545.

ON THE THEORY OF THE SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

[From the *Messenger of Mathematics*, vol. II. (1873), pp. 6—12.]

I CONSIDER a differential equation under the form

$$\phi(x, y, p) = 0,$$

where

1°. ϕ is as to p, rational and integral of the degree n;

2°. it is, or is taken to be, one-valued in regard to (x, y);

3°. it has no mere (x, y) factor;

4°. it is indecomposable as regards p.

Considering (x, y) as the coordinates of a point *in plano*, the differential equation determines a system of curves, in general indecomposable, the system depending on a single variable parameter, and such that through each point of the plane there pass n curves.

Such a system is represented by an integral equation

$$f(x, y, c_1, c_2 \dots c_m) = 0,$$

where

5°. f is rational and integral in regard to the m constants, which constants are connected by an algebraic $(m-1)$ fold relation;

6°. it is, or is taken to be, one-valued in regard to (x, y);

7°. it has no mere (x, y) factor;

8°. it is indecomposable as regards (x, y);

9°. Considering (x, y) as given, the equation $f=0$, together with the $(m-1)$ fold relation between the constants, must constitute a m-fold relation of the order n, that is, must give for the constants n sets of values. We may, if we please, take f to be linear in regard to the constants $c_1, c_2, \ldots, c_m$, and then the condition simply is, that the $(m-1)$ fold relation shall be of the order n.

I give in regard to these definitions such explanations as seem necessary.

2°. A one-valued function of (x, y) is either a rational function, or a function such as $e^x \sin y$, &c., which for any given values whatever, real or imaginary, of the variables, has only one value. A function is taken to be one-valued when, either for any values whatever of the variables, or for any class of values (e.g. all real values), we select for any given values of the variables one value, and attend exclusively to such one value, of the function.

Thus, U a rational function of (x, y), $\phi = p^2 - U = 0$, ϕ is one-valued in regard to (x, y). But if, U not being the square of a rational function, we take $\sqrt{(U)}$ to be a one-valued function (consider it as denoting, say for all real values of x, y for which U is positive, the positive square-root of U), then, $\phi = p + \sqrt{(U)} = 0$, ϕ is taken to be one-valued in regard to (x, y).

3°. The meaning is, that the equation $\phi = 0$ is not satisfied irrespectively of the value of p, by any relation between the variables x, y.

4°. The meaning is that ϕ is not the product of two factors, each rational and integral in regard to p, and being or being taken to be one-valued in regard to (x, y). Thus, if as before U is a rational function of (x, y) but is not the square of a rational function, and if we do not take any one-valued function, then the equation $\phi = p^2 - U = 0$ is indecomposable; but if we take $\sqrt{(U)}$ as one-valued, then we have $\phi = \{p - \sqrt{(U)}\}\{p + \sqrt{(U)}\} = 0$, and the equation breaks up into the two equations $p - \sqrt{(U)} = 0$ and $p + \sqrt{(U)} = 0$. I assume as an axiom, that the curves represented by the indecomposable differential equation are in general indecomposable; for supposing the differential equation satisfied in regard to a system of curves, the general curve breaking into two curves, each depending on the arbitrary parameter, then we have two distinct systems of curves; either the differential equation is satisfied in regard to each system separately, and in this case they are the same system twice repeated; or the differential equation is satisfied in regard to one system only, and in this case the other system is not part of the solution, and it is to be rejected. As an instance, take the equation $\phi = px + y = 0$, $(x\,dy + y\,dx = 0)$; if we choose to integrate this in the form $x^2y^2 - c = 0$, this equation represents the two hyperbolas $xy + \sqrt{(c)} = 0$, $xy - \sqrt{(c)} = 0$, but considering each of these separately, and giving to the constant theory any value whatever, we have simply the system of hyperbolas $xy - c = 0$ twice repeated. But if by any (faulty) process of integration the solution had been obtained in a form such as $(c+x)(c-xy)=0$, then the differential equation is not satisfied in regard to the system $c + x = 0$; and the factor $c + x$ is to be rejected. Observe that it is said, that the curves are *in general* indecomposable; particular curves of the system may very well be decomposable; thus in the foregoing example, where the system of curves is $xy - c = 0$, in the particular case $c = 0$, the hyperbola breaks up into the two lines $x = 0$, $y = 0$.

5°. It is *necessary* to consider a form $f=0$ involving the m constants connected by the $(m-1)$ fold relation. For taking such a system of constants, imagine an equation $f(x, y, c_1, c_2, \ldots, c_m)=0$ rational and integral in regard to (x, y), and representing an indecomposable curve; such an integral equation leads to a differential equation of the form $\phi(x, y, p)=0$, rational in regard to (x, y); whence, conversely, a differential equation of the form last referred to *may have* for its integral the equation $f(x, y, c_1, c_2, \ldots, c_m)=0$. And we cannot *in a proper form* exhibit this integral in terms of a single constant. For first consider for a moment $c_1, c_2, \ldots, c_m$ as the coordinates of a point in m-dimensional space; the curve is not in general unicursal, and unless it be so, we cannot express the quantities $c_1, c_2, \ldots, c_m$ rationally in terms of a parameter; that is, we cannot in general express $c_1, c_2, \ldots, c_m$ rationally in terms of a parameter. Secondly, if by means of the $(m-1)$ fold relation we sought to eliminate from the equation $f=0$ all but one of the m constants, we should indeed arrive at an equation $F(x, y, c)=0$ rational and integral as regards x and y, and also as regards c; but this equation *would not represent an indecomposable curve.*

6°. It is important to remark that, even in the case where $\phi(x, y, p)$ is one-valued in regard to (x, y) (2°), there is not in every case a form $f=0$ one-valued in regard to (x, y). A simple example shows this; let α, β be incommensurable (e.g. $\alpha=e$, $\beta=\pi$), then the equation $\phi=\beta xp+\alpha y=0\ (\alpha y\,dx+\beta x\,dy=0)$ has for its integral $c=x^\alpha y^\beta$, where $x^\alpha y^\beta$ is not a one-valued function of x, y, and we cannot in any way whatever transform the integral so as to express it in terms of one-valued functions of x and y. But taking $x^\alpha y^\beta$ to be a one-valued function—if e.g. for all real values of x, y we consider $x^\alpha y^\beta$ as representing the real value of $(\pm x)^\alpha(\pm y)^\beta$—, we shall have, without any loss of generality, the integral of the differential equation; the *whole* system of curves $c=x^\alpha y^\beta$, c any value whatever, is the same whether we attribute to $x^\alpha y^\beta$ its infinite series of values, or only one of these values.

7°. The meaning is, that the equation $f=0$ is not satisfied irrespectively of the values of $c_1, c_2, \ldots, c_m$ by any relation between x, y only.

8°. The meaning is, that the function f is not the product of two factors, rational or irrational in regard to $c_1, c_2, \ldots, c_m$, but each of them one-valued or taken to be one-valued in regard to x, y. Thus the function

$$f=x^2y^2-c, \quad =\{xy+\sqrt{(c)}\}\{xy-\sqrt{(c)}\},$$

is decomposable; but if we do not take any one-valued function, then $f=xy-c$ is indecomposable; if we take $\sqrt{(xy)}$ to be one-valued, then it is decomposable.

The case $f=x^\alpha y^\beta-c$ (α and β incommensurable) is to be noticed; starting from the differential equation $\alpha y\,dx+\beta x\,dy=0$, there is no reason for writing the integral $f=x^\alpha y^\beta-c=0$ rather than in either of the forms $f=x^{m\alpha}y^{m\beta}-c=0$, $f=x^{\frac{\alpha}{m}}y^{\frac{\beta}{m}}-c=0$, ($m$ an integer), (α, β), $(m\alpha, m\beta)$, $\left(\dfrac{\alpha}{m}, \dfrac{\beta}{m}\right)$ are, each pair as well as the others, two incommensurable magnitudes. If we choose to take $x^\alpha y^\beta$ but not $x^{\frac{\alpha}{m}}y^{\frac{\beta}{m}}$ as one-valued, then $f = x^\alpha y^\beta - c$ and $f = x^{m\alpha} y^{m\beta} - c$ are each one-valued, and the former is, the latter is not, indecomposable; they would be each decomposable if we chose to take $x^{\frac{\alpha}{m}} y^{\frac{\beta}{m}}$ as one-valued; and if only $x^{m\alpha} y^{m\beta}$ were taken to be one-valued, then $f = x^{m\alpha} y^{m\beta} - c$ would be indecomposable.

9°. This is a mere statement of the condition in order that the system of curves represented by the integral equation may be such that, through a given point of the plane, there may pass n of these curves. Since the number of constants is unlimited, there is clearly no loss of generality in assuming that the equation is linear in regard to the several constants.

I consider now the differential equation

$$\phi(x, y, p) = 0,$$

(as already stated of the degree n as regards p), and its integral equation

$$f(x, y, c_1, c_2, \ldots, c_m) = 0.$$

I take (x, y) to be the coordinates of a point, say in the horizontal plane, and I use C to refer to the constants $c_1, c_2, \ldots, c_m$ collectively, thus for given values of x, y I speak of the n values of C, meaning thereby the n values of the set $(c_1, c_2, \ldots, c_m)$; of C having a two-fold value, meaning thereby that two of the sets $c_1, c_2, \ldots, c_m$ become identical; and so on.

The case $C = c$, where there is only a single constant c, is interesting as affording an easier geometrical conception; we may take $c = z$ to be a third coordinate; the equation $f(x, y, z) = 0$ thus represents a surface, such that its plane sections $z = c$, or say these sections projected by vertical ordinates on the horizontal plane $z = 0$ are the series of curves $f(x, y, c) = 0$. But the case is not really distinct from the general one.

The theory of singular solutions depends on the following considerations:

To a given point P on the horizontal plane belong n values of C, each determining a curve $f(x, y, C) = 0$ through P; and also n values of p, viz. these give the directions at P of the n curves respectively.

The curve $f(x, y, C) = 0$ may be such as to have in general a certain number of nodes and of cusps (either or each of these numbers being $= 0$): we may imagine C determined, say $C = C_0$, so that the curve shall have one additional node: this node I call a "level point." Take P at the level point, there are n values of C, viz. C_0 and $n - 1$ other values; that is, there are through P the nodal curve, and $n - 1$ other curves, and therefore $2 + (n - 1), = n + 1$ directions of the tangent; but the directions are determined by the equation $\phi = 0$ of the order n: and the only way in which we can have more than n values is when this equation becomes an identity $0 = 0$; that is, P at the level point, the function $\phi(x, y, p)$ will vanish identically, irrespectively of the value of p.

The ordinary nodes (if any) on the curves $f(x, y, C)=0$ form a locus called the "nodal locus," and the cusps (if any) a locus called the "cuspidal locus." Take the point P a given point on the nodal locus, C has a two-fold value answering to the curve in regard to which P is a node, and $n-2$ other values; that is, the n curves through P are the nodal curve reckoned twice and $n-2$ other curves; the directions are the directions at the node, and the $n-2$ other directions, in all n directions, which are the directions given by the equation $\phi=0$; there is no peculiarity in regard to this equation.

Similarly, take P anywhere on the cuspidal locus: C has a two-fold value, answering to the curve in regard to which P is a cusp, and $n-2$ other values; that is, the n curves through P are the cuspidal curve reckoned twice and $n-2$ other curves; the directions are the direction at the cusp reckoned twice and $n-2$ other directions: in all n directions, which are the directions given by the equation $\phi=0$; this equation thus gives a two-fold value of p.

There is a locus (distinct from the nodal and cuspidal loci) which may be called the "envelope locus," such that taking P anywhere on this locus C has a two-fold value; for such position of P the n values of C are the value in question reckoned twice and $n-2$ other values; the n curves through P are that belonging to the two-fold value of C, or say the two-fold curve, and $n-2$ other values; and the n directions are the direction along the two-fold curve counted twice, and $n-2$ other directions; these are the n directions given by the equation $\phi=0$, viz. this equation gives a two-fold value of p.

The envelope locus may be an indecomposable curve, or it may break up into two or more curves; and it may happen that either the whole curve or one or more of the component curves may coincide with a particular curve or curves of the system $f(x, y, C)=0$.

There is a locus (distinct from the cuspidal and envelope loci) which may be called the tac-locus, such that taking P anywhere on this locus p has a two-fold value; for such position of P, there is no peculiarity as regards C, viz. C has n distinct values giving rise to n curves through P; but as the directions are given by the equation $\phi=0$, two of the curves touch each other, viz. the tac-locus is the locus of points, such that at any one of them two of the curves $f(x, y, C)=0$ through the point touch each other.

We may by an extension of the received notation write

$$\operatorname{disct}_C f(x, y, C)=0$$

to denote the equation between (x, y), such that, for any values of (x, y), which satisfy the condition, or say for any position of P on the C-discriminant locus, there is a two-fold value of C. By what precedes it appears that the C-discriminant locus is made up of the nodal, cuspidal, and envelope loci, and without going into the proof I infer that it is in fact made up of the nodal locus twice, the cuspidal locus three times, and the envelope locus once.

Writing moreover

$$\text{disct}_p\, \phi(x, y, p) = 0$$

to denote the equation between (x, y), such that for any values of (x, y) which satisfy the condition, or say for any position of P on the p-discriminant locus, there is a two-fold value of p. By what precedes, it appears that the p-discriminant locus is made up of the envelope locus, cuspidal locus, and the tac-locus; as I infer, each of them once.

The foregoing are the abstract principles: I consider the singular solution to be that given by the equation which belongs to the envelope-locus (viz. *I do not recognise any singular solution which is not of the envelope species*); and the result of the investigation is, when we seek in the ordinary way to obtain the singular solution, whether from the integral equation or from the differential equation, that we account for the extraneous factors which present themselves in the two processes respectively. I reserve for another communication the discussion of particular examples.