502.

ON THE SURFACES DIVISIBLE INTO SQUARES BY THEIR CURVES OF CURVATURE.

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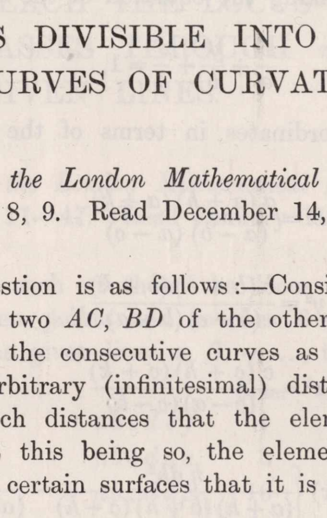
GEOMETRICALLY, the question is as follows:—Consider any two curves of curvature AB, CD of one set, and any two AC, BD of the other set, as shown by the continuous lines of the figure: drawing the consecutive curves as shown by dotted lines, the curve consecutive to AB at an arbitrary (infinitesimal) distance from AB, the other three curves may be drawn at such distances that the elements at A, B, and C shall be each of them a square; but this being so, the element at D will not be in general a square, and it is only for certain surfaces that it is so. But if (whatever the curves



of curvature AB, CD, AD, BC may be) the element at D is a square, then it is clear that the whole surface can be, by means of its curves of curvature, divided into infinitesimal squares.

Analytically, if for a given surface the equations of its curves of curvature are expressed in the form $h=f(x, y, z)$, $k=\phi(x, y, z)$; then the coordinates x, y, z can be expressed each of them as a function of the parameters h, k, and we have for the element of distance between two consecutive points on the surface

$$dx^2 + dy^2 + dz^2 = A dh^2 + C dk^2,$$

where A, C are in general each of them a function of h and k. The condition for the divisibility into squares is that the quotient $A \div C$ shall be of the form function h $\div$ function k.

It was shown by M. Bertrand that, in a triple system of orthotomic isothermal surfaces, each surface possesses the property in question of divisibility into squares by means of its curves of curvature. But in such a triple system, each surface of the system is necessarily a quadric; so that the theorem comes to this, that a quadric surface is, by means of its curves of curvature, divisible into squares. The analytical verification is at once effected: taking the equation of the surface to be

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

then the expressions for the coordinates in terms of the parameters h, k of a curve of curvature are

$$x^2 = \frac{a\,(a+h)\,(a+k)}{(a-b)\,(a-c)},$$

$$y^2 = \frac{b\,(b+h)\,(b+k)}{(b-c)\,(b-a)},$$

$$z^2 = \frac{c\,(c+h)\,(c+k)}{(c-a)\,(c-b)},$$

and we have

$$4\,(dx^2 + dy^2 + dz^2) = (h-k)\left\{\frac{h\,dh^2}{(a+h)\,(b+h)\,(c+h)} - \frac{k\,dk^2}{(a+k)\,(b+k)\,(c+k)}\right\};$$

so that $A \div C$ is of the required form.

But there is nothing to show that the property is confined to quadric surfaces; and the question of the determination of the surfaces possessing the property appears to be one of considerable difficulty, and which has not hitherto been examined.