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NOTE ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS BY SERIES.

[From the *Oxford, Cambridge, and Dublin Messenger of Mathematics,* vol. v. (1869), pp. 77—82.]

THERE is a speciality in the integration of certain differential equations by series, which (though evidently quite familiar to those who have written on the subject—Ellis, Boole, Hargreave) has not, it appears to me, been exhibited in the clearest form. To fix the ideas, consider a linear differential equation of the second order integrable in the form $y = Ax^\lambda + Bx^{\lambda+1} + \ldots$; λ is determined by a quadratic equation, and for each value of λ the coefficients $B, C, \ldots$ are given multiples of A; we have thus the general solution

$$y = A\left(x^\alpha + \frac{B}{A}\,x^{\alpha+1}\,\ldots\right) + K\left(x^\beta + \frac{L}{K}\,x^{\beta+1} + \text{\&c.}\,\ldots\right).$$

The speciality referred to is, when the two roots differ by an integer number; suppose α is the smaller root, and $\beta = \alpha + k$ (k a positive integer) the larger. Then, inasmuch as the series

$$y = Ax^\alpha + Bx^{\alpha+1} + \text{\&c.}$$

is identical in form with the general solution as above written down, it is clear that, starting with the root $\lambda = \alpha$, the coefficients beginning with that of x^β, $= x^{\alpha+k}$, ought not to be any longer determinate multiples of A, but should contain a new arbitrary constant K, and thus that the series derived from the root $\lambda = \alpha$ should be the general solution containing two arbitrary constants. The most simple case is when the substitution of the series in the differential equation leads to a relation between *two* consecutive coefficients of the series. Here the values $\frac{B}{A}$, $\frac{C}{A}$, &c. are fractions the numerators and denominators of which are factorial functions of α such that, for some coefficient $\frac{F}{A}$ preceding $\frac{K}{A}$ (if K is the coefficient of $x^{\alpha+k}$), and for all the succeeding coefficients $\frac{G}{A}$, &c. there is in the *numerators* one and the same evanescent factor; this being so, it is allowable to write $F=0$, $G=0$, &c. giving for the differential equation the finite solution

$$y = A\left(x^\alpha + \frac{B}{A}x^{\alpha+1} \ldots + \frac{E}{A}x^{\alpha+e}\right):$$

but if, notwithstanding the evanescent factor, we carry on the series, then in the coefficient of $x^{\alpha+k}$ there occurs in the *denominator* the same evanescent factor, so that the coefficient of this term presents itself in the form $A\,\frac{P}{Q}\cdot\frac{0}{0}$, = an arbitrary constant K (since the $\frac{0}{0}$ is essentially indeterminate), and the solution is thus obtained in the form

$$y = A\left(x^\alpha + \frac{B}{A}x^{\alpha+1} \ldots + \frac{E}{A}x^{\alpha+e}\right) + K\left(x^{\alpha+k} + \frac{L}{K}x^{\alpha+k+1} + \text{\&c.}\right),$$

viz. there is one particular solution which is finite.

Take for example the equation

$$\frac{d^2y}{dx^2} + q\frac{dy}{dx} - \frac{2}{x^2}y = 0 \qquad \text{(I)},$$

mentioned *Cambridge Math. Journal*, t. II. p. 176 (1840). If the integral is assumed to be

$$y = Ax^\alpha + Bx^{\alpha+1} + Cx^{\alpha+2} + \text{\&c.},$$

then we find

$$\begin{aligned}
(\alpha+1)(\alpha-2)\,A &= 0,\\
(\alpha-1)(\alpha+2)\,B + q\alpha A &= 0,\\
\alpha\,(\alpha+3)\,C + q\,(\alpha+1)\,B &= 0,\\
(\alpha+1)(\alpha+4)\,D + q\,(\alpha+2)\,C &= 0,\\
&\text{\&c.}
\end{aligned}$$

Hence $\alpha = -1$, or else $\alpha = 2$;

$$B = \frac{-q\alpha}{(\alpha-1)(\alpha+2)}A,$$

$$C = \frac{q^2\cdot\alpha(\alpha+1)}{(\alpha-1)\,\alpha\,(\alpha+2)(\alpha+3)}A,$$

$$D = \frac{-q^3\alpha(\alpha+1)(\alpha+2)}{(\alpha-1)\,\alpha\,(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}A,$$

&c.

Here taking $\alpha=-1$, we are at liberty to make C and all the following coefficients $=0$: in fact, if we commence by assuming

$$y = Ax^{\alpha} + Bx^{\alpha+1},$$

the equations

$$\begin{aligned} (\alpha+1)(\alpha-2)\,A &= 0,\\ (\alpha-1)(\alpha+2)\,B + \quad q\alpha A &= 0,\\ q(\alpha+1)\,B &= 0, \end{aligned}$$

are all satisfied if only $\alpha=-1$, $B=\dfrac{-q\alpha A}{(\alpha-1)(\alpha+2)}$; and we have thus the finite solution $y=A\left(\dfrac{1}{x}-\dfrac{q}{2}\right)$; but if we continue the series, retaining D to represent the indeterminate quantity $\dfrac{-q^3\alpha(\alpha+1)(\alpha+2)}{(\alpha-1)\,\alpha\,(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}A$, we have the solution

$$y = A\left(\frac{1}{x}-\tfrac{1}{2}q\right) + D\left(x^2 - \tfrac{1}{2}qx^3 + \&c.\right),$$

the second member of which is in fact the series derived from the root $\alpha=2$. This series is expressible by means of an exponential, viz. we have

$$x^2 - \tfrac{1}{2}qx^3 + \&c. = \frac{12}{q^3}\left\{\left(\frac{1}{x}+\tfrac{1}{2}q\right)e^{-qx} - \left(\frac{1}{x}-\tfrac{1}{2}q\right)\right\},$$

and the complete integral is thus

$$y = A\left(\frac{1}{x}-\tfrac{1}{2}q\right) + B\left(\frac{1}{x}+\tfrac{1}{2}q\right)e^{-qx},$$

but this result is not immediately connected with the investigation. It may however be noticed that, writing $z=ye^{qx}$, the equation in z is

$$\frac{d^2z}{dx^2} - q\frac{dz}{dx} - \frac{2}{x^2}z = 0,$$

which only differs from the original equation in that it has $-q$ in place of q: there is consequently the particular solution $z=\dfrac{1}{x}+\tfrac{1}{2}q$, giving for y the particular solution $y=\left(\dfrac{1}{x}+\tfrac{1}{2}q\right)e^{-qx}$, and we have thence the complete solution as above.

Consider, secondly, the differential equation

$$\frac{d^2y}{dx^2} - \left(\tfrac{1}{4}q^2 + \frac{2}{x^2}\right)y = 0 \qquad \text{(II)},$$

(derived from the equation (I) by writing therein ye^{-qx} in place of y); this equation is satisfied by the series

$$y = Ax^{\alpha} + Bx^{\alpha+2} + Cx^{\alpha+4} + \&c.$$

Then $\alpha = -1$ or $\alpha = +2$, but here the series belonging to $\alpha = -1$ contains only odd powers, the other contains only even powers of x; hence the two series do not coalesce as in the former case, and the first series is obtained without the indeterminate symbol $\frac{0}{0}$ in any of the coefficients. We have in fact

$$\begin{gathered}(\alpha+1)(\alpha-2)\,A = 0,\\ (\alpha+3)(\alpha\quad\;)\,B - \tfrac{1}{4}q^2 A = 0,\\ (\alpha+5)(\alpha+2)\,C - \tfrac{1}{4}q^2 B = 0,\\ \vdots\end{gathered}$$

and there is not in the series in question (or in the other series) any evanescent factor, either in the numerators or in the denominators.

But consider, thirdly, the differential equation

$$\frac{d^2y}{dx^2} + (q-2\theta)\frac{dy}{dx} + \left(\theta^2 - q\theta - \frac{2}{x^2}\right)y = 0 \qquad \text{(III)},$$

(derived from (I) by writing therein $ye^{-\theta x}$ in place of y). This is satisfied by the series

$$y = Ax^{\alpha} + Bx^{\alpha+1} + Cx^{\alpha+2} + Dx^{\alpha+3} + \&\text{c.},$$

where $\alpha = -1$ or $\alpha = +2$ as before; in the series belonging to $\alpha = -1$, the coefficient D should become indeterminate. The relation between the coefficients is here a relation between three consecutive coefficients, viz. we have

$$\begin{aligned}&(\alpha+1)(\alpha-2)\,A = 0,\\ &(\alpha+2)(\alpha-1)\,B + (q-2\theta)(\quad\alpha\quad)\,A = 0,\\ &(\alpha+3)(\quad\alpha\quad)\,C + (q-2\theta)(\alpha+1)\,B + (\theta^2 - q\theta)\,A = 0,\\ &(\alpha+4)(\alpha+1)\,D + (q-2\theta)(\alpha+2)\,C + (\theta^2 - q\theta)\,B = 0,\\ &(\alpha+5)(\alpha+2)\,E + (q-2\theta)(\alpha+3)\,D + (\theta^2 - q\theta)\,C = 0,\\ &\&\text{c.}\end{aligned}$$

It is to be shown that in the series for $\alpha = -1$, the expression $(q-2\theta)(\alpha+2)\,C + (\theta^2 - q\theta)\,B$ contains the evanescent factor $(\alpha+1)$, and consequently that D is indeterminate; we have in fact

$$B = -\frac{(q-2\theta)\,\alpha}{(\alpha+2)(\alpha-1)}\,A,$$

$$C = \frac{1}{\alpha(\alpha+3)}\left\{\frac{(q-2\theta)^2\,\alpha\,(\alpha+1)}{(\alpha+2)(\alpha-1)} - (\theta^2 - q\theta)\right\},$$

and thence

$$(q-2\theta)(\alpha+2)\,C + (\theta^2 - q\theta)\,B$$

$$= \frac{1}{\alpha(\alpha+3)}\left\{-\frac{(q-2\theta)^3\,\alpha\,(\alpha+1)(\alpha+2)}{(\alpha+2)(\alpha+1)} - (q-2\theta)(\theta^2 - q\theta)(\alpha+2)\right\}A - \frac{(\theta^2 - q\theta)(q-2\theta)\,\alpha}{(\alpha+2)(\alpha-1)},$$

and then

$$\frac{\alpha+2}{\alpha(\alpha+3)}+\frac{\alpha}{(\alpha+2)(\alpha-1)}=\frac{2\alpha^3+6\alpha^2-4}{(\alpha-1)\,\alpha\,(\alpha+2)(\alpha+3)}=\frac{2(\alpha+1)(\alpha^2+2\alpha-2)}{(\alpha-1)\,\alpha\,(\alpha+2)(\alpha+3)},$$

so that the whole expression contains the factor $\alpha+1$. But observe that in the present case, if (as is allowable) we write $D=0$, the next coefficient E (depending not on D only, but on D and C) will not vanish; so that the solution obtained on the assumption $D=0$ will go on to infinity: and if instead of assuming $D=0$, we assume $D=$ an arbitrary quantity D', then E and the subsequent coefficients will contain terms depending on D'; and the complete form of the series belonging to $\alpha=-1$ will be

$$y=A\left(x^{-1}+\frac{B}{A}+\frac{C}{A}x+\frac{D}{A}x^2+\frac{E}{A}x^3+\&\text{c.}\right)+D'\left(x^2+\frac{E}{D'}x^3+\&\text{c.}\right),$$

where the second member is in fact the series belonging to $\alpha=2$. It is hardly necessary to remark that the solution thus obtained can be expressed by means of exponentials, viz. that the solution is

$$y=A\left(\frac{1}{x}-\tfrac{1}{2}q\right)e^{\theta x}+D'\frac{12}{q^3}\left\{\left(\frac{1}{x}+\tfrac{1}{2}q\right)e^{-qx}-\left(\frac{1}{x}-\tfrac{1}{2}q\right)\right\}e^{\theta x}.$$