526.

ON A SURFACE OF THE EIGHTH ORDER.

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I REPRODUCE in an altered form, so as to exhibit the application thereto of the theory of the six coordinates of a line, the analysis by which Dr Hierholzer obtained the equation of the surface of the eighth order, the locus of the vertex of a quadricone which touches six given lines.

I call to mind that if $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ are the coordinates of any two points on a line, then the quantities (a, b, c, f, g, h), which denote respectively

$$(\beta\gamma' - \beta'\gamma,\ \gamma\alpha' - \gamma'\alpha,\ \alpha\beta' - \alpha'\beta,\ \alpha\delta' - \alpha'\delta,\ \beta\delta' - \beta'\delta,\ \gamma\delta' - \gamma'\delta),$$

and which are such that $af + bg + ch = 0$, are the six coordinates of the line[1].

Consider the given point (x, y, z, w) and the given line (a, b, c, f, g, h), and write for shortness

$$\begin{aligned} P &= \quad\ \ hy - gz + aw, \\ Q &= -hx \quad\ \ + fz + bw, \\ R &= \quad gx - fy \quad\ \ + cw, \\ S &= -ax - by - cz \quad\ \ , \end{aligned}$$

then taking (X, Y, Z, W) as current coordinates, the equation of the plane through the given point and line is

$$PX + QY + RZ + SW = 0.$$

Considering in like manner the given point (x, y, z, w) and the three given lines $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, \ldots)$, $(a_3, \ldots)$, then we have the three planes

$$\begin{aligned} P_1X + Q_1Y + R_1Z + S_1W &= 0, \\ P_2X + Q_2Y + R_2Z + S_2W &= 0, \\ P_3X + Q_3Y + R_3Z + S_3W &= 0, \end{aligned}$$

[1] Cayley, "On the six coordinates of a line," *Camb. Phil. Trans.* vol. XI. (1869), [435], pp. 290—323.

and if these planes have a common line, the point (x, y, z, w) is in a line meeting each of the three given lines; that is, the locus of the point is the hyperboloid through the three given lines. It follows that the equations

$$\left\| \begin{matrix} P_1, & Q_1, & R_1, & S_1 \\ P_2, & Q_2, & R_2, & S_2 \\ P_3, & Q_3, & R_3, & S_3 \end{matrix} \right\| = 0$$

reduce themselves to a single equation, that of the hyperboloid in question.

I write for shortness

$$\begin{aligned}(000) = (agh)\,x^2 + (bhf)\,y^2 + (cfg)\,z^2 + (abc)\,w^2 & \\ + [(abg) - (cah)]\,xw & \\ + [(bch) - (abf)]\,yw & \\ + [(caf) - (bcg)]\,zw & \\ + [(bfg) + (chf)]\,yz & \\ + [(cgh) + (afg)]\,zx & \\ + [(ahf) + (bgh)]\,xy, &\end{aligned}$$

viz. (123) will mean $(a_1 g_2 h_3)\,x^2 +$ etc. where $(a_1 g_2 h_3)$ etc. denote as usual the determinants

$$\left| \begin{matrix} a_1, & g_1, & h_1 \\ a_2, & g_2, & h_2 \\ a_3, & g_3, & h_3 \end{matrix} \right| \text{ etc.;}$$

then the equations in question are found to be $x(123) = 0$, $y(123) = 0$, $z(123) = 0$, $w(123) = 0$, reducing themselves to the single equation $(123) = 0$, which is accordingly that of the hyperboloid through the three lines[1].

Proceeding now to the above-mentioned problem, we have the point (x, y, z, w), and the six lines $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, \ldots)$ etc., say the lines 1, 2, 3, 4, 5, 6: the six planes

$$P_1 X + Q_1 Y + R_1 Z + S_1 W = 0, \text{ etc.}$$

must be tangents to the same quadricone; that is, considering the sections by the plane $W = 0$, the six lines

$$P_1 X + Q_1 Y + R_1 Z = 0, \text{ etc.}$$

must be tangents to the same conic, and the condition for this is

$$[1\,2\,3\,4\,5\,6] = 0,$$

[1] This equation is given in the paper above referred to, § 54.

where the symbol stands for the determinant

$$\begin{vmatrix} P_1^2, & Q_1^2, & R_1^2, & Q_1R_1, & R_1P_1, & P_1Q_1 \\ P_2^2 \ldots \\ \vdots \end{vmatrix}.$$

But as is well known this equation may be written

$$(*) \qquad (126)(346)(145)(235) - (146)(236)(125)(345) = 0,$$

where (126) etc. denote the determinants

$$\begin{vmatrix} P_1, & Q_1, & R_1 \\ P_2, & Q_2, & R_2 \\ P_3, & Q_3, & R_3 \end{vmatrix} \text{ etc.;}$$

or, what is the same thing, they denote the functions above represented by the like symbols $(126) = (a_1 g_2 h_6) x^2 + \text{etc.}$ The equation (∗) just obtained is Hierholzer's equation for the surface of the eighth order, the locus of the vertex of a quadricone which touches six given lines.

I remark that in my "Memoir on Quartic Surfaces," *Proc. Lond. Math. Soc.* vol. III. (1870), [445], pp. 19—69, I obtained the equation of the surface under the foregoing form $[1\,2\,3\,4\,5\,6] = 0$ or say $[(P, Q, R)^2] = 0$, noticing that there was a factor w^4, so that the order of the surface is $= 8$; and further that the equation might be written

$$w^8 \exp. \frac{1}{w} \{x(g\partial_c - h\partial_b) + y(h\partial_a - f\partial_c) + z(f\partial_b - g\partial_a)\} [(a, b, c)^2] = 0,$$

where exp. Θ (read exponential) denotes e^Θ, and $[(a, b, c)^2]$ denotes

$$\begin{vmatrix} a_1^2, & b_1^2, & c_1^2, & b_1c_1, & c_1a_1, & a_1b_1 \\ a_2^2, \\ \vdots \end{vmatrix}.$$

Also that the equation contains the four terms

$$x^8 [(a, -h, g)^2] + y^8 [(h, b, -f)^2] + z^8 [(-g, f, c)^2] + w^8 [(a, b, c)^2] = 0.$$

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