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Thermal waves in inelastic bodies (*)

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A THERMODYNAMIC theory of materials with internal state variables is proposed. In this theory the response of a material depends on the deformation, the temperature and two groups of internal state variables, namely the thermal variables and the mechanical ones. The evolution of these internal variables is governed by the first-order differential equation (evolution equation) in which the temperature gradient as an additional independent variable also occurs. The form of the evolution equation proposed for the thermal variables leads to the functional constitutive equation for the heat flux. The similarity between this equation and the Maxwell-Cattaneo relation is discussed. In the theory the stress, the entropy and the heat flux, too, are determined by the free energy function of a material. This fact is the consequence of the second law of thermodynamics. The theory constructed is used for the investigation of one-dimensional acceleration and shock waves. The general equations for the velocity of these two kinds of waves propagating into a material at an equilibrium state are obtained. The particular forms of the constitutive and the evolution equations for an elastic-viscoplastic body are proposed in the discussion of the shock waves. The conditions under which uncoupled mechanical and thermal waves propagate into the general material with internal state variables are derived. The corresponding velocities are determined.

Zaproponowano termodynamiczną teorię materiałów z parametrami wewnętrznymi. W teorii tej reakcja materiału zależy od deformacji, temperatury i dwóch grup parametrów (wewnętrznych zmiennych stanu): mechanicznych i termicznych. Ewolucją parametrów wewnętrznych rządzi równanie różniczkowe pierwszego rzędu (równanie ewolucji), w którym występuje gradient temperatury jako dodatkowa zmienna niezależna. Zaproponowana postać równania ewolucji dla parametrów termicznych prowadzi do funkcjonalnego równania konstytutywnego dla strumienia ciepła. Przedyskutowano podobieństwo tego równania do związku Maxwella-Cattaneo. W budowanej teorii naprężenie, entropia a także strumień ciepła są wyznaczone przez funkcję energii swobodnej materiału. Jest to konsekwencja drugiego prawa termodynamiki. Teorię zastosowano do badania jednowymiarowych fal uderzeniowych i przyspieszenia. Otrzymano ogólne równania na prędkości tych dwóch rodzajów fal rozprzestrzeniających się w materiale w stanie równowagi. W dyskusji fal uderzeniowych zaproponowano szczególną postać równań konstytutywnych i ewolucji. Wyprowadzono warunki, przy których niesprzężone mechaniczne i termiczne fale rozprzestrzeniają się w ogólnym materiale z parametrami wewnętrznymi. Wyznaczono odpowiednie prędkości.

В настоящей работе предложена термодинамическая теория материалов характеризуемых внутренними параметрами. Согласно этой теории реакция материала зависит от деформации, температуры и от двух совокупностей внутренних параметров — внутренних параметров состояния: механических и теоретических. Эволюция внутренних параметров подчиняется дифференциальному уравнению первого порядка — уравнению эволюции. В этом уравнении присутствует, в качестве дополнительной переменной, градиент температуры. Обсуждено сходство этого уравнения с зависимостью Максвелла-Каттанео. Согласно построенной теории напряжение, энтропия и поток тепла определяются функцией свободной энергии материала. Это свойство является следствием второго принципа термодинамики. Теория применяется к исследованию одномерных ударных волн и волн ускорения. Получены общие уравнения для скоростей этих двух разновидностей волн, распространяющихся в среде в состоянии равновесия. Рассматривая ударные волны предположено конститутивные уравнения и уравнения эволюции особого вида. Получено условия, при исполнении которых несопряженные механические и термические волны распространяются в общей среде с внутренними параметрами. Определено соответствующие скорости.

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1. Second sound

THE MECHANICAL theories of the nineteenth and twentieth centuries taking into account the transport of the heat base on the classical theory of heat conduction. This theory comes down from J. B. J. FOURIER, who in 1822 formulated his proportional relation between the flux of heat and the gradient of the temperature distribution. As a consequence of this relation the temperature distribution in a body is governed by a parabolic partial differential equation, which predicts that the application of a thermal disturbance in a finite region instantaneously affects all points of the body. This fact contradicts of course the physical observations and the particular theory of relativity. As early as 1867, thanks to MAXWELL [1], the modified rate type equation of heat was given. That equation was free of this contradiction. Unfortunately, Maxwell immediately cast out the term involving the rate of the heat flux (1) and gained Fourier's law.

The problem of the existence of the finite speed of propagation of thermal disturbances has been investigated for the last forty years of this century. In that time the *second sound* (the speed of the thermal signals) was observed in the fluid helium. LANDAU [2] suggested that the behaviour of helium may be described by a gas of elementary excitations called *phonons* and that a thermal wave is the propagation of a phonon density disturbance. Landau's theory predicts that this second sound propagates with the speed $v_p/\sqrt{3}$, where v_p is the velocity of sound (the first sound). It was predicted also, that the second sound must exist in any solid since all solids exhibit phonon excitations (see CHESTER [3]), but only just recently experiments by ACKERMAN *et al.* [4, 5] on solid helium have shown that the second sound does indeed occur in solids (2).

From the physical point of view the classical theory of heat conduction does not take into account the short time required to establish a steady-state heat conduction when a temperature gradient is suddenly produced in a body. Of course, there are the experiences, the physical problems, in which by comparing the magnitude of the process time, this short time of heat conduction may be neglected. This neglected short time is called *the thermal relaxation time*. The inclusion of the relaxation time ensures that the corresponding field equations do not imply that thermal signals have an infinite speed of propagation.

In a number of works the phenomenological modification is introduced into the classical Fourier law to obtain a wave-type equation for heat conduction. In [6, 7] CATTANEO assumed that Fourier's law was valid only for a quasi-equilibrium state and in order to generalize this law to include the non-equilibrium state he proposed that it be replaced by the constitutive relation:

$$\tau\dot{\mathbf{q}}+\mathbf{q} = -k\mathbf{g}, \tag{1.1}$$

where $\mathbf{q}$ is the heat flux vector, $\mathbf{g}$ is the temperature gradient, k is the positive constant called the coefficient of thermal conductivity and τ is the thermal relaxation time. This constitutive relation was already obtained by MAXWELL [1], so the Eq. (1.1) can be called the Maxwell-Cattaneo *relation*. Similar suggestions of the modification of Fourier's law

(1) The existence of the rate of the heat flux leads to a hyperbolic heat equation.

(2) For fluid and solid thermal waves were observed at very low temperatures.

were made by VERNOTTE [8], and CHESTER [3]. This problem was treated by KALISKI, who in [9] gave reasons for taking into consideration the rate of the heat and introduced the notion of the thermal inertia of the body. In [10] MAZILU derived the hyperbolic equation of heat conduction based on the notion of an inertial system.

Instead of the Maxwell-Cattaneo relation (1.1) GURTIN and PIPKIN [11] assumed that the response of a rigid heat conductor depends not only upon the temperature, but also the summed histories of temperature gradient and they could obtain the temperature rate waves with finite wave speed. This idea was studied further by CHEN [12], CHEN and GURTIN [13] (3) and McCARTHY [14] (4). In the linearized theory of GURTIN and PIPKIN (Sec. 7 in [11]) the constitutive equation for the heat flux $\mathbf{q}$ has the form

$$\mathbf{q}(t) = -\int_0^\infty a(s)\mathbf{g}(t-s)ds. \tag{1.2}$$

An interesting fact is, that assuming for the kernel $a(s)$ the expression

$$a(s) = k\tau^{-1}e^{-s\tau^{-1}}, \tag{1.3}$$

one may obtain, after differentiating of the Eq. (1.2) with respect to time t, the Maxwell-Cattaneo relation (1.1).

LORD and SHULMAN provided in [15] further arguments justifying the modification of Fourier's law and investigated the implications of the modification for a problem of one-dimensional wave propagation in the coupled thermoelasticity (5). For the same medium the propagation of discontinuities of the stress and the temperatures was studied by ACHENBACH [17] in a case of one-dimensional theory. In [18] NAYFEH and NEMAT-NASSER analysed the behaviour of thermo-elastic waves in a solid half-space. In their analysis the thermal relaxation time of heat conduction was included. For the rate-type plastic material thermo-mechanical acceleration waves were investigated by TOKUOKA in [19].

The aim of the present paper is to formulate the thermodynamic theory of materials with internal state variables (internal parameters) in which the thermal acceleration and shock waves exist and the Maxwell-Cattaneo relation may occure. Although in [20] the analysis of acceleration waves in a material with internal parameters was carried out the consequences of the assumed constitutive evolution equation were not derived. Furthermore, it was not shown that the Maxwell-Cattaneo relation might occur in that theory (6).

In Secs. 3 and 4 a thermodynamic theory of a material is formulated. In this theory the response of a material depends on the deformation, the temperature and the internal state variable vector. For the internal state variables we have (we postulate) a vector differential equation of order one in which the temperature gradient takes place additionally. The theory constructed is used for the analysis of acceleration and shock waves in Secs. 5 and 6.

(3) For the case of deformable solids.

(4) In [12 and 14] the variation of the amplitude of temperature rate wave was discussed.

(5) A similar investigation was carried out by POPOV [16].

(6) In [21] SULICIU discusses the use of the Maxwell-Cattaneo relation in the theory with internal state variables.

2. Thermomechanical material with internal state variables

According to the aim of this section we deal here with the formulation of a thermodynamic theory of materials with internal state variables (internal parameters) in which the thermal waves of the second and the first order as well as the mechanical waves may appear.

By the waves of the second order (the waves of the weak discontinuities) we mean the acceleration mechanical waves and the temperature rate waves, i.e. the singular surfaces on which the acceleration of particle and the rate of the temperature have jump discontinuities. By the waves of the first order (the shock waves) we mean the surfaces of the strain and temperature discontinuities.

In order to formulate the thermodynamic theory of the material (of a body $\mathscr{B}$), we assume the following properties of the body $\mathscr{B}$:

A1. The body $\mathscr{B}$ can deform in elastic and inelastic manners.

A2. The body $\mathscr{B}$ can conduct the heat.

A3. There exists a homogeneous (⁷) reference configuration $\varkappa$, i.e. such a configuration, that the response of each particle of the body $\mathscr{B}$ is the same.

A4. The material of the body can be described by the use of rational thermodynamics (⁸).

A5. The dissipation of the body $\mathscr{B}$ may be described by the appropriate set of the internal state variables (internal parameters).

A6. The memory of the material of the body $\mathscr{B}$ is particularly sensitive to the past history of the nonhomogeneous temperature distributions.

Now, we want to express in mathematical language the above set of the assumptions. To describe deformations and changes in the temperature we introduce the deformation gradient $\mathbf{F}$ and the absolute temperature ϑ. If $\mathbf{x} = \boldsymbol{\chi}(X, t)$ is the place of a particle X of the body $\mathscr{B}$ at time t, then $\operatorname{Grad}\boldsymbol{\chi}(X, t) \equiv \mathbf{F}(X, t)$. The temperature gradient $\mathbf{g}(X, t) \equiv \operatorname{grad}\vartheta(X, t)$ describes the nonhomogeneity in the distribution of the temperature in $\mathscr{B}$. The set of the internal state variables is denoted by $\boldsymbol{\alpha}$. The internal variables are of a different geometrical nature and may have a different physical interpretation.

The quantities introduced ($\mathbf{F}, \vartheta, \mathbf{g}, \boldsymbol{\alpha}$) form the set of the independent variables, which can be called the state variables. Their values at a given time t are called the state.

Now in the further formulations we may, of course, use the notions of the state space, the process space, the method of preparation space, the evolution function and the others, which were introduced by KOSIŃSKI and PERZYNA in [22], but in this paper it would be superfluous.

As the dependent variables we take the Cauchy stress tensor $\mathbf{T}_c$ (or the first Piola-Kirchhoff stress tensor $\mathbf{T}$, which is related to $\mathbf{T}_c$ by $\mathbf{T} = \varrho^{-1}\mathbf{T}_c(\mathbf{F}^T)^{-1})$), the free energy ψ, the entropy η and the heat flux $\mathbf{q}$. The laws of motions and thermodynamics have then forms, in the presence of the body force $\mathbf{b}$ and the rate of heat supply r

$$
\begin{gathered}
\operatorname{div}\mathbf{T}_c+\varrho\mathbf{b} = \varrho\ddot{\boldsymbol{\chi}}, \quad \mathbf{T}_c = \mathbf{T}_c^T, \\
\varrho(\dot{\psi}+\dot{\eta}\vartheta+\eta\dot{\vartheta})-\mathbf{T}_c\cdot\mathbf{L}+\operatorname{div}\mathbf{q} = \varrho r, \qquad (2.1)\\
-\dot{\psi}-\eta\dot{\vartheta}+\frac{1}{\varrho}\mathbf{T}_c\cdot\mathbf{L}-\frac{1}{\varrho\vartheta}\mathbf{q}\cdot\mathbf{g} \geqslant 0,
\end{gathered}
$$

(⁷) This assumption may be neglected in consideration of this section.

(⁸) i.e. thermodynamics based on the Clausius-Duhem inequality.

where $\mathbf{L} = \text{grad}\,\dot{\boldsymbol{\chi}}$ is the velocity gradient, $\mathbf{T}_c \cdot \mathbf{L}$ represents the inner product of $\mathbf{T}_c$ and $\mathbf{L}$, i.e. $\mathbf{T}_c \cdot \mathbf{L} = \text{tr}(\mathbf{T}_c \mathbf{L}^T)$ with $\mathbf{L}^T$ as the transpose of the tensor $\mathbf{L}$. Here ϱ denotes the mass density.

To specify the material structure of the body $\mathscr{B}$ we shall introduce the following constitutive assumptions:

K1. The response of the material in particle X at time t depends on the values of the deformation $\mathbf{F}(X, t)$, the temperature $\vartheta(X, t)$ and the internal state variables $\boldsymbol{\alpha}(X, t)$.

K2. The evolution of the internal state variables $\boldsymbol{\alpha}$ during the thermodynamic process is governed by a vector differential equation of the first order, in which the temperature gradient as the additional variable takes place.

Now we express in the mathematical form these assumptions bearing the properties A1 to A6 in mind. The postulates K1 together with A3 and A5 take the form of the constitutive equations:

$$
\begin{aligned}
\mathbf{T}(X,t) &= \mathscr{T}(\mathbf{F}(X,t), \vartheta(X,t), \boldsymbol{\alpha}(X,t)),\\
\psi(X,t) &= \Psi(\mathbf{F}(X,t), \vartheta(X,t), \boldsymbol{\alpha}(X,t)),\\
\eta(X,t) &= N(\mathbf{F}(X,t), \vartheta(X,t), \boldsymbol{\alpha}(X,t)),\\
\mathbf{q}(X,t) &= \mathbf{Q}(\mathbf{F}(X,t), \vartheta(X,t), \boldsymbol{\alpha}(X,t)).
\end{aligned}
\tag{2.2}
$$

The following evolution equation for the internal state variables $\boldsymbol{\alpha}$ reflects the postulate K2 and the property A6

$$
\dot{\boldsymbol{\alpha}}(X,t) = \mathbf{A}(\mathbf{F}(X,t), \vartheta(X,t), \mathbf{g}(X,t), \boldsymbol{\alpha}(X,t)). \tag{2.3}
$$

The constitutive equation assumed should verify the laws of mechanics and thermodynamics. The second law of thermodynamics $(2.1)_4$ requires that for each differentiable deformation and temperature field and for differentiable free energy function the inequality (⁹)

$$
(\mathbf{T} - \partial_{\mathbf{F}}\Psi) \cdot \dot{\mathbf{F}} - (\partial_\vartheta \Psi + \eta)\dot{\vartheta} - \partial_{\boldsymbol{\alpha}} \Psi \cdot \dot{\boldsymbol{\alpha}} - \frac{1}{\varrho\vartheta}\mathbf{q} \cdot \mathbf{g} \geqslant 0, \tag{2.4}
$$

holds.

The main problem in the thermodynamics of continuum is that of defining the restrictions which the second law of thermodynamics imposes on constitutive functions describing the response of material. In 1967 COLEMAN and GURTIN in their paper [23] and VALANIS in his paper [24] have formulated the restrictions imposed by the second law. These restrictions are made though for another type of constitutive equations.

Their results may be derived in this case, too. Since the third and the fourth terms in (2.4) are independent of $\dot{\mathbf{F}}$ and $\dot{\vartheta}$, so the method formulated in [23] adapted in the present case implies the following identities in each termodynamic process

$$
\begin{aligned}
\mathscr{T}(\mathbf{F}, \vartheta, \boldsymbol{\alpha}) &= \partial_{\mathbf{F}} \Psi(\mathbf{F}, \vartheta, \boldsymbol{\alpha}),\\
N(\mathbf{F}, \vartheta, \boldsymbol{\alpha}) &= -\partial_\vartheta \Psi(\mathbf{F}, \vartheta, \boldsymbol{\alpha}),
\end{aligned}
\tag{2.5}
$$

(⁹) Replacing $\mathbf{T}_c$ by $\mathbf{T}$ in (2.1) we obtain for the mechanical power $\mathbf{T}_c \cdot \mathbf{L} = \varrho \mathbf{T} \cdot \dot{\mathbf{F}}$ and, instead of $(2.1)_4$, the inequality

$$
-\dot{\psi} - \dot{\vartheta}\eta + \mathbf{T} \cdot \dot{\mathbf{F}} - \frac{1}{\varrho\vartheta}\mathbf{q} \cdot \mathbf{g} \geqslant 0.
$$

and the following inequality

$$(2.5)_3 \qquad -\partial_{\alpha}\Psi\cdot\dot{\boldsymbol{\alpha}}-\frac{1}{\varrho\vartheta}\mathbf{q}\cdot\mathbf{g}\geqslant 0.$$

They mean that the stress, the entropy are determined by the free energy function and the inequality of the general dissipation must be satisfied.

Now, we formulate the fundamental for further investigations, assumptions concerning the influence of the temperature gradient upon the response of the material:

P1. There exist two groups of the internal state variables $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{\mathrm{H}}, \boldsymbol{\alpha}_{\mathrm{q}})$, namely: *the mechanical variables* $\boldsymbol{\alpha}_{\mathrm{H}}$ and *the thermal variables* $\boldsymbol{\alpha}_{\mathrm{q}}$.

P2. The forms of the evolution equations for them are as follows:

$$(2.6) \qquad \begin{aligned} \dot{\boldsymbol{\alpha}}_{\mathrm{H}} &= \mathbf{B}_{\mathrm{H}}(\mathbf{F},\vartheta,\boldsymbol{\alpha}_{\mathrm{H}}),\\ \dot{\boldsymbol{\alpha}}_{\mathrm{q}} &= \mathbf{g}+\mathbf{B}_{\mathrm{q}}(\mathbf{F},\vartheta,\boldsymbol{\alpha}_{\mathrm{q}}).\end{aligned}$$

The form of the function **A** in (2.2) assumed here denotes that we cast out the influence of the temperature gradient upon the rate of the mechanical variables $\dot{\boldsymbol{\alpha}}_{\mathrm{H}}$ keeping the linear relation between thermal variables $\dot{\boldsymbol{\alpha}}_{\mathrm{q}}$ and the temperature gradient.

Since we know the right sides of the evolution equations we are able to derive the next consequences of the second law. These consequences are true only in the theory constructed. We put the right sides of (2.6) into $(2.5)_3$:

$$(2.7) \qquad 0\leqslant -\left(\partial_{\alpha_{\mathrm{q}}}\Psi(\mathbf{F},\vartheta,\boldsymbol{\alpha})+\frac{1}{\varrho\vartheta}\mathbf{Q}(\mathbf{F},\vartheta,\boldsymbol{\alpha})\right)\cdot\mathbf{g}-\partial_{\alpha_{\mathrm{q}}}\Psi(\mathbf{F},\vartheta,\boldsymbol{\alpha})\cdot\mathbf{B}_{\mathrm{q}}(\mathbf{F},\vartheta,\boldsymbol{\alpha}_{\mathrm{q}}) -\partial_{\alpha_{\mathrm{H}}}\Psi(\mathbf{F},\vartheta,\boldsymbol{\alpha})\cdot\mathbf{B}_{\mathrm{H}}(\mathbf{F},\vartheta,\boldsymbol{\alpha}_{\mathrm{H}}).$$

If we take into account the fact that the solution of $\boldsymbol{\alpha}_{\mathrm{q}}$ in (2.6) does not depend on the actual value of the temperature gradient ([10]), then we see that (2.7) holds for all $(\mathbf{F},\vartheta,\boldsymbol{\alpha},\mathbf{g})$, only if the coefficient of **g** vanishes and the sum of the remaining terms is not negative:

$$(2.8) \qquad \begin{gathered} Q(\mathbf{F},\vartheta,\boldsymbol{\alpha}) = -\varrho\vartheta\partial_{\alpha_{\mathrm{q}}}\Psi(\mathbf{F},\vartheta,\boldsymbol{\alpha}),\\ 0\leqslant -\partial_{\alpha_{\mathrm{q}}}\Psi(\mathbf{F},\vartheta,\boldsymbol{\alpha})\cdot\mathbf{B}_{\mathrm{q}}(\mathbf{F},\vartheta,\boldsymbol{\alpha}_{\mathrm{q}})-\partial_{\alpha_{\mathrm{H}}}\Psi(\mathbf{F},\vartheta,\boldsymbol{\alpha})\cdot\mathbf{B}_{\mathrm{H}}(\mathbf{F},\vartheta,\boldsymbol{\alpha}_{\mathrm{H}}).\end{gathered}$$

In the present theory, an interesting fact is that in contrast to the results obtained by Coleman and Gurtin [23] and Valanis [24], in addition to the stress and entropy, the heat flux is determined by the free energy function. There are the thermodynamic theories of materials in which similar results take place [11, 13, 26]. We can gather the consequences of the second law of thermodynamic in the following theorem.

Theorem 1. *In the thermomechanical material with internal state variables for which the relations* (2.2), (2.3) *and* (2.6) *hold, the stress, the entropy and the heat flux are determined by the free energy function*:

$$(2.9) \qquad \mathbf{T} = \partial_{\mathbf{F}}\Psi(\mathbf{F},\vartheta,\boldsymbol{\alpha}), \quad \eta = -\partial_{\vartheta}\Psi(\mathbf{F},\vartheta,\boldsymbol{\alpha}), \quad \mathbf{q} = -\varrho\vartheta\,\partial_{\alpha_{\mathrm{q}}}\Psi(\mathbf{F},\vartheta,\boldsymbol{\alpha}),$$

and the functions Ψ, $\mathbf{B}_{\mathrm{q}}$ *and* $\mathbf{B}_{\mathrm{H}}$ *satisfy the internal dissipation inequality* $(2.8)_2$.

([10]) It is in consequence of the existence and uniqueness of the solution. Cf. [25].

3. Maxwell-Cattaneo relation

After defining the restriction which the second law of thermodynamics imposes on the constitutive equation we are going to answer the following question: under which assumptions does the thermodynamic theory constructed contain the Maxwell-Cattaneo relation of heat conduction? The answer this question is one of the main aims of this paper. In order to make replay we assume additional postulates,

P3. The free energy function in $(2.2)_2$ depends on $\boldsymbol{\alpha}_{\mathbf{q}}$ in the following way

$$\Psi(\mathbf{F}, \vartheta, \boldsymbol{\alpha}) \equiv \Psi_{\mathbf{R}}(\mathbf{F}, \vartheta, \boldsymbol{\alpha}_{\mathbf{H}}) + \frac{1}{2}\frac{k}{\varrho\vartheta\tau}\boldsymbol{\alpha}_{\mathbf{q}}\cdot\boldsymbol{\alpha}_{\mathbf{q}}, \tag{3.1}$$

where ([11]) Ψ_R is a function of the deformation $\mathbf{F}$, the temperature ϑ and the mechanical internal state variables $\boldsymbol{\alpha}_{\mathbf{H}}$, only.

P4. The function $\mathbf{B}_{\mathbf{q}}$ of the right side of $(2.6)_2$ has the form

$$\mathbf{B}_{\mathbf{q}}(\mathbf{F}, \vartheta, \boldsymbol{\alpha}) \equiv -\frac{1}{\tau}\boldsymbol{\alpha}_{\mathbf{q}}. \tag{3.2}$$

The last assumption implies the following evolution equation for the thermal internal state variables $\boldsymbol{\alpha}_{\mathbf{q}}$:

$$\dot{\boldsymbol{\alpha}}_{\mathbf{q}} = \mathbf{g} - \frac{1}{\tau}\boldsymbol{\alpha}_{\mathbf{q}}, \quad \boldsymbol{\alpha}_{\mathbf{q}}(0) = \mathbf{0}, \tag{3.3}$$

where the vanishing initial conditions are assumed.

By the assumption P3 and Theorem 1 we find that the constitutive equation for the heat flux takes the form

$$\mathbf{q} = -\frac{k}{\tau}\boldsymbol{\alpha}_{\mathbf{q}}. \tag{3.4}$$

THEOREM 2. *For a material with internal state variables, which has the properties* A1–A6 *and fulfils the assumptions* K1, K2, P1–P4 *its heat flux is given by the functional equation* ([12])

$$\mathbf{q}(t) = -\frac{k}{\tau}\int_0^t e^{-\frac{1}{\tau}(t-s)}\mathbf{g}(s)\,ds. \tag{3.5}$$

This equation, in the class of differentiable functions $\mathbf{q}(t)$ *is equivalent to the Maxwell-Cattaneo relation*

$$\tau\dot{\mathbf{q}}(t) + \mathbf{q}(t) = -k\mathbf{g}(t). \tag{3.6}$$

P r o o f. The solution of the evolution equation (3.3) is given by the integral

$$\boldsymbol{\alpha}_{\mathbf{q}}(t) = \int_0^t e^{-\frac{1}{\tau}(t-s)}\mathbf{g}(s)\,ds.$$

In view of the heat flux constitutive equation (3.4) the proof of (3.5) is complete. To prove (3.6) one needs to differentiate $\mathbf{q}(t)$ with respect to time t in (3.5) and use some properties of integrals with parameters.

([11]) As previously, τ denotes the thermal relaxation time of the material.

([12]) Cf. ACHENBACH [17].

A material for which Theorem 2 holds we shall call *the thermomechanical material with internal state variables.*

It is interesting when the relation (3.5) turns out to be classic example of Fourier's law. It is evident that for $\tau = 0$ the Eq. (3.6) reduces to Fourier's law. This case corresponds to a definite conductor for which the thermal relaxation does not exist in material and, consequently, to the infinite speed of propagation of thermal disturbance. Clearly Fourier's law is not a special case of (3.5). However [13] when the temperature gradient $\mathbf{g}(s)$ is constant for all time $s \in (-\infty, t)$, then our functional relation for the heat flux (3.5) reduces to

$$\mathbf{q} = -k\mathbf{g}. \tag{3.7}$$

4. Equilibrium state

For further considerations it will be convenient to have some properties of the constitutive functions at an equilibrium state.

We say, that the system of quantities $(\mathbf{F}^{\#}, \vartheta^{\#}, \mathbf{0}, \boldsymbol{\alpha}^{\#})$ is a local *thermodynamic state of equilibrium* of the thermomechanical material with internal state variables, if the functions $\mathbf{B_H}$ and $\mathbf{B_q}$ in (2.6) vanish, i.e. $\mathbf{B_H}(\mathbf{F}^{\#}, \vartheta^{\#}, \mathbf{0}, \boldsymbol{\alpha}^{\#})$ and $\mathbf{B_q}(\mathbf{F}^{\#}, \vartheta^{\#}, \mathbf{0}, \alpha^{\#})$.

Since, by (3.2), $\mathbf{B_q}(\mathbf{F}, \vartheta, \boldsymbol{\alpha}_\mathbf{q}) = -\dfrac{1}{\tau}\boldsymbol{\alpha}_\mathbf{q}$, this condition follows $\boldsymbol{\alpha}_\mathbf{q}^{\#} = 0$. We see, that the thermodynamical equilibrium state requires the zero temperature gradient and the vanishing time derivatives of the internal state variable vector.

Let us notice that the function $\boldsymbol{\alpha}(t) = (\boldsymbol{\alpha}_\mathbf{H}^{\#}, \mathbf{0})$ for $t \in [0, t_k)$, $t_k \leqslant \infty$, is the solution of

$$\begin{aligned} \dot{\boldsymbol{\alpha}}_\mathbf{H}(t) &= \mathbf{B_H}\big((\mathbf{F}^{\#}, \vartheta^{\#}, \boldsymbol{\alpha}_\mathbf{H}(t)\big), \\ \dot{\boldsymbol{\alpha}}_\mathbf{q}(t) &= -\frac{1}{\tau}\boldsymbol{\alpha}_\mathbf{q}(t), \end{aligned} \tag{4.1}$$

with the initial value $\big(\boldsymbol{\alpha}_\mathbf{H}(0), \boldsymbol{\alpha}_\mathbf{q}(0)\big) = (\boldsymbol{\alpha}_\mathbf{H}^{\#}, \mathbf{0})$. This solution is called *the equilibrium solution.*

The properties of the free energy and the heat flux constitutive functions will be proved in the case of quasi-asymptotical stable equilibrium states.

We say that an equilibrium solution $\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}^{\#}$ $\big(\boldsymbol{\alpha}(t) = (\boldsymbol{\alpha}_\mathbf{H}^{\#}, \mathbf{0})\big)$ of (4.1) is *quasi-asymptotically stable* [14], if

$$\bigvee_{\delta<0} \; \bigwedge_{\alpha(\cdot)-\text{solution of (4.1)}} \|\boldsymbol{\alpha}(t_0) - \boldsymbol{\alpha}^{\#}\| < \delta \Rightarrow \lim_{t\to\infty} \boldsymbol{\alpha}(t) = \boldsymbol{\alpha}^{\#}. \tag{4.2}$$

A thermodynamic equilibrium state is called *the quasi-asymptotically stable equilibrium state* if the corresponding equilibrium solution is quasi-asymptotically stable.

For such states the following theorem is true.

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[13] cf. [11].

[14] Cf. Antosiewicz [27].

THEOREM 3. *In a quasi-asymptotically stable equilibrium state of a thermomechanical material with internal state variables the free energy has a minimum and the heat flux vanishes.*

Proof. Let us consider the thermodynamic process on $[0, \infty)$ for which $\mathbf{F}(t) = \mathbf{F}^{\#}$, $\vartheta(t) = \vartheta^{\#}$, $\mathbf{g}(t) = \mathbf{0}$ for $t \geqslant 0$, and $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0$, where the set $(\mathbf{F}^{\#}, \vartheta^{\#}, \mathbf{0}, \boldsymbol{\alpha}^{\#})$ forms a quasi-asymptotically stable equilibrium state. For such a process $\dot{\mathbf{F}}(t) = \mathbf{0}$ and $\dot{\vartheta}(t) = 0$, and $\boldsymbol{\alpha}(t)$ is given as the solution of

$$
\begin{aligned}
&\dot{\boldsymbol{\alpha}}_{\mathrm{H}}(t) = \mathbf{B}_{\mathrm{H}}(\mathbf{F}^{\#}, \vartheta^{\#}, \boldsymbol{\alpha}_{\mathrm{H}}(t)), \qquad \boldsymbol{\alpha}_{\mathrm{H}}(0) = \boldsymbol{\alpha}_{\mathrm{H}}^0,\\
&\dot{\boldsymbol{\alpha}}_{\mathbf{q}}(t) = -\frac{1}{\tau}\boldsymbol{\alpha}_{\mathbf{q}}, \qquad \boldsymbol{\alpha}_{\mathbf{q}}(0) = \boldsymbol{\alpha}_{\mathrm{H}}^0.
\end{aligned}
\tag{4.3}
$$

In view of $(2.1)_4$ $\dot{\psi}(t) \leqslant 0$ for $t \geqslant 0$. The continuity of the function $\psi(t) = \Psi(\mathbf{F}^{\#}, \vartheta^{\#}, \boldsymbol{\alpha}(t))$ in t yields

$$
\Psi(\mathbf{F}^{\#}, \vartheta^{\#}, \boldsymbol{\alpha}^0) \geqslant \Psi(\mathbf{F}^{\#}, \vartheta^{\#}, \boldsymbol{\alpha}(t)) \quad \text{for} \quad t \geqslant 0 \tag{4.4}
$$

because $\Psi(\mathbf{F}^{\#}, \vartheta^{\#}, \boldsymbol{\alpha}(t))$ is a non-increasing function of t. Let us assume that the initial vector $\boldsymbol{\alpha}^0$ is arbitrarily chosen from the neighbourhood of $\boldsymbol{\alpha}^{\#}$, i.e. $||\boldsymbol{\alpha}^0 - \boldsymbol{\alpha}^{\#}|| < \delta$, where δ is taken from (4.2). Then, since Ψ is continuous, and the equilibrium state is quasi-asymptotically stable, it follows that

$$
\lim_{t\to\infty} \Psi\big(\mathbf{F}^{\#}, \vartheta^{\#}, \boldsymbol{\alpha}(t)\big) = \Psi(\mathbf{F}^{\#}, \vartheta^{\#}, \boldsymbol{\alpha}^{\#}). \tag{4.5}
$$

Combining (4.4) and (4.5) we obtain

$$
\Psi(\mathbf{F}^{\#}, \vartheta^{\#}, \boldsymbol{\alpha}) \geqslant \Psi(\mathbf{F}^{\#}, \vartheta^{\#}, \boldsymbol{\alpha}^{\#}), \tag{4.6}
$$

for all vectors $\boldsymbol{\alpha}$ in the neighbourhood of $\boldsymbol{\alpha}^{\#}$. The Eq. (4.6) expresses the minimum property of the free energy at the quasi-asymptotically stable equilibrium state. Consequently

$$
\partial_{\boldsymbol{\alpha}} \Psi(\mathbf{F}^{\#}, \vartheta^{\#}, \boldsymbol{\alpha}^{\#}) = \mathbf{0}. \tag{4.7}
$$

The last equation, called the equation of internal equilibrium (cf. [23], thanks to (2.9)), completes the proof of the theorem.

The last proven fact will be used in the investigation of the waves in the material under consideration. It will be the subject of the next two Sections.

5. One-dimensional acceleration waves

As was said before, results concerning the thermodynamic theory of the material under consideration would be used in the wave investigation.

Firstly, the acceleration waves will be analyzed. It will be done in the case of one-dimensional theory.

The one-dimensional motion of the body $\mathscr{B}$ is described by the scalar function $\chi: \mathscr{B} \times R \to R$ giving the location $x = \chi(X, t)$ at time t of the material point (partical) of the body whose position in the reference configuration $\varkappa$ is X. We shall, as is customary, identify each material point with its position in the reference configuration. The configuration $\varkappa$ is assumed to be homogeneous with a mass density $\varrho_\varkappa$.

The displacement of a material point is defined by the function $u\colon \mathscr{B}\times R \to R$

$$u(X,t) = \chi(X,t)-X \quad \text{for} \quad (X,t)\in\mathscr{B}\times R, \tag{5.1}$$

and, assuming that the necessary derivatives of u exist, we further define

$$\begin{aligned} \dot{u}(X,t) &= \frac{\partial}{\partial t}u(X,t) = \frac{\partial}{\partial t}\chi(X,t),\\ E(X,t) &= \frac{\partial}{\partial X}u(X,t) = \frac{\partial}{\partial X}\chi(X,t)-1, \end{aligned} \tag{5.2}$$

as the particle velocity and the strain ([15]), respectively. To describe thermal effects, we introduce the absolute temperature $\vartheta(X,t)>0$ and the temperature gradient ([16]) $g_\varkappa(X,t) = \frac{\partial}{\partial X}\vartheta(X,t)$.

We assume that the motion of the body contains *an acceleration wave* — i.e. a curve $\Sigma \subset \mathscr{B}\times[0,\infty)$ on which the second derivatives of χ, i.e.: $\ddot{u}$, $\dot{E}$, $\partial_X E$ and the first derivatives of ϑ — namely $\dot{\vartheta}$ and $g_\varkappa$ (and also higher derivatives of them) suffer jump discontinuities.

If $(Y(t),t)\in\Sigma$, then the value $Y(t)$ is the material point at which the wave is to be found at time t. The wave moves with the speed

$$U(t) = \frac{d}{dt}Y(t). \tag{5.2}$$

The following kinematical condition of compatibility [28]

$$\frac{d}{dt}[\![f]\!] = [\![\partial_X f]\!]\frac{d}{dt}Y(t)+[\![\dot{f}]\!], \tag{5.4}$$

with $f=\dot{u}$, E, and ϑ implies that

$$-U[\![\dot{E}]\!] = U^2[\![\partial_X E]\!] = [\![\ddot{u}]\!], \quad [\![\dot{\vartheta}]\!] = -U[\![g_\varkappa]\!], \tag{5.5}$$

where we use the well-known notation for the jump in a function $f(X,t)$ across Σ at t.

The Theorems 1 from both [20] and [25], and the Eq. (3.3) solve the question concerning the jump of the time derivatives of the internal state variables $\boldsymbol{\alpha}_H$ and α_q.

Lemma. *On the acceleration wave Σ in the motion of the thermomechanical material with internal state variables, the mechanical variables, with its derivatives, and the thermal variables have no jump discontinuities, i.e.* $[\![\boldsymbol{\alpha}_H]\!] = [\![\dot{\boldsymbol{\alpha}}_H]\!] = \mathbf{0}$, $[\![\alpha_q]\!]=0$, *but the jump in the time derivative of α_q exists and is given by*

$$[\![\dot{\alpha}_q]\!] = g_\varkappa, \tag{5.6}$$

provided the function $\mathbf{B}_H$ in $(2.6)_1$ is continuous in all variables and Lipschitz continuous with respect to $\boldsymbol{\alpha}$.

([15]) In the one-dimensional case the strain E is often used instead of the deformation gradient $F=E+1$.

([16]) Here we have introduced the material gradient of the temperature. It is connected with the spatial gradient by $g_\varkappa \equiv \frac{\partial\vartheta}{\partial X} = \frac{\partial\vartheta}{\partial x}\frac{\partial\chi}{\partial X} = (E+1)g$.

The laws of balance of momentum and energy are equivalent in the case of acceleration waves, to the assertion that ([29])

(5.7) (17) $$[\![\partial_X T]\!] = \varrho_\varkappa [\![\ddot{u}]\!], \quad \varrho_\varkappa [\![\dot{\psi} + \dot{\eta}\vartheta + \eta\dot{\vartheta}]\!] = T[\![\dot{E}]\!] - [\![\partial_X q]\!].$$

Using the Eqs. (2.2), $(2.8)_1$ and (5.6) and the compatibility condition (5.4) with f equal α_q we obtain (18), from (5.7), the system of two algebraic equations with respect to $[\![\ddot{u}]\!]$ and $[\![\dot{\vartheta}]\!]$:

(5.8) $$\begin{gathered}(\partial_E\mathscr{T} - \varrho_\varkappa U^2)[\![\ddot{u}]\!] + (\partial_{\alpha_q}\mathscr{T} - U\partial_\vartheta\mathscr{T})[\![\dot{\vartheta}]\!] = 0,\\ \varrho_\varkappa\vartheta\,\partial_E N[\![\ddot{u}]\!] + \left(\varrho_\varkappa\vartheta U\partial_{\alpha_q}N - \vartheta^{-1}UQ - \varrho_\vartheta\vartheta U^2\partial_\vartheta N + \frac{k}{\tau}\right)[\![\dot{\vartheta}]\!] = 0.\end{gathered}$$

This set has non-trivial solutions if its determinant vanishes. But we notice that the coefficients of these equations are formed by the values of the constitutive functions and their derivatives on the wave Σ.

To simplify the computation let us suppose that the wave is propagating into the material being at a quasi-asymptotically stable equilibrium state $(E^\#, \vartheta^\#, 0, \boldsymbol{\alpha}^\#)$. Then in view of results of the previous Section $\alpha_q^\# = 0$ and the heat flux vanishes at this state.

The condition of the vanishing determinant of (5.8) gives (19)

(5.9) $$\left(\frac{U}{c_0}\right)^4 - \left(\frac{U}{c_0}\right)^2(b_0 + d_0 + 1) + b_0 = 0,$$

where the coefficients are following:

(5.10) $$c_0^2 = \frac{\partial_E\mathscr{T}}{\varrho_\varkappa}, \quad b_0 = \frac{k}{\tau\vartheta^\#\partial_E\mathscr{T}\,\partial_\vartheta N}, \quad d_0 = \frac{(\partial_\vartheta\mathscr{T})^2}{\varrho_\varkappa\partial_E\mathscr{T}\,\partial_\vartheta N},$$

where the values of $\mathscr{T}$, N, should be taken at $(E^\#, \vartheta^\#, \boldsymbol{\alpha}^\#)$.

Notice that the expression $\varrho_\varkappa^{-1}\partial_E\mathscr{T}(E^\#, \vartheta^\#, \boldsymbol{\alpha}^\#)$ is the velocity of the acoustical (mechanical) wave in the material. Since such a wave must be real we have $\partial_E\mathscr{T} > 0$. Additionally, the term $\vartheta^\#\partial_\vartheta N(E^\#, \vartheta^\#, \boldsymbol{\alpha}^\#)$ is called in thermodynamics the specific heat of the material, which also should be positive. These facts imply b_0 and d_0 positive, and the proof of the following theorem.

THEOREM 4. *In the thermomechanical material with internal state variables an acceleration wave propagating into a quasi-asymptotically stable equilibrium state has four real and symmetric velocities* (20).

We study the Eq. (5.9) for two particular cases.

Case 1. No thermomechanical coupling in the material, i.e. $\partial_\vartheta\mathscr{T} = 0$. Then $d_0 = 0$ and we have two pairs of solutions

(5.11) $$\begin{aligned} U_{1,2}^2 &= c_0^2, \quad \text{i.e. } U_{1,2}^2 = \partial_E\mathscr{T}(E^\#, \vartheta^\#, \boldsymbol{\alpha}^\#)/\varrho_\varkappa.\\ U_{3,4}^2 &= c_0^2 b, \quad \text{i.e. } U_{3,4}^2 = k/\varrho_\varkappa\tau\vartheta^\#\partial_\vartheta N(E^\#, \vartheta^\#, \boldsymbol{\alpha}^\#).\end{aligned}$$

(17) We assume here the continuity of the body force and the rate of the heat supply and the continuous differentiability of the constitutive functions.

(18) See derivation in [20].

(19) A similar result for materials with memory has been obtained by CHEN and GURTIN [13]; cf. also [12, 19, 21].

(20) Because of the biquadratic equation (5.9).

COROLLARY 1. *For the material which has no thermomechanical coupling the acceleration waves are separated into two parts: one is a purely mechanical wave and has the velocity* $(5.11)_1$ *as a material without thermal influence, and the other is a purely thermal wave* ([21]) *and has the velocity* $(5.11)_2$.

Let us notice that in the case of the thermal wave velocity the thermal relaxation time tending to zero (it denotes that the Maxwell-Cattaneo relation reduces to Fourier's law) implies the infinity velocity of propagation of thermal disturbances, i.e.

$$\text{from } \tau \to 0 \text{ follows } U^2_{3,4} \to \infty . \tag{5.12}$$

When the material has no thermomechanical coupling and is a non-conductor, the coefficient of thermal conductivity k vanishes, then

$$U_{3,4} = 0, \tag{5.13}$$

which yields for a non-conductor (with absence of thermomechanical coupling) that the wave is purely mechanical and there is no thermal wave but there is a static curve (i.e. a point, in a one-dimensional case) of thermal jump.

Case 2. A non-conductor with thermomechanical coupling, i.e. $k = 0$. Then $b_0 = 0$ and we have

COROLLARY 2. *For a non-conductor there are only two symmetric coupled* ([22]) *thermomechanical waves with the velocity*

$$(5.14)\qquad \begin{aligned} &U^2_{1,2} = c_0^2(d_0+1), \quad \text{i.e.} \quad U^2_{1,2} = \frac{\partial_S \mathcal{T}(E^\#, \vartheta^\#, \boldsymbol{\alpha}^\#)}{\varrho_\varkappa} + \frac{(\partial_\vartheta \mathcal{T}(E^\#, \vartheta^\#, \boldsymbol{\alpha}^\#))^2}{\varrho_\varkappa^2 \partial_\vartheta N(E^\#, \vartheta^\#, \boldsymbol{\alpha}^\#)} .\\ &U_{3,4} = 0. \end{aligned}$$

6. One-dimensional shock waves

Coming to a shock wave, we define it as a curve Σ [cf. (5.1)] on which the strain E and the temperature ϑ suffer jump discontinuities, i.e. $[\![E]\!] \neq 0$ and $[\![\vartheta]\!] \neq 0$.

Let us formulate the fundamental theorem.

THEOREM 5. *In a shock wave in the thermomechanical material with internal state variables the following relations hold*:

$$(6.1)\qquad \begin{aligned} &[\![\boldsymbol{\alpha}_H]\!] = \mathbf{0}, \quad [\![\dot{\boldsymbol{\alpha}}_H]\!] = \mathbf{B}_H(E^-, \vartheta^-, \boldsymbol{\alpha}_H) - \mathbf{B}(E^+, \vartheta^+, \boldsymbol{\alpha}_H),\\ &[\![\alpha_q]\!] = V^{-1}[\![\vartheta]\!], \quad [\![\dot{\alpha}_q]\!] = [\![g_\varkappa]\!] + (\tau V)^{-1}[\![\vartheta]\!], \end{aligned}$$

provided the function $\mathbf{B}_H$ *in the Eq.* (2.6) *is continuous in all variables and Lipschitz continuous with respect to* $\boldsymbol{\alpha}$. *Here* $V(t) = \dfrac{d}{dt} Y(t)$ *is the velocity of the shock wave.*

P r o o f. The part of the proof concerning $\boldsymbol{\alpha}_H$ and $\dot{\boldsymbol{\alpha}}_H$ in $(6.1)_{1,2}$ is the same as in

([21]) Cf. TOKUOKA [19].

([22]) A similar result has been obtained in elastic non-conductors. This is an example of a homotropic wave; cf. [12, 20, 29].

[30, 31]. To prove the relation $(6.1)_{3,4}$ let us notice that the Eq. (3.3) in the case of shock waves is equivalent to the integral equation

$$\alpha_q(X,t) = \int_0^t e^{-\frac{1}{\tau}(t-s)} g_\varkappa(X,s)\,ds. \tag{6.2}$$

Therefore, in view of $g_\varkappa(X,s) = \partial_X \vartheta(X,s)$, the equation

$$\alpha_q(X,t) = \frac{\partial}{\partial X}\int_0^t e^{-\frac{1}{t}(t-s)} \vartheta(X,s)\,ds \tag{6.3}$$

is equivalent to (6.2) on either side of the wave.

Even if the temperature is discontinuous the integral in (6.3) will be continuous on the wave. Now, we use the condition of compatibility (5.4) with f equal to the integral. Then we obtain

$$\frac{d}{dt}Y(t)\left[\!\!\left[\frac{\partial}{\partial X}\int_0^t e^{-\frac{1}{\tau}(t-s)}\vartheta(X,s)\,ds\right]\!\!\right] = -\left[\!\!\left[\frac{\partial}{\partial t}\int_0^t e^{-\frac{1}{\tau}(t-s)}\vartheta(X,s)\,ds\right]\!\!\right] =$$
$$= -\left[\!\!\left[\vartheta(X,t) - \frac{1}{\tau}\int_0^t e^{-\frac{1}{\tau}(t-s)}\vartheta(X,s)\,ds\right]\!\!\right] = -[\![\vartheta]\!]. \tag{6.4}$$

Hence

$$-[\![\alpha_q]\!] = V^{-1}[\![\vartheta]\!],$$

because $V(t) = \frac{d}{dt}Y(t)$. Furthermore, since $\dot{\alpha}_q = g_\varkappa - \frac{1}{\tau}\alpha_q$ in (3.3), thus

$$[\![\dot{\alpha}_q]\!] = [\![g_\varkappa]\!] + \frac{1}{\tau V}[\![\vartheta]\!],$$

which completes the proof.

As the consequence of the theorem we have for the heat flux

$$[\![q]\!] = \frac{k}{\tau V}[\![\vartheta]\!]. \tag{6.5}$$

The laws of balance of momentum and energy are equivalent in the case of shock waves, to the assertion that ([28])

$$[\![T]\!] = -\varrho_\varkappa V[\![\dot{u}]\!], \quad [\![\partial_X T]\!] = \varrho_\varkappa[\![\ddot{u}]\!],$$
$$-\varrho_\varkappa V[\![\psi + \eta\vartheta + \frac{1}{2}\dot{u}^2]\!] = [\![T\dot{u}]\!] - [\![q]\!]. \tag{6.6}$$

As the consequences of these equations and the compatibility condition (5.4) we obtain

$$V^2 = \frac{[\![T]\!]}{[\![E]\!]}, \quad -\varrho_\varkappa V\left\{[\![\psi]\!] + [\![\eta\vartheta]\!] - \frac{T^- + T^+}{2\varrho_\varkappa}[\![E]\!]\right\} + [\![q]\!] = 0. \tag{6.7}$$

The Eq. $(6.7)_2$ is the Hugoniot relation; it gives all possible thermodynamical states $(E^-, \vartheta^-, \boldsymbol{\alpha}^-)$ which can be reached across a shock wave from an initial state $(E^+, \vartheta^+, \boldsymbol{\alpha}^+)$.

The general constitutive equations considered up to this time make it impossible for us to determine the velocity of the shock wave from the Eq. $(6.7)_1$. This is the reason for introducing the particular form of the constitutive equation. It will be done for an elastic-viscoplastic body with one mechanical internal state variable α_H and a thermal one α_q.

Assuming very small strains and very small temperature increments, the free energy may be taken in the form of a biquadratic expression

$$(6.8)\qquad \Psi(E,\vartheta,\boldsymbol{\alpha}) = \frac{1}{\varrho}\{a_{02}(\vartheta-\vartheta_0)^2+a_{20}(E-\alpha_H)^2+a_{11}(E-\alpha_H)(\vartheta-\vartheta_0) + a_{30}\alpha_H^2+a_{40}\alpha_q^2\},$$

where the mechanical variable α_H may be treated here as the permanent inelastic strain. Assuming that the elastic part of the strain $E-\alpha_H$ fulfils Hooke's law and by considering certain simple deformation and heating processes the constants are determined as

$$(6.9)\qquad a_{20} = -\varrho_\varkappa c_w/2\vartheta_0, \quad a_{20} = (\lambda+2\mu)/2, \quad a_{11} = -(3\lambda+2\mu)K, \quad a_{40} = k/2\tau\vartheta_0,$$

where λ and μ are Lamé constant, c_w is the specific heat at constant volume, K is the coefficient of thermal expansion and ϑ_0 is the reference temperature of at quasi-asymptotical stable equilibrium state in the front of the shock wave.

Theorem 1 implies then

$$(6.10)\qquad \begin{aligned} T &= (\lambda+2\mu)(E-\alpha_H)-(3\lambda+2\mu)(\vartheta-\vartheta_0)K,\\ \eta &= c_w(\vartheta-\vartheta_0)/\vartheta_0+(3\lambda+2\mu)(E-\alpha_H)K/\varrho_\varkappa,\\ q &= -k\alpha_q/\tau, \end{aligned}$$

where we invoked the basic assumption of small strains and small temperature increments.

In order to give an evolution equation for the inelastic strain α_H, we assume that the model of the material under consideration has linear elastic properties up to static yield condition $|T| \leqslant \sigma$ and above it — i.e. for $|T| \geqslant \sigma$ — viscoplastic ones [32]. It follows the form of the evolution equation

$$(6.11)\qquad \dot{\alpha}_H = \begin{cases} \gamma(\vartheta)\left(\dfrac{T}{\sigma(\vartheta)}-\operatorname{sgn} T\right) & \text{for} \quad |T| \geqslant \sigma(\vartheta),\\ 0 & \text{for} \quad |T| < \sigma(\vartheta). \end{cases}$$

We have assumed a dependence of the yield stress σ and the viscosity coefficient γ on the temperature ϑ. For the thermal variable we have, as in the general case, the equation

$$(6.12)\qquad \dot{\alpha}_q = g_\varkappa - \frac{1}{\tau}\alpha_q.$$

The material ahead of the wave must be in a quasi-asymptotically stable equilibrium state $(E_0, \vartheta_0, 0, \alpha^0)$. It will be satisfied when the stress at this state $T_0 = (\lambda+2\mu)(E_0-\alpha_H^0) - (3\lambda+2\mu)(\vartheta_0-\vartheta_0)K$ does not exceed the yield stress $\sigma(\vartheta_0)$.

Taking into account our constitutive equations we are able to derive from the Eqs. (6.7) the following two algebraic equations:

$$(6.13)\qquad \begin{aligned} (\varrho_\varkappa V^2-2a_{20})[\![E]\!]-a_{11}[\![\vartheta]\!] &= 0,\\ a_{11}V^2[\![E]\!]-2(a_{02}V^2+a_{40})[\![\vartheta]\!] &= 0. \end{aligned}$$

In order to have non-trivial solutions the determinant of (6.13) must vanish. Hence we have

$$\left(\frac{V}{c}\right)^4-\left(\frac{V}{c}\right)^2(b+d+1)+b=0, \tag{6.14}$$

where $c^2=\frac{\lambda+2\mu}{\varrho_\varkappa}$ is the velocity of dilatation waves and

$$b=k/(\lambda+2\mu)c_w\tau,\qquad d=(3\lambda+2\mu)^2K^2\vartheta_0/(\lambda+2\mu)\varrho_\varkappa c_w. \tag{6.15}$$

It is seen that the coefficients ([23]) b and d are positive.

THEOREM 6. *In the elastic-viscoplastic body governed by the Eqs.* (6.8)–(6.12) *a shock wave has four real and symmetric velocities* [17].

As in the previous case two particular cases will be considered.

Case 1. No thermomechanical coupling in the material, i.e. $K=0$, the coefficient of thermal expansion vanishes. Then $d=0$ and we have two pairs of velocities

$$V_{1,2}^2=\frac{\lambda+2\mu}{\varrho_\varkappa},\qquad V_{3,4}^2=\frac{k}{\varrho_\varkappa c_w\tau}. \tag{6.16}$$

COROLLARY 1. *For the case considered there are two separated waves: the mechanical dilatation waves with the speed* $\pm\sqrt{\frac{\lambda+2\mu}{\varrho_\varkappa}}$ *and the purely thermal waves with the velocity* $\pm\sqrt{\frac{k}{\varrho_\varkappa c_w\tau}}$. *If the thermal relaxation time vanishes* ($\tau\to 0$) *the finite velocity of the thermal wave goes to infinity* ($V_{3,4}^2\to\infty$).

Case 2. A non-conductor, i.e. $k=0$ (the coefficient of thermal conductivity vanishes). Then $b=0$, and we have

COROLLARY 2. *For a non-conductor there are only two symmetric coupled thermomechanical shock waves with the velocity*

$$V_{1,2}^2=\frac{(3\lambda+2\mu)^2}{\varrho_\varkappa^2}\,\frac{K^2\vartheta_0}{c_w},\qquad V_{3,4}=0. \tag{6.17}$$

References

1. J. C. MAXWELL, Phil. Trans. Royal. Soc. Lond., **157**, 49, 1867.
2. L. LANDAU, J. Phys. USSR, **5**, 71, 1941.
3. M. CHESTER, Phys. Review, **131**, 2013, 1963.
4. C. C. ACKERMAN, B. BERTMAN, H. A. FAIRBANK and R. A. GUYER, Phys. Rev. Letters, **16**, 789, 1966.
5. C. C. ACKERMAN and W. C. OVERTON, Jr., Phys. Rev. Letters, **22**, 764, 1969.
6. C. CATTANEO, Atti del. semin. math. e fisic della Univ. di Modena, **3**, 3, 1948.
7. C. CATTANEO, Compt. Rend. Acad. Sci., **247**, 431, 1958.
8. P. VERNOTTE, Compt. Rend. Acad. Sci., **246**, 3154, 1958.
9. S. KALISKI, Bull. Acad. Polon. Sci., Série Sci. Techn., **13**, 211, 253, 1965.

([23]) In the computation we assumed that $[\![\vartheta]\!]\,\frac{\vartheta_0+\vartheta^-}{2\vartheta_0}\approx[\![\vartheta]\!]$.

10. P. MAZILU, *The equation of heat* (in prepar.).
11. M. E. GURTIN, A. C. PIPKIN, Arch. Rat. Mech. Anal., **31**, 113, 1968.
12. P. J. CHEN, Arch. Rat. Mech. Anal., **35**, 1, 1969.
13. P. J. CHEN and M. E. GURTIN, Zeit. ang. Math. Phys., **21**, 232, 1970.
14. M. C. MCCARTY, Int. J. Engng. Sci., **9**, 163, 1971.
15. H. W. LORD and Y. SHULMAN, J. Mech. Phys. Solids, **15**, 299, 1967.
16. E. B. POPOV, Prikl. Mat. Mech. (J. Appl. Math. Mech. USSR), **31**, 349, 1967.
17. J. D. ACHENBACH, J. Mech. Phys. Solids, **16**, 273, 1968.
18. A. H. NAYFEH and S. NEMAT-NASSER, Zeit. ang. Math. Phys., **23**, 50, 1972.
19. T. TOKUOKA, Trans. Japan. Soc. Aero. Space Sci., **16**, 102, 143, 1973.
20. W. KOSIŃSKI and P. PERZYNA, Arch. Mech., **24**, 629, 1972.
21. I. SULICIU, in this issue.
22. W. KOSIŃSKI and P. PERZYNA, Bull. Acad. Polon. Sci., Série Sci. Techn., **21**, 647, 655, 1973.
23. B. D. COLEMAN and M. E. GURTIN, J. Chem. Phys., **47**, 597, 1967.
24. K. C. VALANIS, Proc. Symp. Mech. Beh. Mat. Dyn. Loads, Sept. San Antonio, **343**, 1968.
25. W. KOSIŃSKI and W. WOJNO, Arch. Mech., **25**, 709, 1973.
26. W. KOSIŃSKI, Arch. Mech., **25**, 441, 1973.
27. H. A. ANTOSIEWICZ, Ann. Math. Studies, **41** (Contr. theory non-linear oscill., 4), 141, 1958.
28. P. J. CHEN, Handbuch der Physic, VIa/3, 303, 1973.
29. B. D. COLEMAN and M. E. GURTIN, Arch. Rat. Mech. Anal., **19**, 266, 1965.
30. W. KOSIŃSKI, Proc. Vibr. Probl., **15**, 205, 1974.
31. W. KOSIŃSKI, Int. J. Non-Lin. Mech., **9**, 481, 1974.
32. P. PERZYNA, Advances in App. Mech., **11**, 313, 1971.

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