w^2).\end{aligned}$$

Then $I=3a^2w^2I'$, $J=a^3w^3J'$, and the equation is $I'^3-J'^2=0$, viz. this is

$$\{4x^2+3y^2-4axw+4a^2w^2+3(z^2-k^2w^2)\}^3$$
$$-\{(2aw-x)[8x^2+9y^2+4axw-4a^2w^2+9(z^2-k^2w^2)]-27ay^2w\}^2=0,$$

or, as this may also be written,

$$\{4x^2+3y^3-4axw+4a^2w^2+3(z^2-k^2w^2)\}^3$$
$$-\{-8x^3-9xy^2+12ax^2w+12a^2xw^2-8a^3w^3-9(x-2aw)(z^2-h^2w^2)\}=0.$$

Developing, the whole divides by 27, and the equation of the ring finally is

$$(y^2-4axw)^2\{y^2+(x-aw)^2\}$$
$$+\{3y^4+y^2(2x^2-2axw+20a^2w^2)+8ax^3w+8a^2x^2w^2-32a^3xw^3+16a^4w^4\}(z^2-k^2w^2)$$
$$+(3y^2+x^2+8axw-8a^2w^2)(z^2-k^2w^2)^2$$
$$+(z^2-k^2w^2)^3=0.$$

Elliptic ring, or envelope of a sphere of constant radius having its centre on an ellipse.

Taking k for the radius of the sphere, and $z=0$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for the equations of the ellipse, the coordinates of a point on the ellipse are $a\cos\theta$, $b\sin\theta$; hence the equation of the variable sphere is

$$(x-aw\cos\theta)^2+(y-bw\sin\theta)^2+z^2-k^2w^2=0.$$

The reciprocal of this is

$$k^2(X^2+Y^2+Z^2)-(aX\cos\theta+bY\sin\theta+W)^2=0,$$

viz. writing this under the form

$$aX\cos\theta+bY\sin\theta+W+k\sqrt{(X^2+Y^2+Z^2)}=0,$$

and taking the envelope in regard to θ, the equation of the reciprocal surface is

$$a^2X^2+b^2Y^2=\{W+k\sqrt{(X^2+Y^2+Z^2)}\}^2,$$

viz. this is

$$(a^2-k^2)X^2+(b^2-k^2)Y^2-k^2Z^2-W^2=2kW\sqrt{(X^2+Y^2+Z^2)},$$

or

$$\{(a^2-k^2)X^2+(b^2-k^2)Y^2-k^2Z^2-W^2\}^2-4k^2W^2(X^2+Y^2+Z^2)=0,$$

that is

$$\{(a^2-k^2)X^2+(b^2-k^2)Y^2-k^2Z^2\}^2-2W^2\{(a^2+k^2)X^2+(b^2+k^2)Y^2+k^2Z^2\}+W^4=0,$$

which is a quartic surface having the nodal conic $W=0$, $(a^2-k^2)X^2+(b^2-k^2)Y^2-k^2Z^2=0$. This singularity alone would only reduce the order of the reciprocal surface to 12; the reciprocal surface or elliptic ring is in fact (as I proceed to show) of the order 8.

For this purpose reverting to the equation

$$(x - aw\cos\theta)^2 + (y - bw\sin\theta)^2 + z^2 - k^2w^2 = 0,$$

this may be written

$$A\cos 2\theta + B\sin 2\theta + C\cos\theta + D\sin\theta + E = 0,$$

where

$$A = \quad (a^2 - b^2)\,w^2,$$
$$B = \quad 0,$$
$$C = -\,4axw,$$
$$D = -\,4byw,$$
$$E = \quad (a^2 + b^2)\,w^2 + 2(x^2 + y^2 + z^2 - k^2w^2),$$

and the equation is

$$\{12(A^2 + B^2) - 3(C^2 + D^2) + 4E^2\}^3$$
$$-\,\{27A(C^2 - D^2) + 54BCD - [72(A^2 + B^2) + 9(C^2 + D^2)]\,E + 8E^3\}^2 = 0,$$

or say

$$\{12A^2 - 3(C^2 + D^2) + 4E^2\}^3 - \{27A(C^2 - D^2) - [72A^2 + 9(C^2 + D^2)]\,E + 8E^3\}^2 = 0.$$

This is of the order 12, but it is easy to see that the terms in E^6 and $E^4(C^2 + D^2)$ disappear from the equation, all the other terms divide by w^4; and the equation is thus of the order 8.

The equation may be obtained somewhat differently as follows. The equation of the variable sphere is

$$(x - \alpha w)^2 + (y - \beta w)^2 + z^2 - k^2w^2 = 0,$$

where (α, β) vary subject to the condition $\dfrac{\alpha^2}{a^2} + \dfrac{\beta^2}{b^2} = 1$. We have therefore

$$x - \alpha w - \lambda\,\frac{\alpha w}{a^2} = 0,$$

$$y - \beta w - \lambda\,\frac{\beta w}{b^2} = 0,$$

and thence

$$\alpha w = \frac{a^2x}{a^2 + \lambda}, \quad x - \alpha w = \frac{\lambda x}{a^2 + \lambda},$$

$$\beta w = \frac{b^2y}{b^2 + \lambda}, \quad y - \beta w = -\,\frac{\lambda y}{b^2 + \lambda}.$$

Consequently

$$\frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2 - k^2w^2}{\lambda^2} = 0,$$

$$\frac{a^2x^2}{(a^2 + \lambda)^2} + \frac{b^2y^2}{(b^2 + \lambda)^2} - \quad w^2 \quad = 0,$$

from which λ is to be eliminated. The second equation may be replaced by

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2 - k^2w^2}{\lambda} - w^2 = 0,$$

which has the first for its derived equation in regard to λ. Hence, writing this last equation in the form

$$w^2(a^2+\lambda)(b^2+\lambda)\lambda - (b^2+\lambda)\lambda x^2 - (a^2+\lambda)\lambda y^2 - (a^2+\lambda)(b^2+\lambda)(z^2-k^2w^2) = 0,$$

we have to equate to zero the discriminant of this cubic function of λ. Calling the equation

$$(A, B, C, D\between \lambda, 1)^3 = 0,$$

we have

$$\begin{aligned} A &= \quad 3w^2, \\ B &= (a^2+b^2)w^2 - x^2 - y^2 - (z^2 - k^2w^2), \\ C &= a^2b^2w^2 - b^2x^2 - a^2y^2 - (a^2+b^2)(z^2-k^2w^2), \\ D &= -3a^2b^2(z^2-k^2w^2). \end{aligned}$$

The required equation then is

$$A^2D^2 + 4AC^3 + 4B^3D - 3B^2C^2 - 6ABCD = 0.$$

The developed equation (Salmon's *Conic Sections*, Ed. v., p. 325) is

$$\begin{aligned} &(b^2x^2+a^2y^2-a^2b^2w^2)^2\{(x-cw)^2+y^2\}\{(x+cw)^2+y^2\} \\ &- \left\{\begin{matrix} 2b^2(a^2-2b^2)x^6 - 2(a^4-a^2b^2+3b^4)x^4y^2 - 2(3a^4-a^2b^2+b^4)x^2y^4 + 2a^2(b^2-2a^2)y^6 \\ -b^2(6a^4-10a^2b^2+6b^4)x^4w^2 + (4a^6-6a^4b^2-6a^2b^4+4b^6)x^2y^2w^2 \\ -a^2(6a^4-10a^2b^2+6b^4)y^4w^2 \\ +2c^2(3a^4-a^2b^2+b^4)x^2w^4 - 2c^2(a^4-3a^2b^2+3b^4)y^2w^4 - 2a^2b^2c^4(a^2+b^2)w^6 \end{matrix}\right\}(z^2-k^2w^2) \\ &+ \left\{\begin{matrix} (a^4-6a^2b^2+6b^4)x^4 + (6a^4-10a^2b^2+6b^4)x^2y^2 + (6a^4-6a^2b^2+b^4)y^4 \\ -2c^2(a^4-a^2b^2+3b^4)x^2w^2 + 2c^2(3a^4-a^2b^2+b^4)y^2w^2 + c^4(a^4+4a^2b^2+b^4)w^4 \end{matrix}\right\}(z^2-k^2w^2)^2 \\ &+ \left\{\begin{matrix} (a^2-2b^2)x^2 + (2a^2-b^2)y^2 \\ + c^2(a^2+b^2)w^2 \end{matrix}\right\} 2c^2(z^2-k^2w^2)^3 \\ &+ c^4(z^2-k^2w^2)^4 = 0. \end{aligned}$$

I remark that the before-mentioned nodal conic $W = 0$, $(a^2-k^2)X^2 + (b^2-k^2)Y^2 - k^2Z^2 = 0$ is the reciprocal of a quadric cone, which is a bitangent cone of the ring: this is a cone, vertex at the centre of the ring, and which is the envelope of the right cone, vertex the same point, circumscribed about the variable sphere which generates the ring.

Centro-surface of a paraboloid.

For the paraboloid $\frac{X^2}{a} + \frac{Y^2}{b} - 2ZW = 0$, it may be shown that the centro-surface is the envelope of the quadric

$$\frac{ax^2}{(a+\theta)^2} + \frac{by^2}{(b+\theta)^2} - 2zw - 2\theta w^2 = 0.$$

The quartic surface is consequently the envelope of the quadric

$$\frac{(a+\theta)^2}{a}X^2 + \frac{(b+\theta)^2}{b}Y^2 + 2\theta Z^2 - 2ZW = 0,$$

viz. this is

$$\theta^2\left(\frac{X^2}{a} + \frac{Y^2}{b}\right) + 2\theta(X^2+Y^2+Z^2) + aX^2 + bY^2 - 2ZW = 0.$$

Hence the quartic surface is

$$\left(\frac{X^2}{a} + \frac{Y^2}{b}\right)(aX^2 + bY^2 - 2ZW) - (X^2+Y^2+Z^2)^2 = 0,$$

or, what is the same thing,

$$X^2Y^2(a-b)^2 - 2ZW(bX^2 + aY^2) - 2abZ^2(X^2+Y^2) - abZ^4 = 0.$$

This has four conic nodes; viz. considering the equations

$$\frac{X^2}{a} + \frac{Y^2}{b} = 0, \quad aX^2 + bY^2 - 2ZW = 0, \quad X^2+Y^2+Z^2 = 0,$$

these give the point $X=0$, $Y=0$, $Z=0$ four times, and four other points which are the nodes in question; the point $(X=0, Y=0, Z=0)$ is a singular point of a higher order; the reduction caused by these singularities should be $= 8 + 19$, so as to make the order of the surface of centres $= 9$; that is the reduction on account of the point $(X=0, Y=0, Z=0)$ must be $= 19$; but it is not by any means obvious how this is so.

Parallel surface of the paraboloid.

This is given, Salmon's *Solid Geometry*, 2nd Edit., pp. 146 and 148, [Ed. 4, p. 180], for the paraboloid $aX^2 + bY^2 + 2rZW = 0$, as the envelope of the quadric surface

$$\frac{\theta a x^2}{\theta a + 1} + \frac{\theta b y^2}{\theta b + 1} + 2\theta rzw - (\theta^2 r^2 + k^2)\,w^2 = 0.$$

The reciprocal quartic is thus the envelope of

$$\frac{\theta a + 1}{\theta a}X^2 + \frac{\theta b + 1}{\theta b}Y^2 + \frac{\theta^2 r^2 + k^2}{\theta^2 r^2}Z^2 + \frac{2}{\theta r}ZW = 0,$$

that is

$$(X^2+Y^2+Z^2) + \frac{1}{\theta}\left(\frac{X^2}{a} + \frac{Y^2}{b} + \frac{2}{r}ZW\right) + \frac{1}{\theta^2}\frac{k^2}{r^2}Z^2 = 0,$$

whence the equation is

$$4\frac{k^2}{r^2}Z^2(X^2+Y^2+Z^2) - \left(\frac{X^2}{a} + \frac{Y^2}{b} + \frac{2}{r}ZW\right)^2 = 0,$$

viz. this is a quartic having the nodal line-pair $Z=0$, $\frac{X^2}{a} + \frac{Y^2}{b} = 0$; and a further singularity at the point $X=0$, $Y=0$, $Z=0$. It would require some consideration to show that the order of the parallel surface is thence $= 10$, as it should be.

Envelope of the planes through the points of an ellipsoid at right angles to the radius vectors from the centre.

This is given in my paper "Sur la surface &c." in the *Annali di Matematica*, t. II. (1859), [250], as the envelope of the quadric surface

$$\frac{x^2}{2-\dfrac{\theta}{a^2}}+\frac{y^2}{2-\dfrac{\theta}{b^2}}+\frac{z^2}{2-\dfrac{\theta}{c^2}}-\theta w^2=0.$$

The reciprocal quartic surface is thus the envelope of

$$\left(2-\frac{\theta}{a^2}\right)X^2+\left(2-\frac{\theta}{b^2}\right)Y^2+\left(2-\frac{\theta}{c^2}\right)Z^2-\frac{1}{\theta}W^2=0,$$

or, what is the same thing,

$$\theta\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)-2(X^2+Y^2+Z^2)+\frac{1}{\theta}W^2=0,$$

viz. this is

$$W^2\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)-(X^2+Y^2+Z^2)^2=0,$$

which is in fact the inverse surface

$$\left(\frac{X}{X^2+Y^2+Z^2},\ \frac{Y}{X^2+Y^2+Z^2},\ \frac{Z}{X^2+Y^2+Z^2}\ \text{for}\ X, Y, Z\right)$$

of the ellipsoid $\dfrac{X^2}{a^2}+\dfrac{Y^2}{b^2}+\dfrac{Z^2}{c^2}=1$; this is obvious geometrically inasmuch as the reciprocal of the variable plane is the inverse of the point on the ellipsoid.

The quartic surface has the nodal conic

$$W=0,\quad X^2+Y^2+Z^2=0;$$

and also the node $X=0$, $Y=0$, $Z=0$; there is consequently in the order of the reciprocal surface a reduction $24+2=26$, or the order of the reciprocal surface is $=10$.

Centro-surface of the ellipsoid.

Writing the equation of the ellipsoid in the form $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-w^2=0$, the centro-surface is given as the envelope of the quadric surface

$$\frac{a^2x^2}{(\theta+a^2)^2}+\frac{b^2y^2}{(\theta+b^2)^2}+\frac{c^2z^2}{(\theta+c^2)^2}-w^2=0,$$

(Salmon, [Ed. 2], p. 400, [Ed. 4, p. 179]), and hence the reciprocal quartic surface is the envelope of

$$\left(a+\frac{\theta}{a}\right)^2X^2+\left(b+\frac{\theta}{b}\right)^2Y^2+\left(c+\frac{\theta}{c}\right)^2Z^2-W^2=0,$$

in regard to the variable parameter θ, viz. the equation is

$$\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)(a^2X^2+b^2Y^2+c^2Z^2-W^2)-(X^2+Y^2+Z^2)^2=0,$$

(see Salmon, [Ed. 2], p. 144 [Ed. 4, p. 172]). It hence at once appears, that the quartic surface has 12 nodes, viz. these are the four angles of the fundamental tetrahedron $(XYZW)$, and the eight points

$$\left\{\begin{aligned}&\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=0,\\&X^2+Y^2+Z^2=0,\\&a^2X^2+b^2Y^2+c^2Z^2-W^2=0,\end{aligned}\right.$$

or writing as it is convenient to do

$$(\alpha,\ \beta,\ \gamma)=(b^2-c^2,\ c^2-a^2,\ a^2-b^2);$$

and therefore

$$\alpha+\beta+\gamma=0,\quad a^2\alpha+b^2\beta+c^2\gamma=0,\quad a^4\alpha+b^4\beta+c^4\gamma=-\alpha\beta\gamma;$$

these are the eight points

$$\frac{X^2}{W^2}=-\frac{a^2}{\beta\gamma},\quad \frac{Y^2}{W^2}=-\frac{b^2}{\gamma\alpha},\quad \frac{Z^2}{W^2}=-\frac{c^2}{\alpha\beta};$$

the order of the reciprocal of the quartic surface is thus $36-2.12$, $=12$, which is in fact the order of the surface of centres.

The equation of the centro-surface is given, Salmon, [Ed. 2], p. 151, and *Quart. Math. Jour.*, t. II. (1858), p. 220, in the form

$$(\alpha,\ \beta,\ \gamma)^6(\xi,\ \eta,\ \zeta,\ \omega)^{12}=0,$$

where $\xi,\ \eta,\ \zeta,\ \omega$ stand for $ax,\ by,\ cz,\ iw$; it is therefore of the degree 18 in regard to $a,\ b,\ c$.

Parallel surface of the ellipsoid.

This is given, Salmon, [Ed. 2], p. 148 [Ed. 4, p. 176], as the envelope of the quadric surface

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-\left(1+\frac{k^2}{\theta}\right)w^2=0.$$

The reciprocal quartic is thus the envelope of

$$(a^2+\theta)X^2+(b^2+\theta)Y^2+(c^2+\theta)Z^2-\frac{\theta W^2}{k^2+\theta}=0,$$

or writing $k^2+\theta=\lambda$, this is

$$(a^2-k^2+\lambda)X^2+(b^2-k^2+\lambda)Y^2+(c^2-k^2+\lambda)Z^2-\left(1-\frac{k^2}{\lambda}\right)W^2=0,$$

or, what is the same thing,

$$\lambda^2(X^2+Y^2+Z^2)+\lambda\left[(a^2-k^2)X^2+(b^2-k^2)Y^2+(c^2-k^2)Z^2-W^2\right]+k^2W^2=0,$$

whence the equation is

$$\{(a^2-k^2)X^2+(b^2-k^2)Y^2+(c^2-k^2)Z^2-W^2\}^2-4k^2W^2(X^2+Y^2+Z^2)=0,$$

viz. this is a quartic having the nodal conic

$$W=0,\quad (a^2-k^2)X^2+(b^2-k^2)Y^2+(c^2-k^2)Z^2=0.$$

The order of the reciprocal or parallel surface is thus $36-24, =12$, as it should be. The nodal conic of the quartic surface is the reciprocal of a bitangent or node-couple quadric cone, vertex the centre, in the parallel surface: this cone is imaginary for the ellipsoid, but real for either of the hyperboloids, and its existence in the case of the hyperboloid is readily perceived.

Reverting to the equation

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-\left(1+\frac{k^2}{\theta}\right)w^2=0,$$

or say

$$(a^2+\theta)(b^2+\theta)(c^2+\theta)(k^2+\theta)w^2$$
$$-x^2(b^2+\theta)(c^2+\theta)\theta-y^2(c^2+\theta)(a^2+\theta)\theta-z^2(a^2+\theta)(b^2+\theta)\theta=0,$$

this is

$$(A, B, C, D, E\between\theta, 1)^4=0,$$

where putting for shortness

$$\begin{aligned}\alpha&=a^2+b^2+c^2+k^2,\\ \beta&=b^2c^2+c^2a^2+a^2b^2+k^2(a^2+b^2+c^2),\\ \gamma&=a^2b^2c^2+k^2(b^2c^2+c^2a^2+a^2b^2),\\ \delta&=a^2b^2c^2k^2,\end{aligned}$$

and

$$\begin{aligned}p&=x^2+y^2+z^2,\\ q&=(b^2+c^2)x^2+(c^2+a^2)y^2+(a^2+b^2)z^2,\\ r&=b^2c^2x^2+c^2a^2y^2+a^2b^2z^2,\end{aligned}$$

we have

$$\begin{aligned}A&=12w^2,\\ B&=3\alpha w^2-3p,\\ C&=2\beta w^2-2q,\\ D&=3\gamma w^2-3r,\\ E&=12\delta w^2.\end{aligned}$$

The equation of the parallel surface is of course

$$(AE-4BD+3C^2)^3-27(ACE-AD^2-B^2E+2BCD-C^3)^2=0.$$

It is remarked (Salmon, [Ed. 2], p. 148 [Ed. 4, p. 176]) that there is in the plane $z=0$, a nodal conic $\frac{x^2}{a-c}+\frac{y^2}{b-c}-\left(1+\frac{k^2}{c}\right)w^2=0$, the complete section being made up of this conic twice, and of the curve of the eighth order which is the parallel curve of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-w^2=0$; the like is of course the case as to the sections by the other two principal planes $x=0$ and $y=0$. For the section by the plane $w=0$ (or plane infinity) we have at once $p^2r^2(4pr-q^2)=0$, where observe that

$$\begin{aligned}q^2-4pr &= \{(b^2+c^2)x^2+(c^2+a^2)y^2+(a^2+b^2)z^2\}^2-4(x^2+y^2+z^2)(b^2c^2x^2+c^2a^2y^2+a^2b^2z^2),\\ &=(1, 1, 1, -1, -1, -1\between(b^2-c^2)x^2, (c^2-a^2)y^2, (a^2-b^2)z^2)^2\\ &=\text{norm. }\{x\sqrt{(b^2-c^2)}+y\sqrt{(c^2-a^2)}+z\sqrt{(a^2-b^2)}\}.\end{aligned}$$

The section is thus made up of two conics, each twice, and of four right lines: viz. the conics are $x^2+y^2+z^2=0$, the circle at infinity and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$, the section at infinity of the ellipsoid; and the lines are

$$x\sqrt{(b^2-c^2)}\pm y\sqrt{(c^2-a^2)}\pm z\sqrt{(a^2-b^2)}=0,$$

viz. these are the common tangents of the two conics. The circle at infinity is a nodal conic on the surface, which has thus 4 nodal conics.