553.

TWO SMITH'S PRIZE DISSERTATIONS.

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WRITE dissertations:

1. On the condition of the similarity of two dynamical systems.

2. On orthogonal surfaces.

1. We may consider two particles m, m', describing similarly two similar paths (which for convenience may be taken to be similarly situate in regard to two sets of rectangular axes respectively), viz. this means that the times t, t' of passage through corresponding arcs s, s' are proportional. The ratios $\frac{s'}{s}$, $\frac{t'}{t}$, are thus each of them constant; and this must also be the case with the ratio $\frac{v'}{v}$, of the velocities v, v' at corresponding points; since it is clear that we must have $\frac{s'}{s} = \frac{v'}{v} \cdot \frac{t'}{t}$.

Now in order that the two particles may move as above under the action of any forces upon the two particles respectively, it is clearly necessary that the forces F, F' at corresponding points shall act in the same direction, and be in a constant ratio of magnitude. To obtain this ratio, imagine the two particles, masses m, m', moving as above, in the corresponding infinitesimal elements of time τ, τ', with the velocities v, v' through the infinitesimal arcs σ, σ' respectively, $\left(\frac{\tau}{\tau'} = \frac{t}{t'},\ \frac{v}{v'} = \frac{v}{v'},\ \frac{\sigma}{\sigma'} = \frac{s}{s'}\right)$; the deflections from the tangent will be $\frac{1}{2}\frac{F}{m}\tau^2$, $\frac{1}{2}\frac{F'}{m'}\tau'^2$ respectively, and these must be in the ratio of the corresponding arcs σ, σ', viz. we must have

$$\frac{F\tau^2}{m} : \frac{F'\tau'^2}{m'} = \sigma : \sigma',$$

or, what is the same thing,

$$\frac{Ft^2}{m} : \frac{F't'^2}{m'} = s : s',$$

that is,

$$\frac{F'}{F} = \frac{m'}{m}\,\frac{s'}{s} \div \left(\frac{t'}{t}\right)^2,$$

and this relation subsisting, and the velocities at the beginnings of the elements of time τ, τ' being in the assumed ratio, it is clear that the velocities at the ends of these elements of time will be in the same ratio; and thus the two particles will go on moving in the manner in question.

All that has been said as to two particles, applies without alteration to any two systems of particles moving under the like geometrical conditions, and we thus arrive at the conclusion; given two similarly constituted systems, which at any instant are in a given magnitude-ratio $\frac{s'}{s}$, their component particles being in a given ratio $\frac{m'}{m}$ (the same for each pair of component particles), then if the particles of the two systems respectively are to move in similar paths of the same magnitude-ratio $\frac{s'}{s}$, the times of describing corresponding arcs being in a given constant ratio $\frac{t'}{t}$ $\Big($this implying as above that the ratio of the velocities at corresponding points is $\frac{v'}{v}$, $=\frac{s'}{s} \div \frac{t'}{t}\Big)$, it is necessary that the forces on corresponding particles in corresponding positions shall act in the same directions, and shall be in the constant magnitude-ratio

$$\frac{F'}{F} = \frac{m'}{m} \cdot \frac{s'}{s} \div \left(\frac{t'}{t}\right)^2,$$

and this being so, the motion of the two systems will in fact be similar as above explained.

Taking $\frac{m'}{m}$, $=\mu$ for the mass-ratio, $\frac{s'}{s}$, $=\sigma$ for the length-ratio, and $\frac{t'}{t}$, $=\tau$ for the time-ratio; also $\frac{F'}{F}$, $=\phi$ for the force-ratio, the condition determining the force-ratio ϕ is thus

$$\phi = \frac{\mu\sigma}{\tau^2}.$$

It is to be observed, that if the forces are entirely internal, and proportional to homogeneous functions of the same order, say $-n$, of the coordinates of all or any of the particles; e.g. if they are central forces varying as the inverse nth power of the distances; then the condition as to the action of the forces in the two systems respectively can always be satisfied by giving a proper constant value to the ratio of the absolute forces (or forces at unity of distance); thus, if in the first system we have two particles m_1, m_2 attracting or repelling each other with a force $\frac{km_1m_2}{r^n}$, and if in the second system the force is $\frac{k'm_1'm_2'}{r'^n}$; then the condition as to the direction of the forces at corresponding positions is satisfied *ipso facto;* and the condition as to magnitude is

$$\frac{k'm_1'm_2'}{r'^n} \times \frac{r^n}{km_1m_2} = \frac{m'}{m}\frac{s'}{s}\frac{t^2}{t'^2},$$

that is,

$$\frac{k'}{k} = \frac{m'}{m_1'm_2'}\frac{m_1m_2}{m}\frac{s'r'^n}{sr^n}\frac{t^2}{t'^2}$$

$$= \frac{m}{m'}\left(\frac{s'}{s}\right)^{n+1}\frac{t^2}{t'^2},$$

or, say

$$\frac{t'}{t} = \left(\frac{s'}{s}\right)^{\frac{n+1}{2}}\left(\frac{km}{k'm'}\right)^{\frac{1}{2}}.$$

In the case $n = 2$, the present theorem (applying however only to the case of two elliptic orbits of the same eccentricity) agrees with Kepler's third law, or say with the theorem

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{(\mu)}},$$

that is,

$$T \propto \frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}},$$

where observe that the μ, or mass in the sense of the formula, is the km, or attractive force on a unit of mass, of the theorem as above written down.

2. In a family of surfaces $F(x, y, z, p) = 0$, containing a single variable parameter p, there is through any given point of space, a surface or surfaces of the family; or (if more than one, confining the attention to one of these surfaces) we may say that there is, through any given point of space, a surface of the family.

Considering now two other families of surfaces, there will be through any given point of space, three surfaces, one of each family; and if (for every given point of space whatever) these intersect each other at right angles, we have a system of orthogonal surfaces.

Supposing the equations of the three families to be

$$F(x, y, z, p) = 0,$$

$$\Phi(x, y, z, q) = 0,$$

$$\Psi(x, y, z, r) = 0,$$

then the requisite conditions are

$$\frac{dF}{dx}\frac{d\Phi}{dx}+\frac{dF}{dy}\frac{d\Phi}{dy}+\frac{dF}{dz}\frac{d\Phi}{dz}=0,$$

$$\frac{dF}{dx}\frac{d\Psi}{dx}+\ldots\ldots\ldots\ldots\ldots\ldots=0,$$

$$\frac{d\Phi}{dx}\frac{d\Psi}{dx}+\ldots\ldots\ldots\ldots\ldots\ldots=0,$$

viz. these equations must be satisfied, not in general identically, but in virtue of the given equations $F=0$, $\Phi=0$, $\Psi=0$.

Or, what is more convenient, we may take the equations of the three families to be

$$p-f(x,\ y,\ z)=0,\quad q-\phi(x,\ y,\ z)=0,\quad r-\psi(x,\ y,\ z)=0\,;$$

and write the conditions in the form

$$\frac{dp}{dx}\frac{dq}{dx}+\frac{dp}{dy}\frac{dq}{dy}+\frac{dp}{dz}\frac{dq}{dz}=0,$$

$$\frac{dp}{dx}\frac{dr}{dx}+\ldots\ldots\ldots\ldots\ldots\ldots=0,$$

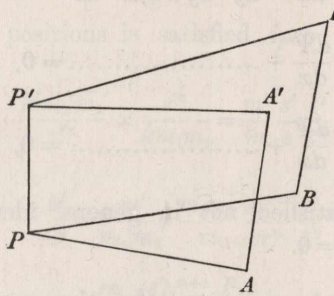
$$\frac{dq}{dx}\frac{dr}{dx}+\ldots\ldots\ldots\ldots\ldots\ldots=0,$$

where of course p, q, r stand for their given functional values, $p=f(x,\ y,\ z)$, &c.; the equations in this form contain only $(x,\ y,\ z)$, and not the parameters p, q, r; so that, if satisfied at all, they must be satisfied identically; and the required conditions therefore are that the last-mentioned system of equations shall be satisfied identically by the values p, q, r considered as given functions of $(x,\ y,\ z)$.

The last-mentioned conditions lead to the theorem known as Dupin's; viz. it follows from them that the surfaces intersect along their curves of curvature; or more definitely, each surface of one family is intersected by the surfaces of the other two families in its two sets of curves of curvature respectively.

To indicate the geometrical ground of the theorem, consider on a surface of one family a point P, and at this point the normal meeting the consecutive surface in P'; the surfaces through P of the other two families respectively will pass through P', and meet the given surface in two curves PA, PB (viz. PA, PB represent infinitesimal arcs on these two curves respectively), the angle at P being a right angle.

Drawing at A, B normals to the given surface to meet the consecutive surface in the points A', B' respectively, the same two surfaces will meet the consecutive surface in the arcs $P'A'$, $P'B'$ respectively; and (the system being orthogonal), we must have the angle at P' a right angle. This imposes a condition upon the direction (in the tangent plane at P) of the orthogonal directions PA, PB; viz. it is found that these must be such that the normals PP', AA' intersect, or, what is the same



thing, the normals PP', BB' (one of these conditions implying the other); that is, that the lines PA, PB shall be the directions of the two curves of curvature through P on the given surface.

Observe that PP', AA' intersecting each other, the four points P, P', A, A' are in the same plane, that is, PA, $P'A'$ intersect, these lines being the normals at P, P' respectively of the surface through P of one of the other two families; and similarly PP', BB' intersecting each other, the lines PB, $P'B'$ intersect; these being the normals at P, P' respectively of the surface through P of the other two families. We have through PP' two planes at right angles to each other; and these are met by a plane $A'P'B'$, in two lines $A'P'$, $B'P'$, the inclinations of which to the line PP' differ only infinitesimally from a right angle, say they are $90° - a$ and $90° - b$ respectively; hence if the angle $A'P'B'$ is $= 90° - c$, this is the hypotenuse of a right-angled spherical triangle, the sides whereof are $90° - a$, $90° - b$; wherefore $\sin c = \sin a \sin b$, viz. $\sin c$ is an infinitesimal of a higher order which may be neglected, or the angle P' will be $= 90°$; that is, the surfaces through P of the other two families, intersecting the given surface in the directions PA, PB of the two curves of curvature, will intersect the consecutive surface at P' in the two directions $P'A'$, $P'B'$ *at right angles to each other;* which is an *a posteriori* verification of Dupin's theorem.

In what precedes the given surface through P may be regarded as a surface assumed at pleasure; and it in effect appears that taking the consecutive surface also at pleasure (but varying only infinitesimally from the given surface), the condition in order that the two surfaces, which at P intersect each other and the given surface at right angles, shall at P' intersect the consecutive surface in two directions at right angles to each other, is that they shall intersect the given surface in the directions PA, PB of the two curves of curvature. But if we thus take the consecutive surface at pleasure,—or say if we construct it by measuring off along the normal at *each* point P of the given surface an infinitesimal distance PP', $= \rho$, where ρ an arbitrary function of the coordinates of the point P,—then although on the consecutive surface the lines $P'B'$, $P'A'$ are at right angles to each other, there is nothing to show, and it is not in fact the case, that these lines $P'A'$, $P'B'$ are the directions of the curves of curvature on the consecutive surface. In the orthogonal system they must be so; and this imposes upon the infinitesimal normal distance ρ, a condition; viz. it is found that ρ considered as a function of (x, y, z) must satisfy a certain partial differential equation of the second order.

It hence appears that no one of the three families of surfaces can be assumed at pleasure; for taking the equation of a family to be $p-f(x, y, z)=0$, then p being the value of the parameter for the given surface of the foregoing investigation, and $p+\delta p$ the value of the parameter for the consecutive surface, the normal distance at the point (x, y, z) between the two surfaces is

$$=\delta p \div \sqrt{\left\{\left(\frac{dp}{dx}\right)^2+\left(\frac{dp}{dy}\right)^2+\left(\frac{dp}{dz}\right)^2\right\}};$$

viz. δp is here a constant; and we have

$$1 \div \sqrt{\left\{\left(\frac{dp}{dx}\right)^2+\left(\frac{dp}{dy}\right)^2+\left(\frac{dp}{dz}\right)^2\right\}},$$

satisfying the foregoing partial differential equation; or, what is the same thing, p considered as a function of x, y, z must satisfy a certain partial differential equation of the third order; viz. this is the condition to be satisfied in order that a family of surfaces $p-f(x, y, z)=0$ may belong to an orthogonal system.