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An integral equation method for the solution of time-dependent problems in linearized kinetic theory

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A METHOD of solution of the linearized BGK equation for one-dimensional time-dependent flow at constant temperature and density is analyzed. It is assumed that on the line $x = 0$ the distribution function of the particle velocities is given explicitly and that at time $t = 0$ the velocity distribution is prescribed. The uniqueness and existence of the solution for certain class of functions and initial-boundary conditions is demonstrated. The differential-integral equation is reduced to the integral equation which in turn is solved analytically. The method of solution is based on the particular form equivalent in our case to the Neuman series of the integral equation. By use of this method we solved the problem which was previously examined by Cercignani by means of the "elementary solutions" and by other authors by use of the approximate methods.

W pracy omówiona jest metoda rozwiązania zlinearyzowanego równania BGK dla przepływów niestacjonarnych, jednowymiarowych ze stałą temperaturą i gęstością. Przyjmuje się, że na linii $x = 0$ funkcja rozkładu prędkości molekuł jest dana w sposób jawny, a także że dany jest rozkład prędkości w chwili $t = 0$. Dowodzi się, że dla pewnej klasy funkcji oraz warunków brzegowych i początkowych rozwiązanie jest jednoznaczne. Równanie różniczkowe całkowe sprowadzone jest do równania całkowego, dla którego znaleziono rozwiązanie analityczne. Metoda rozwiązania polega na wykorzystaniu szczególnej postaci, jaką przyjmuje w tym przypadku szereg Neumana równania całkowego. Stosując tę metodę rozwiązuje się pewne zagadnienie badane przez Cercignaniego metodą "rozwiązań elementarnych", a także przez innych autorów metodami przybliżonymi.

В работе обсужден метод решения линеаризованного уравнения BGK для нестационарных, одномерных течений с постоянной температурой и плотностью. Принимается, что на линии $x = 0$ функция распределения скорости молекул дана явным образом, а также, что дано распределение скорости в момент $t = 0$. Доказывается, что для некоторого класса функций, а также граничных и начальных условий решение единственно. Интегро-дифференциальное уравнение сводится к интегральному уравнению, для которого найдено аналитическое решение. Метод решения заключается в использовании частного вида, какой принимает в этом случае ряд Неймана интегрального уравнения. Применяя этот метод решается некоторая задача исследования Церциниани методом "элементарных решений", а также другими авторами приближенными методами.

THE GREAT difficulties encountered in solving Boltzmann's equation have very often led to the use of the model BGK equation. However, this model equation is also difficult to solve and wherever possible its linearized form is used. Some physical problems such as the well known of COUETTE, KRAMER or RAYLEIGH are one-dimensional and then for some flow conditions the linear one-dimensional BGK equation is relevant. In this case the superposition law permits to consider separately different simplified cases which describe specific physical flows [1, p. 157]. Mostly rarefied gas dynamic problems described in the literature are stationary and there are very few analytic solutions [1, 2, 3, 4,]. Some non-stationary problems were solved by CERCIGNANI [3] using the method of elementary solutions and Laplace's transformation.

A method of solving one of the simplified, linearized, one-dimensional BGK equation describing a non-stationary flow with constant temperature and density in the absence of external forces will be considered. The relevant equation [1, pp. 174] is

$$(1)\qquad \frac{\partial\varphi_1}{\partial t}+c_x\frac{\partial\varphi_1}{\partial x} = -\varphi_1(x,t,c_x)+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-c_x^2}\varphi_1(x,t,c_x)dc_x,$$

where $f=f_{00}(1+c_z\varphi_1(x,t,c_x))$ — particle velocity distribution function, $f_{00} = n_0\pi^{-\frac{2}{3}}e^{-c^2}$ — absolute Maxwellian function, $\mathbf{c}$ — molecular velocity in $\sqrt{2kT/m}$ unit, $\mathbf{u}$—average velocity in $\sqrt{2kT/m}$ unit,

$$(2)\qquad u_z=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-c_x^2}\varphi_1(x,t,c_x)dc_x - \mathbf{u}\text{ component in } z \text{ direction.}$$

Introducing the notation

$$(3)\qquad \varphi_1(x,t,c_x)=\psi(x,t,c_x)e^{-t}$$

and using the method of characteristics and Laplace transformation, a formal solution of (1) yields the following integral equation:

$$(4)\qquad \psi(x,t,c_x)=\Omega\left(\frac{x}{c_x}-t,c_x\right)+\frac{1}{c_x\sqrt{\pi}}\int_0^x\int_{-\infty}^{\infty} e^{-u^2}\psi\left(s,t-\frac{x-s}{c_x},u\right)du\,ds.$$

Multiplying this equation by $\frac{1}{\sqrt{\pi}}e^{-c_x^2}$ and integrating with respect to c_x leads to:

$$(5)\quad g_1(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\Omega\left(\frac{x}{c_x}-t,c_x\right)dc_x+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_0^x\frac{1}{c_x}e^{-c_x^2}g_1\left(s,t-\frac{x-s}{c_x}\right)ds\,dc_x,$$

where

$$(6)\qquad g_1(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-c_x^2}\psi(x,t,c_x)dc_x.$$

$\Omega(v,c_x)$ is an arbitrary function differentiable with respect to v. The function $\Omega(v,c_x)$ will be henceforth considered as known.

Neumann's series for (5) can be written in the form:

$$K^{(1)}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-c_x^2}\Omega\left(\frac{x}{c_x}-t,c_x\right)dc_x=\int_{-\infty}^{\infty}K_1\left(\frac{x}{c_x}-t,c_x\right)dc_x,$$

$$K^{(2)}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{u}{c_x-u}e^{-c_x^2-u^2}\left[\Omega_1\left(\frac{x}{u}-t,u\right)-\Omega_1\left(\frac{x}{c_x}-t,u\right)\right]du\,dc_x$$

$$=\int_{-\infty}^{\infty}K_2\left(\frac{x}{c_x}-t,c_x\right)dc_x,$$

$$K^{(3)}=\int_{-\infty}^{\infty}K_3\left(\frac{x}{c_x}-t,c_x\right)dc_x.$$

It can be noticed that the expressions under the integral can be reduced to the same form, and this suggests that the solution must be of the form:

$$g_1(x,t) = \int_{-\infty}^{\infty} F\left(\frac{x}{c_x}-t, c_x\right) dc_x. \tag{7}$$

Putting (7) into (5), an equation for the function $F(v, c_x)$ is obtained

$$\int_{-\infty}^{\infty}\left\{\frac{\partial F_1}{\partial v}-\frac{1}{\sqrt{\pi}}e^{-c_x^2}\Omega\left(\frac{x}{c_x}-t, c_x\right)-\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{c_x e^{-u}}{u-c_x}F_1\left(\frac{x}{c_x}-t, c_x\right)du - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{ue^{-c_x^2}}{u-c_x}F_1\left(\frac{x}{c_x}-t, u\right)du\right\}dc_x = 0 \tag{8}$$

with the condition

$$\left.\frac{\partial F_1}{\partial v}\right|_{v=0} = \frac{1}{\sqrt{\pi}}e^{-c_x^2}\Omega(0, c_x),$$

where the following notation was used

$$\frac{\partial F_1(v, c_x)}{\partial v} = F(v, c_x). \tag{9}$$

A sufficient condition for (8) to be satisfied is that the expression in the brackets is zero.

Using the uniqueness theorem for the solution of (4) (see Appendix I) we observe that this is also a necessary condition.

Hence F_1 must satisfy the following equation:

$$\frac{\partial F_1}{\partial v} = \frac{1}{\sqrt{\pi}}e^{-c_x^2}\Omega(v, c_x)+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f_u(c_x)e^{-c_x^2}F_1(v,u)du - F_1(v, c_x), \tag{10}$$

where

$$f_u(c_x) = \frac{u}{u-c_x}+p(c_x)\,\delta(c_x-u), \tag{11}$$

$$p(x) = e^{x^2}\int_{-\infty}^{\infty}\frac{te^{-t^2}}{t-x}dt = \sqrt{\pi}\left(e^{x^2}-2x\int_0^x e^{x^2}dx\right). \tag{12}$$

CERCIGNANI has shown (2) that the function $f_u(c_x)$ has the following property:

$$\int_{-\infty}^{\infty} f_{u_1}(x)f_{u_2}(x)xe^{-x^2}dx = c(u_1)\,\delta(u_1-u_2), \tag{13}$$

where

$$c(u) = ue^{-u^2}\{[p(u)]^2+\pi^2 u\},$$

$$\int_{-\infty}^{\infty} xe^{-x^2}f_u(x)dx = 0, \tag{14}$$

$$\int_{-\infty}^{\infty} x^2e^{-x^2}f_u(x)dx = 0. \tag{15}$$

Making use of the orthogonality properties the Eq. (10) can be solved analytically (Appendix II) as follows:

$$(16)\quad F(v, c_x) = \frac{\partial F_1}{\partial v} = \frac{1}{\sqrt{\pi} c_x e^{-c_x^2}} \int_0^v e^{\left(\frac{1}{\sqrt{\pi}} e^{-c_x^2} p(c_x) - 1\right)(v - v_1)} \times$$

$$\times \left\{ \left(\frac{1}{\sqrt{\pi}} e^{-c_x^2} p(c_x) - 1 \right) \sin \left[\sqrt{\pi} c_x e^{-c_x^2} (v - v_1) \right] + \sqrt{\pi} c_x e^{-c_x^2} \cos \left[\sqrt{\pi} c_x e^{-c_x^2} (v - v_1) \right] \right\} \times$$

$$\times G(v_1, c_x) dv_1 + \frac{\left(\frac{1}{\sqrt{\pi}} e^{-c_x^2} p(c_x) - 1 \right) \Omega(0, c_x)}{\pi c_x} \sin \left(\sqrt{\pi} c_x e^{-c_x^2} v \right)$$

$$+ \frac{\Omega(0, c_x) e^{-c_x^2}}{\sqrt{\pi}} \cos \left(\sqrt{\pi} c_x e^{-c_x^2} v \right) e^{\left(\frac{1}{\sqrt{\pi}} e^{-c_x^2} p(c_x) - 1 \right) v},$$

where

$$G(v, z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left[\Omega(v, z) + \frac{\partial \Omega(v, z)}{\partial v} - \frac{1}{z} \int_{-\infty}^{\infty} u e^{-u^2} \Omega(v, u) f_z(u) du \right].$$

Now, the following general solution of (4) can be written:

$$(17)\quad \psi(x, t, c_x) = \Omega\left(\frac{x}{c_x} - t, c_x \right) + \frac{1}{c_x} \int_0^x \int_{-\infty}^{\infty} \int_0^{\frac{s}{z} - t + \frac{x-s}{c_x}} \frac{1}{\sqrt{\pi} z e^{-z^2}} \exp\left\{ \left(\frac{1}{\sqrt{\pi}} e^{-z^2} p(z) - 1 \right) \times \right.$$

$$\times \left[\left(\frac{s}{z} - t + \frac{x-s}{c_x} \right) - v_1 \right] \Bigg\} \left\{ \left(\frac{1}{\sqrt{\pi}} e^{-z^2} p(z) - 1 \right) \sin \left[\sqrt{\pi} z e^{-z^2} \left(\frac{s}{z} - t + \frac{x-s}{c_x} - v_1 \right) \right] \right.$$

$$\left. + \sqrt{\pi} z e^{-z^2} \cos \left[\sqrt{\pi} z e^{-z^2} \left(\frac{s}{z} - t + \frac{x-s}{c_x} - v_1 \right) \right] \right\} G(v_1, z) dv_1 \, dz \, ds$$

$$+ \frac{1}{c_x} \int_0^x \int_{-\infty}^{\infty} \left\{ \frac{\frac{1}{\sqrt{\pi}} \left(e^{-z^2} p(z) - 1 \right) \Omega(0, z)}{\pi z} \sin \left[\sqrt{\pi} z e^{-z^2} \left(\frac{s}{z} - t + \frac{x-s}{c_x} \right) \right] \right.$$

$$\left. + \frac{\Omega(0, z) e^{-z^2}}{\sqrt{\pi}} \cos \left[\sqrt{\pi} z e^{-z^2} \left(\frac{s}{z} - t + \frac{x-s}{c_x} \right) \right] \right\} \exp\left\{ \left(\frac{1}{\sqrt{\pi}} e^{-z^2} p(z) - 1 \right) \left(\frac{s}{z} - t + \frac{x-s}{c_x} \right) \right\} dz \, ds.$$

A particular solution for given boundary and initial conditions can be deduced from the general solution. Practically, great difficulties appear as it is not easy to deduce $\Omega(v, c_x)$ for given boundary and initial conditions. When this can be done, an analytical solution of the problem is obtained.

For illustration, the following problem will be solved:

1. The gas fills the whole infinite space.

2. Up to $t = 0$ the flow is stationary and the distribution function (the disturbance φ_1) is a known function of x and c_x.

3. The disturbance function $\varphi_1(x, t, c_x)$ is antisymmetric with respect to the plane $x = 0$.

4. For $t > 0$ the function φ_1 is continuous at $x = 0$.

At the moment $t = 0$ the factors maintaining the stationary state disappear.

The flow for $t > 0$ for constant temperature and density is to be determined.

To be more specific, the case when the initial conditions are

$$\varphi_1(x, t, c_x)|_{t \leqslant 0} = Ax \pm B, \quad x \gtrless 0$$

will be considered.

The continuity and antisymmetry conditions indicate that

$$\varphi_1(x, t, c_x)|_{x=0} = 0, \quad t > 0.$$

In this case, $\Omega(v, c_x)$ can be found. If $|x/c_x| < t$, then the particle which is at point x will be in contact with the wall at t, and the boundary conditions must be used to determine the function $\Omega(v, c_x)$. When $|x/c_x| \geqslant t$ the initial conditions should be used. Substituting the value of ψ at $t = 0$ and $x = 0$ in (4) it follows:

$$\Omega(v, c_x) = \begin{cases} 0, & v < 0, \quad \left|\dfrac{x}{c_x}\right| < t, \\ Ac_x(1-e^{-v}) + Be^{-v}, & v \geqslant 0, \quad x > 0, \quad c_x > 0, \quad \left|\dfrac{x}{c_x}\right| \geqslant t, \\ -Ac_x(1-e^{-v}) - Be^{-v}, & v \geqslant 0, \quad x < 0, \quad c_x < 0, \quad \left|\dfrac{x}{c_x}\right| \geqslant t. \end{cases} \tag{18}$$

If the values of x and c_x are of different signs, the antisymmetry condition for $x = 0$ should be used. Putting (18) in the general solution (17) the analytic expression for the function φ_1 is obtained.

Now the hydrodynamic magnitudes can be deduced. The average velocity follows from (2), (3), (6), (16):

$$\begin{aligned} u_z &= \frac{1}{2} e^{-t} g_1 = \frac{1}{2} e^{-t} \int_{-\infty}^{\infty} F\left(\frac{x}{c_x} - t, c_x\right) dc_x \\ &= \operatorname{sgn} x \frac{2e^{-t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} H\left(\frac{c_x}{x} - \frac{1}{t}\right) e^{(q(c_x)-1)\left(\frac{x}{c_x} - t\right)} \left\{ \left[\frac{(q(c_x)-1)B}{\sqrt{\pi} c_x} + \frac{A}{\sqrt{\pi}} \right] \times \right. \\ &\quad \left. \times \sin\left[\sqrt{\pi} c_x e^{-c_x^2} \left(\frac{x}{c_x} - t\right)\right] + Be^{-c_x^2} \cos\left[\sqrt{\pi} c_x e^{-c_x^2} \left(\frac{x}{c_x} - t\right)\right] \right\} dc_x, \end{aligned} \tag{19}$$

where

$$q(c_x) = \frac{1}{\sqrt{\pi}} e^{-c_x^2} p(c_x), \quad H(x) = \begin{cases} 1, & x > 0, \\ \dfrac{1}{2}, & x = 0, \\ 0, & x < 0. \end{cases}$$

$A = 0$ corresponds to the physical case of a gas, initially in two semi-spaces $x \gtrless 0$, of constant temperature T_0 and number density n_0 having an average velocity $+B/2$ at $x > 0$ and

$-B/2$ at $x < 0$. One can imagine that the two semi-spaces are divided, say, by a membrane which is suddenly removed at $t = 0$ and therefore at $t > 0$ a diffusion of velocity occurs. The resulting perturbation function at $t > 0$ is continuous at $x = 0$.

The problem was solved by CERCIGNANI and TAMBI [3] using the method of elementary solutions and Laplace transformation. The resulting mean velocity after retransformation is

$$u_z = \operatorname{sgn} x \left\{ \frac{B}{2} - \frac{B}{\sqrt{\pi}} \int_{-\infty}^{\infty} H\left(\frac{c_x}{x} - \frac{1}{t}\right) e^{(q(c_x)-1)\left(\frac{x}{c_x}-t\right)} \left[\frac{q(c_x)-1}{\sqrt{\pi} c_x} \sin\left[\sqrt{\pi} c_x e^{-c_x^2}\left(\frac{x}{c_x} - t\right)\right] \right.\right.$$
$$\left.\left. + e^{-c_x^2} \cos\left[\sqrt{\pi} c_x e^{-c_x^2}\left(\frac{x}{c_x} - t\right)\right]\right] dc_x \right\}.$$

The Eq. (19) after transformations (Appendix III) can be reduced to this form.

The solution (17) obtained is an exact analytic solution of (1) of greater theoretical than practical value. There is no method of finding particular solutions, except the one given above satisfying boundary and initial conditions required, i.e. corresponding to a given physical problem.

Appendix I

The uniqueness theorem

THEOREM 1. *Let the function $\psi(x, t, c_x)$ satisfy the Eq.* (4). *Let us assume*:

$$\psi(x, t, c_x)|_{x=0} = W_b(t, c_x), \quad t > 0,$$
$$\psi(x, t, c_x)|_{t=0} = W_p(x, c_x).$$

Function $\psi(x, t, c_x)$ is continuous with respect to t. There exists such $\delta > 0$ that the function $\psi(x, t, c_x)$ is analytical with respect to t in $\langle 0, \delta \rangle$ interval. Then $\psi(x, t, c_x)$ is the unique function of this class which satisfies the Eq. (4).

Proof. Let us suppose that there exist two solutions of the Eq. (4) which satisfy boundary and initial conditions required. In this case difference of these solutions $\psi_0 = \psi_1 - \psi_2$ satisfies the Eq. (4) as well as zero boundary and initial conditions.

Let $\Omega_0(v, c_x)$ correspond to ψ_0 solution. Then

$$\Omega_0(-t, c_x) = 0, \tag{1.1}$$

$$\Omega_0\left(\frac{x}{c_x}, c_x\right) + \frac{1}{c_x\sqrt{\pi}} \int_x^0 \int_{-\infty}^{\infty} e^{-u^2} \psi_0\left(s, -\frac{x-s}{c_x}, u\right) du\, ds = 0. \tag{1.2}$$

Taking into account that $\psi_0(x, t, c_x)$ satisfies the Eq. (4) and relations (1.1) and (1.2) hold, we receive:

$$\psi_0(x, t, c_x) = \frac{1}{c_x\sqrt{\pi}} \int_x^0 \int_{-\infty}^{\infty} e^{-u^2} \psi_0\left(s, t - \frac{x-s}{c_x}, u\right) du\, ds, \quad t \geq \left|\frac{x}{c_x}\right|, \tag{1.3}$$

$$\psi_0(x, t, c_x) = \frac{1}{c_x\sqrt{\pi}} \int_{x-tc}^{x} \int_{-\infty}^{\infty} e^{-u^2} \psi_0\left(s, t - \frac{x-s}{c_x}, u\right) du\, ds, \quad 0 \leq t < \left|\frac{x}{c_x}\right|. \tag{1.4}$$

If any function satisfies the Eqs. (1.3) and (1.4), then the derivative with respect to time of this function will satisfy these equations, too.

For given x and c_x and $t < |x/c_x|$, the Eq. (1.4) is valid and consequently $\partial\psi_0/\partial t|_{t=0} = 0$. It can be shown inductively that $\partial^n\psi/\partial t^n|_{t=0} = 0$. Expanding $\psi_0(x, t, c_x)$ into series with respect to t we find that $\psi_0(x, t, c_x) \equiv 0$ at least in $\langle 0, \delta)$ interval. We consider $t > \delta$.

Using mean value theorem and definition (6) in the Eq. (1.4) leads to

$$\psi_0(x, t, c_x) = \frac{1}{c_x}\int\limits_{x-tc_x}^{x} g_1\left(s, t-\frac{x-s}{c_x}\right)ds = tg_1\left(s_1, t-\frac{x-s_1}{c_x}\right)$$

$$= \frac{t}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} e^{-u^2}\psi_0\left(s_1, t-\frac{x-s_1}{c_x}, u\right)du,$$

where $s_1 = s_1(x, t, c_x) \in (x-tc_x, x)$ or $s_1 \in (x, x-tc_x)$.

Let δ_1 means the right end of the greatest interval where for all $t \in (\delta, \delta_1)$ the condition $t-\dfrac{x-s_1}{c_x} \leqslant \delta$ is satisfied. As $s_1 < x$ and c_x is bounded $\left(|c_x| < \dfrac{|x|}{t}\right)$, so $\delta_1 > \delta$. Also for the Eq. (1.3) the interval (δ, δ_1), where $\psi_0(x, t, c_x) \equiv 0$ could be found.

In this way the interval $\langle 0, \delta\rangle$ in which $\psi_0(x, t, c_x) \equiv 0$ can be extended to interval $\langle 0, \delta_1\rangle$. Proceeding in the same way we receive a sequence of intervals $\langle 0, \delta_n\rangle$ where function $\psi_0(x, t, c_x) \equiv 0$. Utilizing continuity of $\psi_0(x, t, c_x)$ with respect to t it can be shown that $\delta_n \to \infty$. It means that $\psi_0(x, t, c_x) \equiv 0$ for $t \in \langle 0, +\infty)$. Hence, the Eq. (4) has at least one solution which satisfies the required boundary and initial conditions.

THEOREM 2. *Let function* $\psi^+(x, t, c_x > 0)$ *satisfy the Eq.* (4) *and function* $\psi^-(x, t, c_x < 0)$ *satisfy the Eq.* (4a)

$$\text{(4a)}\quad \psi(x, t, c_x) = \Omega\left(\frac{x-x_0}{c_x} - t, c_x\right) + \frac{1}{c_x\sqrt{\pi}}\int\limits_{0}^{x-x_0}\int\limits_{-\infty}^{\infty} e^{-u^2}\times$$

$$\times\psi\left(s+x_0, t-\frac{x-x_0-s}{c_x}, u\right)du\,ds.$$

We assume that

$$\psi(x, t, c_x > 0)|_{x=0} = \psi^+(0, t, c_x) = W_b^+(t, c_x), \qquad c_x > 0,$$

$$\psi(x, t, c_x < 0)|_{x=x_0} = \psi^-(x_0, t, c_x) = W_b^-(t, c_x), \qquad c_x < 0,$$

$$\psi(x, t, c_x)|_{t=0} = \psi(x, 0, c_x) = W_p(x, c_x).$$

Function $\psi(x, t, c_x)$ *is continuous with respect to* t. *There exists such* $\delta > 0$ *that function* $\psi(x, t, c_x)$ *is analytical with respect to* t *in the interval* $\langle 0, \delta\rangle$. *Then*

$$\psi(x, t, c_x) = \begin{cases} \psi^+(x, t, c_x), & c_x > 0,\\ \psi^-(x, t, c_x), & c_x < 0\end{cases}$$

is the only function of this class which satisfies the solutions (4) *and* (4a).

The proof is similar to the proof of Theorem 1.

THEOREM 3. *Let the function* $\psi(x, t, c_x)$ *satisfy the Eqs.* (4) *and* (4a). *Let us assume that*

$$\psi(x, t, c_x)|_{x=0} = W_b^+(t, c_x > 0),$$
$$\psi(x, t, c_x)|_{x=x_0} = W_b^-(t, c_x < 0),$$
$$\psi(x, t, c_x)|_{t\leqslant 0} = W_p(x, c_x).$$

Function $\psi(x, t, c_x)$ *is continuous with respect to* t *at* $t > 0$. *Then* $\psi(x, t, c_x)$ *is the only function of this class which satisfies the Eqs.* (4) *and* (4a) *for* $c_x > 0$ *and* $c_x < 0$, *respectively.*

Proof. Similarly to the previous proofs, let us introduce a function $\psi_0 = \psi_1 - \psi_2$. The function ψ_0 must satisfy zero boundary and initial conditions.

1. Let us assume that $c_x > 0$ and consider the Eq. (4). If $\frac{x}{c_x} - t \leqslant 0$ then $\Omega_0\left(\frac{x}{c_x} - t, c_x\right) = 0$ (boundary conditions). Simultaneously, from initial conditions it follows that

$$\Omega_0\left(\frac{x}{c_x} - t, c_x\right) = 0, \quad \frac{x}{c_x} - t > 0.$$

2. For $c_x < 0$, proceeding is the same, but it is connected with the Eq. (4a).

Hence, the function $\psi_0(x, t, c_x)$ must satisfy the equations

$$\psi_0(x, t, c_x) = \frac{1}{c_x\sqrt{\pi}} \int_0^x \int_{-\infty}^{\infty} e^{-u^2}\psi\left(s, t - \frac{x-s}{c_x}, u\right) du\, ds, \quad c > 0, \tag{1.5}$$

$$\psi_0(x, t, c_x) = \frac{1}{c_x\sqrt{\pi}} \int_0^{x-x_0} \int_{-\infty}^{\infty} e^{-u^2}\psi\left(s + x_0, t - \frac{x-x_0-s}{c_x}, u\right) du\, ds, \quad c_x < 0 \tag{1.6}$$

and the condition $\psi_0(x, t, c_x) \equiv 0$ at $t \leqslant 0$.

The proof is similar to the proof of Theorem 1.

Appendix II

Solution of the Eq. (10)

The condition (9) and the Eq. (10) involve the following property:

$$F_1(v, c_x)|_{v=0} = 0. \tag{9a}$$

Proof. Substituting (9) into the Eq. (10) in which $v = 0$, we obtain

$$F_1(0, c_x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-c_x^2} f_u(c_x) F_1(0, u)\, du. \tag{2.1}$$

Multiplying this equation by $c_x f_z(c_x)$ and integrating over c_x we find

$$\int_{-\infty}^{\infty} c_x F_1(0, c_x) f_z(c_x)\, dc_x = \frac{1}{\sqrt{\pi}} C(z) F_1(0, z). \tag{2.2}$$

Simultaneously from the Eq. (2.1) we obtain

$$\int_{-\infty}^{\infty} c_x f_u(c_x) F_1(0, u) du = \sqrt{\pi} c_x e^{+c_x^2} F_1(0, c_x). \tag{2.3}$$

It can be noticed that

$$u f_z(u) + z f_u(z) = 2up(u)\,\delta(u-z). \tag{2.4}$$

Let us substitute c_x for u and z for c_x in the expression (2.2). Adding (2.2) and (2.3) and utilizing (2.4), we obtain

$$2c_x p(c_x) F_1(0, c_x) = \frac{1}{\sqrt{\pi}} C(c_x) F_1(0, c_x) + \sqrt{\pi} c_x e^{c_x^2} F_1(0, c_x);$$

it means that

$$F_1(0, c_x)\left[2c_x p(c_x) - \frac{1}{\sqrt{\pi}} C(c_x) F_1(0, c_x) + \sqrt{\pi}^x c e^{c_x^2}\right] = 0.$$

As the expression in brackets is not identically zero, we obtain (9a).

Solving formally (10) and using the condition (9a) we obtain

$$F_1(v, c_x) = \int_0^v \frac{1}{\sqrt{\pi}} e^{-c_x^2}\left[\Omega(s, c_x) + \int_{-\infty}^{\infty} f_u(c_x) F_1(s, u) du\right] e^{-v+s} ds. \tag{2.5}$$

Multiplying this equation by $c_x f_z(c_x)$ and utilizing orthogonality of $f_u(c_x)$ we find

$$\int_{-\infty}^{\infty} c_x f_z(c_x) F_1(v, c_x) dc_x = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} c_x e^{-c_x^2} f_z(c_x) \int_0^v \Omega(s, c_x) e^{-v+s} ds\, dc_x + \frac{1}{\sqrt{\pi}} \int_0^v C(z) F_1(s, z) e^{-v+s} ds. \tag{2.6}$$

Substituting u for c_x and c_x for z we have

$$\int_{-\infty}^{\infty} u f_{c_x}(u) F_1(v, u) du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u e^{-u^2} \int_0^v \Omega(s, u) f_{c_x}(u) e^{-v+s} ds\, du + \frac{1}{\sqrt{\pi}} \int_0^v C(c_x) F_1(s, c_x) e^{-v+s} ds. \tag{2.7}$$

At the same time the Eq. (10) leads to

$$\int_{-\infty}^{\infty} c_x f_u(c_x) F_1(v, u) du = \sqrt{\pi} c_x e^{c_x^2} \frac{\partial F_1(v_1 c_x)}{\partial v} + \sqrt{\pi} c_x e^{c_x^2} F_1(v, c_x) - c_x \Omega(v, c_x). \tag{2.8}$$

Summing (2.7) adn (2.8) and using (2.4) we obtain

$$(2.9)\quad 2c_x p(c_x)F_1(v, c_x) = \sqrt{\pi}c_x e^{c_x^2}\frac{\partial F_1(v, c_x)}{\partial v} + \sqrt{\pi}c_x e^{c_x^2}F_1(v, c_x) + \frac{1}{\sqrt{\pi}}\int_0^v C(c_x)F_1(s, c_x)e^{-v+s}ds - c_x\Omega(v, c_x) + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_0^v ue^{-u^2}\Omega(s, u)f_{c_x}(u)e^{-v+s}ds\,du.$$

This equation can be reduced to linear differential equation with constant coefficients.

In fact, multiplying both sides by e^v and differentiating over v we find

$$(2.10)\quad \sqrt{\pi}c_x e^{c_x^2}\frac{\partial^2 F_1(v, c_x)}{\partial v} + \left[2\sqrt{\pi}c_x e^{c_x^2} - 2c_x p(c_x)\right]\frac{\partial F_1(v, c_x)}{\partial v} + \left[\sqrt{\pi}c_x e^{c_x^2} - 2c_x p(c_x) + \frac{1}{\sqrt{\pi}}C(c_x)\right]F_1(v, c_x) = c_x\Omega(v, c_x) + c_x\frac{\partial\Omega(v, c_x)}{\partial v} - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} ue^{-u^2}\Omega(v, u)f_{c_x}(u)du$$

and conditions

$$F_1(v, c_x)|_{v=0} = 0, \qquad \left.\frac{\partial F_1(v, c_x)}{\partial v}\right|_{v=0} = \frac{1}{\sqrt{\pi}}e^{-c_x^2}\Omega(0, c_x);$$

c_x is treated as parameter.

Applying to this expression a well known fact from differential equations theory [5], we obtain the solution

$$(2.11)\quad F_1(v, c_x) = \frac{1}{\sqrt{\pi}c_x e^{-c_x^2}}\int_0^v e^{\left(\frac{1}{\sqrt{\pi}}e^{-c_x^2}p(c_x)-1\right)(v-v_1)}\sin\left[\sqrt{\pi}c_x e^{-c_x^2}(v-v_1)\right]G(v, c_x)dv, + \frac{\Omega(0, c_x)}{\pi c_x}e^{\left(\frac{1}{\sqrt{\pi}}e^{-c_x^2}p(c_x)-1\right)v}\sin\left(\sqrt{\pi}c_x e^{-c_x^2}v\right);$$

consequently

$$(2.12)\quad F(v, c_x) = \frac{\partial F_1}{\partial v} = \frac{1}{\sqrt{\pi}c_x e^{-c_x^2}}\int_0^v e^{\left(\frac{1}{\sqrt{\pi}}e^{-c_x^2}p(c_x)-1\right)(v-v_1)}\left\{\left(\frac{1}{\sqrt{\pi}}e^{-c_x^2}p(c_x)-1\right)\sin\left[\sqrt{\pi}c_x e^{-c_x^2}(v-v_1)\right] + \sqrt{\pi}c_x e^{-c_x^2}\cos\left[\sqrt{\pi}c_x e^{-c_x^2}(v-v_1)\right]\right\}G(v_1, c_x)dv_1$$

$$+\left[\frac{\left(\frac{1}{\sqrt{\pi}}e^{-c_x^2}p(c_x)-1\right)\Omega(0, c_x)}{\pi c_x}\sin\left(\sqrt{\pi}\,c_x\,e^{-c_x^2}v\right)\right.$$

$$\left.+\frac{\Omega(0, c_x)e^{-c_x^2}}{\sqrt{\pi}}\cos\left(\sqrt{\pi}\,c_x e^{-c_x^2}v\right)\right]e^{\left(\frac{1}{\sqrt{\pi}}e^{-c_x^2}p(c_x)-1\right)v}.$$

Appendix III

It can be noticed that the arbitrary constant C satisfies the Eq. (1). Hence, if this equation is satisfied by function φ_1 then it will also be satisfied by function

$$\varphi_1'(x, t, c_x) = \varphi_1 - C.$$

Let us put $C = B$ at $x > 0$ and $C = -B$ at $x < 0$. Consequently,

$$g(x, t) = B + g'(x, t),$$

where

$$g(x, t) = g_1(x, t)e^{-t} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-c_x^2}\varphi_1(x, t, c_x)dc_x.$$

Function φ_1 satisfies the integral equation

$$\varphi_1(x, t, c_x) = \Omega\left(\frac{x}{c_x}-t, c_x\right)e^{-t} + \frac{1}{c_x}\int_0^x g\left(s, t-\frac{x-s}{c_x}\right)e^{-\frac{x-s}{c_x}}ds,$$

so

$$\varphi' \pm B = \Omega\left(\frac{x}{c_x}-t, c_x\right)e^{-t} \mp Be^{-\frac{x}{c_x}} \pm B + \frac{1}{c_x}\int_0^x g'\left(s, t-\frac{x-s}{c_x}\right)e^{-\frac{x-s}{c_x}}ds \tag{3.1}$$

and, consequently,

$$\varphi'(x, t, c_x) = \Omega'\left(\frac{x}{c_x}-t, c_x\right)e^{-t} + \frac{1}{c_x}\int_0^x g'\left(s, t-\frac{x-s}{c_x}\right)e^{-\frac{x-s}{c_x}}ds, \tag{3.2}$$

where

$$\Omega'\left(\frac{x}{c_x}-t, c_x\right) = \mp Be^{-\left(\frac{x}{c_x}-t\right)} + \Omega\left(\frac{x}{c_x}-t, c_x\right). \tag{3.3}$$

Introducing the notation $\psi'(x, t, c_x) = \varphi' e^t$ we find

$$\psi'(x, t, c_x) = \Omega'\left(\frac{x}{c_x}-t, c_x\right) + \frac{1}{c_x}\int_0^x g_1'\left(s, t-\frac{x-s}{c_x}\right)ds. \tag{3.4}$$

The equation (3.4) is of the same form as the Eq. (4). It means that it has the same solutions.

By (3.3) the function $\Omega'(v, c_x)$ can be found. For the case $A = 0$, we have:

$$\Omega'(v, c_x) = \begin{cases} -Be^{-v}, & v < 0, \quad \left|\frac{x}{c_x}\right| < t, \quad x > 0, \quad c_x > 0, \\ Be^{-v}, & v < 0, \quad \left|\frac{x}{c_x}\right| < t, \quad x < 0. \quad c_x < 0, \\ 0, & v \geqslant 0 \quad \left|\frac{x}{c_x}\right| \geqslant t, \end{cases}$$

According to the general solution of (17) we obtain

$$u_z = \frac{1}{2} g'(x, t) e^{-t} = \operatorname{sgn}(x) e^{-t} \left[e^t \frac{B}{2} + \frac{B}{\sqrt{\pi}} \int_{-\infty}^{\infty} H\left(\frac{c_x}{x} - \frac{1}{t}\right) e^{(q(c_x)-1)\left(\frac{x}{c_x}-t\right)} \times \right.$$

$$\left. \times \left\{ \frac{q(c_x)-1}{\sqrt{\pi} c_x} \sin\left[\sqrt{\pi} c_x e^{-c_x^2}\left(\frac{x}{c_x} - t\right)\right] + e^{-c_x^2} \cos\left[\sqrt{\pi} c_x e^{-c_x^2}\left(\frac{x}{c_x} - t\right)\right]\right\} dc_x \right].$$

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