Archives of Mechanics • Archiwum Mechaniki Stosowanej • 27, 4, pp. 565–575, Warszawa 1975

An analytic approach to some problems of optimal design of beams and plates

V. KOMKOV and N. P. COLEMAN
(LUBBOCK, TEXAS, and ROCK ISLAND, ILLINOIS)

THE METHOD of optimum design of elastic systems, in particular of beams and plates, is analyzed. The abstract methods of differentiation in Hilbert space are applied. Identification of critical points of functionals determines the optimum design of elastic system.

Autor omawia optymalną metodę projektowania systemów sprężystych, zwłaszcza belek i płyt, stosując abstrakcyjne metody różniczkowania w przestrzeni Hilberta. Identyfikacja krytycznych punktów funkcjonałów określa optymalny projekt układu sprężystego.

Обсуждается метод оптимального проектирования упругих систем, особенно балок и плит. Применяется абстрактные методы дифференцирования в гильбертовом пространстве. Идентификация критических точек функционалов определяет оптимальный проект упругой системы.

Introductory remarks

THE PROBLEM of optimum design of beams, simple plates, sandwich plates and structures has been a subject of many recent papers. The optimum weight design for a given deflection at a specified point along a statically determinate beam has been derived by PRAGER in [6]. Several papers by PRAGER and his collaborators (see particularly CHERN) have generalized this result obtaining necessary conditions for minimal weight subject to some *a priori* specified deflection condition [7, 8, 9, 10]. The derivation of PRAGER and associates utilized Betti's principle (which could be regarded as an alternate form of Castigliano's theorem) $Y(x_0) = \int_0^l M(x)\, m(x+x_0)\, S(x)\, dx$. The assumption of statical determinacy resulted in the moments $M(x)$ and $m(x-x_0)$ being independent of the design parameter, which was $S(x) = [EI(x)]^{-1}$. However, even in the case of minimum weight design for a specified deflection at some given point along a beam, the existence of such admissible design or its uniqueness has never been discussed. Moreover, the criteria of admissibility have never been explicitly stated in these papers.

A more general class of optimization theory, incorporating the usual strength or stability criteria has not been solved to the best of our knowledge. An important problem of optimization theory concerns minimization of maximum deflection, or of maximum slope subject to constraints concerning weight and/or strength criteria. The problem of minimizing the maximum deflection arises in some precision mechanisms, and in the design of weapons. A constraint limiting the maximum stress level, or perhaps the magnitude of the Huber-von Mises stress tensor invariant (or other strength criterion), is naturally assumed in engineering analysis. In this class of optimization problems the difficulty in generalizing Prager's result [6] becomes apparent. Moreover, the problem of minimizing the maximum deflection subject to a constant (given) weight of the beam (or plate) is not the dual of Prager's problem of minimizing the weight subject to a given deflection at some specified cross-section of the beam (or plate), in the usual definition of duality, as given for example by LUENBERGER ([5], Sec. 7.12, pages 200–209), and in general entirely different designs would result in these two optimization problems, unless some additional assumptions are made, which appeared to us to be very unrealistic. (However, these assumptions turn out to be justified in an actual engineering case concerning a certain sandwich beam design). In general, we have to regard these two problems as unrelated, and optimality criteria derived for one of them are inapplicable to the other.

In this paper we shall first consider Prager's problem and derive some theoretical results concerning existence and uniqueness of optimal designs. Then we shall consider the optimization problem of minimizing the maximum deflection or slope subject to some weight and strength criteria.

1. Notation and some mathematical preliminaries

The notation is fairly standard: $M(x)$ is the bending moment due to the applied loads, $m(x, x_0)$ is the bending moment at x due to a unit load positioned at x_0,

$$q(x, x_0) = \frac{dm(x, x_0)}{dx_0},$$

E is Young's modulus (considered constant in this paper), $I(x)$ is the moment of inertia of the cross-section about the neutral axis, $A(x)$ denotes the cross-sectional area.

We note that only the product EI matters in our arguments. In the first half of this paper we shall consider exclusively the beam equation

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2y}{dx^2}\right) = -g(x), \tag{1.1}$$

$x \in [0, l]$, subject to boundary conditions which will be specified later, but such that the problem is well posed. $y(x)$ is an element of the Sobolev space $H^2[0, l]$, i.e. it is twice weakly differentiable, and both derivatives and $y(x)$ are elements of $L_2[0, l]$. The "elastica" condition requires $y(x)$ to be an element of $C^1[0, l]$, that is to be once continuously differentiable. E is a positive constant. $I(x)$ is piecewise continuously differentiable function of x, and so is $A(x)$. $g(x)$ is assumed to be of the form

$$g(x) = \sigma(x) + \sum_{i=1}^{n} c_i \delta(x-\xi_i) + \sum_{j=1}^{m} d_j \delta'(x-\xi_j), \tag{1.2}$$

where c_i, $i = 1, 2, \ldots, n$, d_j, $j = 1, 2, \ldots, m$ are constants, $\delta(x)$ denotes the Dirac delta function, $0 < \xi_k < l$, $k = 1, 2, \ldots, r$ $(r = \max(m, n))$, and $\sigma(x)$ is an element of $L_2[0, l]$. For theoretical reasoning justifying restriction of admissible loads to form (1.2) see [4].

We also assume that there exists a functional relation $A(x) = \phi(I(x))$ in our discussion of beam theory, $\phi\colon \mathscr{R}_+ \to \mathscr{R}_+$ and monotonically increasing. We introduce a parameter (or vector) $S(x)$, which uniquely defines our design. We postulate that $S(x)$ is a positive function (vector with positive components) bounded and piecewise continuous on $[0, l]$. For mathematical convenience, S will be regarded as an element of $L_2[0, l]$ (or $L_2^n[0, l]$). We assume that $A(x)$, $I(x)$, and consequently the deflection $y(x)$, are uniquely determined by our choice of $S(x)$. This triple $[S(x), A(x), I(x)]$ will be referred to as the design. $S(x)$ will be required to satisfy some *a priori* given inequalities $0 < S_1 \leqslant \leqslant S(x) \leqslant S_2$, where S_1, S_2 are positive numbers.

Note. Of course pointwise inequalities do not make any sense in the $L_2[0, l]$ setting and we can revise them as essential inequalities, i.e. essentially $S_1 \leqslant S \leqslant S_2$. However, piecewise continuity of $S(x)$ implies that essential inequalities can be replaced by pointwise inequalities and our objections could have been overlooked in the first place. (Essential $\leqslant$ means almost everywhere in the L_2 case).

2. The minimum deflection problem of Prager

We formulate the following problem:

(A) Let $S(x)$ be the design parameter, which will be identified with $[EI(x)]^{-1}$. For a given $x_0 \in [0, l]$, minimize the deflection $|y(x_0)|$ subject to the constant weight constraint

$$\int_0^l A(x)dx \leqslant 1,$$

i.e.

$$\int_0^l \psi([EI(x)]^{-1})dx = \int_0^l \psi(S(x))dx \leqslant 1. \tag{2.1}$$

LEMMA 1. *If the product* $(M(x)\cdot m(x-x_0))$ *changes sign on* $[0, l]$, *then there exists a solution to problem* (A) *but it is not unique.*

Proof. Since $M(x)$ and $m(x-x_0)$ are piecewise continuous functions, the product is also piecewise continuous, and by hypothesis it changes its sign on $[0, l]$. Hence, on some family σ_1 of open subintervals of $[0, l]$, $(M(x)m(x-x_0)) < 0$ and their complement contains a family open intervals σ_2 such that $M(x)\ (x-x_0) > 0$, and $M(x)m(x-x_0) = 0$ or is undefined if $x \in [0, l]/(\sigma_1 \cup \sigma_2)$. We assume that the set of points on which Mm is undefined is of measure zero.

$$\sigma_1 = \bigcup_{i=1}^{m} (x_{\alpha_i}, x_{\alpha_{i+1}}), \quad \sigma_2 = \bigcup_{i=1}^{n} (x_{\beta_i}, x_{\beta_{i+1}}),$$

$(x_{\alpha_i}, x_{\alpha_{i+1}})$, $(x_{\beta_i}, x_{\beta_{i+1}})$ are open, disjoint subintervals of $(0, l)$. We denote by μ_1 the measure of σ_1 and by μ_2, the measure of $\sigma_2 \cdot \phi(x_k, x_l)$ is the characteristic function for an interval (x_k, x_l), i.e.

$$\phi(x)\begin{cases} = 1 & \text{if} \quad x \in (x_k, x_l), \\ = 0 & \text{if} \quad x \in (x_k, x_l). \end{cases}$$

We select the cross-sectional area to be

$$A(x) = \phi^{-1}\Big\{K\sum_{i=1}^{\nu} \tilde{c}_i\phi(x_i, x_{i+1})x^n\Big\},$$

where x_i, $i = 1, 2, \ldots, \nu$, $\nu \leqslant n+m$, are the end points of the intervals of σ_1 and of σ_2; c_i, K are positive constants and $n \neq 0$ is an integer.

Consequently, the moment of inertia $I(x)$ has to be of the form:

$$I(x) = K\sum_{i=1}^{\nu} \tilde{c}_i\phi(x_i,x_{i+1})x^n$$

and

$$\{I(x)\}^{-1} = K^{-1}\sum_{i=1}^{\nu} \tilde{c}_i^{-1}(x_i,x_{i+1})x^{-n}.$$

Denoting

$$E^{-1}\int_{x_i}^{x_{i+1}} M(x)\,m(x-x_0)x^{-n}dx = \gamma_i(x_0),$$

we see that

$$y(x_0) = K^{-1}\sum_{i=1}^{\nu} \gamma_i(x_0)\tilde{c}_i^{-1};$$

$\gamma_i(x_0)$ are negative if $(x_i, x_{i+1}) \subset \sigma_2$ and positive if $(x_i, x_{i+1}) \subset \sigma_1$. Hence we can choose the constants $\tilde{c}_i$ so that $y(x_0) = 0$, independently of the value of K^{-1}. Now it is clear that K can be selected so that $\int_0^l A(x)dx = 1$. Since n was an arbitrary integer, the proof is complete.

To prove Theorem 1, we also need the following lemma:

LEMMA 2. *If $M(x)\,m(x-x_0) > 0$ for all $x \in (0, l)$, then there exists an optimal $L_2[0, l]$ design (which need not be admissible!).*

Outline of the Proof. The proof follows a fairly routine functional analytic argument. Let us introduce an energy product for

$$u, v \in H^2(0, l), \langle u, v\rangle_{x_0} = \int_0^l M(x)\,m(x-x_0)\,u(x)\,v(x)dx$$

($(M(x)\,m(x-x_0))$ was assumed to be positive!), H will denote the completion of the energy space with the product $\langle\ ,\ \rangle$. Let us identify without any loss of generality the design parameter S with $[EI(x)]^{-1/2}$. Then we have $y(x_0) = \langle S, S\rangle = |||S|||^2$, where $|||\cdot|||$ is the energy norm. (See MIKHLIN [12], Chapter 6 for definitions and a discussion of these concepts). S must be uniformly bounded away from zero, and consequently $y(x_0)$ is also uniformly bounded away from zero. Hence there exists a sequence of admissible elements S_i considered as elements of space H such that $\lim_{i\to\infty} |||S_i|||^2 = \tilde{y}(x_0)$, where the greatest lower bound of $y(x_0)$, will be denoted by $\tilde{y}(x_0)$. By completeness of the Hilbert space H, we can conclude that there exists an element $\tilde{S} \in H$ such that $|||\tilde{S}|||^2 = \tilde{y}(x_0)$. (Observe that positiveness of $S_i \in L_2$ means that weak convergence (in L_2) implies convergence in the norm (see DUNFORD and SCHWARTZ [2], Part I, theorem on page 388, Sec. IV). After a standard argument we conclude the existence of $\tilde{S} \in L_2$ $[0, l]$. Of course S may fail to be an admissible design parameter. (However, it will satisfy the basic inequality constraints of an admissible design). Additional assumptions are needed to prove an existence of an addmissible optimal design.

If we redefine admissibility using more realistic engineering criteria, then the existence of an optimal design is not hard to prove.

We say that the design S(or rather the triple (S, A, I)) is addmissible if, in addition to the previously stated requirements, it is also true that for all $x \in [0, l] M(x)\, m(x-x_0)$ $[I(S)]^{-1} \leqslant \sigma_0/c$ for some *a priori* given constants σ_0, c, while the cross-section is restricted to geometric design which depends on parameter S so that the depth of the beam h is determined uniquely by S and that $h \leqslant c$.

THEOREM 1. *We identify S with $[EI]^{-1}$. There exists an optimal admissible design, provided there exists at least one admissible design.*

We comment that it is possible for only a finite number of admissible designs to exist. In fact it is easy to construct a case when only one admissible design exists. However, in the case of only finitely many admissible designs, the conclusion of the theorem is trivial. Hence we only need to consider the case when infinitely many admissible designs exist.

Proof. We duplicate our arguments concerning the existence of an optimal $L_2[0, l]$ design. In this case we need to show that an L_2 design also has to be piecewise continuous. Assume to the contrary that a limit $\tilde{S}$ of admissible designs S_i exists, $\tilde{y}(\tilde{S})$ is minimal, i.e. $|\tilde{y}(\tilde{S})|_{x_0} \geqslant |y(S)|_{x_0}$ for any admissible design (S), and $S \in L_2[0, l]$, but fails to be piecewise continuous. Hence, given $\varepsilon > 0$, there exists a point $\hat{x} \in [0, l]$ such that given any natural number $n > 0$, there exists $N > n$ such that S_i has more than N discontinuities in the ε-neighborhood of $\hat{x}$ for some sufficiently large values of the subscript i. Since M and m are fixed (and independent of the parameter S), the product Mm has some (fixed) number k of discontinuities in the ε-neighborhood of $\hat{x}$, and the product MmS_i has at least $N-k$ discontinuities. Hence $y_i'(S_i(x), x)$ will be discontinuous if $N > k$, which contradicts the admissibility of S_i completing the proof.

We comment that there is no reason for assuming the uniqueness of such optimal design. However, assuming that two such (admissible) optimal designs exist — say $S_1(x)$, $S_2(x)$, $S_1 \neq S_2$ on some subset of $[0, l]$ on which $M(x)\, m(x-x_0) \neq 0$, we compute:

$$\bar{y}(x = x_0) = \int_0^l \bar{S} M(x)\, m(x-x_0)\, dx,$$

where $\bar{S} = \min(S_1, S_2)$, and find that $|\bar{y}(x = x_0)| < y(S_1)_{|x=x_0|}, = y(S_2)_{|x=x_0|}$, which is a contradiction since $y(S_1)$ is optimal. Hence only one such optimal design exists.

3. An optimization problem in the beam theory

Problem A. Given the total weight W of a statically determinate beam, and given an admissible load $g(x)$, find admissible optimal design determined by a parameter S such that the maximum deflection of the beam is minimized; $(S_1 \leqslant S \leqslant S_2)$.

Problem B. Regarding in the problem A the weight W as a parameter, find its smallest value $\overline{W}$ such that the maximum stress does not exceed a given value σ_0.

Comment. The problem of minimizing the maximum deflection subject to constant weight is not (as we have already remarked) the dual of the weight minimization problem subject to either a given deflection at a given point (PRAGER [6]) or to minimization of the weight subject to a given maximum deflection of the beam.

We postulate the existence of influence function $G(x-x_0)$ such that

$$y(x_0) = G(x) \underset{-}{\overset{*}{\ast}} g(x), \tag{3.1}$$

where $g(x)$ is the applied load, and $*$ is the convolution operation. $G(x)$ depends on the properties of the beam and boundary conditions but is independent of the load $g(x)$.

We shall at first discuss the beam optimization problem in detail because of its simplicity, and only later generalize the result to plates, structures, etc., since almost identical approach will give analogous theoretical results.

We shall assume that the beam is statically determinate and satisfies at the end points one of the following conditions:

a) It is freely supported

(displacement y and bending moment are both equal to zero at the end point),

b) It is built in

($y = dy/dx = 0$ at the end point),

c) It is free.

(The bending moment and the shear load both vanish). We now consider the dependence of the influence function $G(x)$ on some design parameter $S(x)$, i.e.

$$G = G\big(S(x), x\big). \tag{3.2}$$

Duhamel's principle (3.1) becomes

$$\begin{aligned} y\big(S(x), x_0\big) &= G\big(S(x), x\big) * g \\ &= G\big(S(x), x\big) * \frac{d^2}{dx^2} M(x), \end{aligned} \tag{3.3}$$

where $M(x)$ is independent of the design parameter S because of our assumption of statical determinacy.

Suppose that $G\big(S(x), x\big)$ is Fréchet differentiable function of the parameter $S(x)$. Then the condition of optimality of the design with respect to the deflection $y\big(S(x), x_0\big)$ is

$$\frac{d^2}{dx^2} G_S\big(S(x(x) * M(x) = 0, \tag{3.4}$$

where G_S is the Fréchet derivative of G with respect to $S(x)$. (See [7], or author's exposition in [3] for rules of Fréchet differentiation and for definitions. Also see [11]).

If we take into account the side condition of constant weight (or similar constraint) $\Phi(S) = C$ and local maximality of $y(x_0)$ as a function of the (deflection point) variable x_0, we obtain, by using basic properties of convolution products:

$$\gamma_0 \frac{d^2}{dx^2} G_S\big(S(x, x_0)\big) * M(x) + \gamma_1 \Phi_S(S) + \gamma_2 \frac{dy(x_0)}{dx_0} = 0. \tag{3.5}$$

Computing the value of the last term we can rewrite this optimality condition as:

$$(3.6)\qquad \left\{\gamma_0 \frac{d^2}{dx^2} G(S(x), x_0)*M(x)+\gamma_1 \Phi(S)+\gamma_2 \left(\frac{dG(S(x), x_0}{dx_0} * \frac{d^2}{dx^2} M(x)\right)\right\}_S = 0,$$

where $\gamma_0, \gamma_1, \gamma_2$ are Lagrangian multipliers.

We can put the formula (3.11) into a more familiar form by observing that:

$$(3.7)\qquad \frac{d^2}{dx^2} G(S(x), x_0) = \frac{m(x-x_0)}{EI(x)},$$

where $m(x-x_0)$ is the moment produced at x_0 by a unit load situated at x.

Since, in a statically determinate case, $m(x, x_0)$ is independent of the design, we have

$$(3.8)\qquad \frac{d^2}{dx^2} G(x, S(x))*M(x) = m(x)*(EM(x))I_S^{-1}(S(x), x).$$

Using rules of Fréchet differentiation we have

$$(I^{-1})_S = -I^{-2}(S(x), x)\cdot I_S(S(x), x),$$

and we have the necessary condition for optimality of design:

$$(3.9)\qquad \lambda_0 E(m(x)*(M(x))\cdot\left(-I^{-2}(s(x), x)I_S(S(x), x))\right) + \lambda_1 \Phi_S + \lambda_2 q(x)*\left(\frac{dG(S(x), x)}{dx}\right)_S = 0.$$

This implies that Φ_S must lie in the two-dimensional subspace spanned by the other two terms, which can be regarded as orthogonality conditions.

Comment. The statical determinacy was used in Fréchet differentiation to ignore the terms $M(x)$ and $m(x)$. However, the basic condition is theoretically unaltered if statical determinacy is not assumed. The general condition of optimality of design subject to $\Phi(S(x)) =$ constant and $\frac{dy(x_0)}{dx_0} = 0$ becomes in that case:

$$(3.10)\qquad \lambda_0 E\left(m(s(x), x)\right)*\left(M(S(x), x)I^{-1}(S(x), x)\right)_S + \lambda_1 \Phi_S + \lambda_2 q(x)*\left(\frac{dG(S(x), x)}{dx}\right)_S = 0.$$

Usually we restrict the class of cross-sectional areas (and possible shapes of the beam) to some specific geometries. We shall say that $A(x)$ is an admissible cross-sectional area if it satisfies all our restrictions.

The cross-sectional area is assumed to depend uniquely on the parameters. The moment of inertia of the beam is assumed to be uniquely determined if the cross-sectional area is given:

$$(3.11)\qquad A = A(S(x)),\ I(x)) = \phi^{-1}(A(x)) = I(S(x)),$$

that is we assume that a choice of the parameter or vector S uniquely determines the area and the moment of inertia (about the neutral axis) of the cross-section.

An example of application

Design a statically determinate rectangular beam of constant width b and length l, subjected to a loading $g(x)$, to minimize the maximum deflection: $\max_{x_\varepsilon[0,l]} |y(x)|$. The total weight of the beam Q is given *a priori*.

Instead of the total weight condition, we can substitute the alternate condition

$$\int_0^l A(x)dx = \text{constant} \quad \text{or} \quad \int_0^l h(x)dx = \text{constant}.$$

Here the width b is constant, the height $h(x)$ is the design parameter (that is $h(x)$ will be our $S(x)$ of the previous discussion).

We shall follow the above theoretical arguments to derive a necessary condition of optimality for the parameter $h(x)$ for this particular design problem.

As before, we make use of the deflection formula

$$y(x_0) = E^{-1}\int_0^l M(x)m(x-x_0)I^{-1}\big(h(x)\big)dx.$$

Since $I(h) = \dfrac{bh^3(x)}{12}$, we have

$$y(x_0) = 12E^{-1}b\int_0^l \frac{M(x)m(x-x_0)}{h^3(x)}dx.$$

A necessary condition for x_0 to be an extremal point of $y(x_0)$ is

$$\frac{dy(x_0)}{dx_0} = \int_0^l \frac{M(x)q(x-x_0)}{h^3(x)}dx = 0, \quad \text{if} \quad \begin{cases} x_0 \neq 0, \\ x_0 \neq l, \end{cases}$$

where $q(x-x_0) = \dfrac{d}{dx_0}m(x-x_0)$, that is q is the moment at x produced by a unit couple at x_0. The constraint of constant weight is

$$\int_0^l h(x)dx = \text{constant}.$$

According to the formula (3.11) we have the necessary condition for optimality of $h(x)$

$$\text{(A)} \qquad \big(h^{-4}(x)M(x)\big)*\big(m(x)+\lambda_1 q(x)\big) = \lambda_2,$$

where λ_1, λ_2 are (constant) Lagrangian multipliers.

The formula above states that a necessary condition for the optimality of the design is as follows. The convolution product on the left-hand side of equation (A) is independent of x. As before, $M(x)$ was the moment produced by the actual loads, while $m(x, x_0)$ was the moment at x produced by a unit load positioned at x_0.

4. Optimization of thin plate design

We proceed along similar lines, except that no condition similar to static determinancy can be established for the plate under any sensible support condition on the boundary.

The deflection function $w(x, y)$ obeys the differential equation:

$$\nabla^2\big(D(x, y)\nabla^2 w\big)+(1-\nu)\Diamond^4(D, w) = g(x, y) \quad \text{in} \quad \Omega \subset R^2 \tag{4.1}$$

(see [1] for a more general form in case of an anisotropic plate).

On the boundary $\partial\Omega$ the plate obeys one of the two conditions

$$w \equiv 0 \text{ and } \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega, \tag{B1}$$

$$w \equiv 0 \text{ and } \frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial \tau^2} = 0 \text{ on } \partial\Omega, \tag{B2}$$

where n is the direction of outward normal to the boundary and τ is the tangential direction.

If $\partial/\partial s$ denotes the symbol of differentiation with respect to the coordinate locally following the boundary, we have the relation

$$\frac{\partial^2}{\partial\tau^2} = \frac{\partial^2}{\partial s^2} + k\frac{\partial}{\partial n},$$

where k is the curvature of the boundary.

Again, for the static plate equation we assert the existence of the influence function $G(x, y, \xi, \eta)$ such that

$$w(x, y) = \int_\Omega G(x-\xi, y-\eta)g(\xi, \eta)d\xi d\eta = G*g. \tag{4.2}$$

We observe that indeed $G(x-\xi, y-\eta)$ is the deflection at (x, y) produced by a unit load (i.e. Dirac delta function) positioned at (ζ, η), since putting $g = \delta(\xi, \eta)$, we have

$$w(x, y) = G(x, y) * \delta(x, y) = G(x, y)$$

(which is hardly a novel development). We are now ready to state the deflection optimization problem for the thin plate theory.

Given a plate of weight Q, occupying a precompact region $\Omega \subset R^2$ whose boundary $\partial\Omega$ is a Liapunov curve, subjected to a given loading $g(x, y)$, determine the thickness $h(x, y)$ so that the maximum deflection of the plate is minimized. The constant weight constraint is

$$\int_\Omega h(x, y)dxdy = \text{constant}. \tag{4.3}$$

Again we minimize $w(x, y) = G*g$ subject to constraint (4.3) and to condition $\text{grad}\,\big(w(x, y)\big) = 0$.

Assuming that G, h and w depend on a design vector $\mathbf{S}$, we have as the necessary condition of optimality of design for a minimum of the maximal deflection at an interior point

$$G_{\mathrm{S}}*g + \mathbf{1}*h_{\mathrm{S}} + G_{\mathrm{S}}*\text{grad}\,g = 0, \tag{4.4}$$

where $\mathbf{1}$ is the function identically equal to 1.

We comment that grad$(w) = 0$ is *not* a necessary condition for maximality of the plate if it is free at some position of the boundary, since then the maximum of $|w(x, y)|$ may occur on the boundary at a point at which grad $(w) \neq 0$.

Maximum stress level constraint

We observe that the optimization of maximum deflection subject to a constant weight constraint produces necessary criteria (3.6) or (4.4) for beams and plates, respectively, independently of the particular value of Q representing the total weight of the beam or plate. The additional constraint of not exceeding certain maximum stress level, is $\tau_{\max} = \max_{x \in [0,l]} \frac{M(x)c(x)}{I(x)} \leqslant \sigma_0$ in the beam theory, where σ_0 is given and $c(x)$ denotes maximum distance (at a given cross-section) from the neutral axis. $I(x)/c(x)$ is a known function of the design parameter $S(x)$, and this constraint can be handled in the usual manner by introducing an additional Lagrangian multiplier. However, another look at the problem offers the following approach. We observe that $\tau_{\max}$ is uniquely determined by our choice of the parameter $S(x)$ provided $S(x)$ determined uniquely the choice of cross-section. The formulas (A) and (4.4) determine the design only within an arbitrary multiplicative constant, and the constraint $\Phi(S) = \text{constant} = Q$ must be used to determine the design. The constant Q can now be decided upon to satisfy the additional constraint $\tau_{\max} \leqslant \tau_0$.

In this paper we have deliberately avoided computational problems and restricted ourselves to the problem of establishing a theoretical background on which such future computations can be based. An alternate approach utilizing variational techniques of the type given in [3] should be attempted, and is quite likely to give a different set of necessary criteria for optimization.

We finally observe that the method generalizes easily to more complex cases (inertia terms included in dynamic cases, shells of revolution, etc.) but it fails in the case of multi-member structures where some basis changes are needed in the approach.

References

1. S. A. AMBARTSUMIAN, *Theory of anisotropic plates* [in Russian], Nauka, Moscow 1967.
2. N. DUNFORD and J. SCHWARTZ, *Linear operators*, P. II, Interscience, New York 1963.
3. V. KOMKOV, *On a variational formulation of problems in the classical continuum mechanics of solids*, Int. J. Engg. Sci., **6**, 695–720, 1968.
4. V. KOMKOV, *The class of admissible loads of the linear plate and beam theory*, Inst. Lomb. Acad. Sci. e Lett. Series A, **105**, 329–335, 1971.
5. D. G. LUENBERGER, *Optimization by vector space methods*, J. Wiley, New York 1969.
6. W. PRAGER, *Optimal design of statically determinate beams for a given deflection*, Int. J. Mech. Sci., **13**, 893–895, 1971.
7. R. T. SHIELDS and W. PRAGER, *Optimal structural desing for a given deflection*, J. Appl. Math. Phys, **21**, 513–516, 1970.
8. W. PRAGER, *Optimal thermoelastic design for given deflection*, Int. J. Mech. Sci., **12**, 705–709, 1970.
9. J. M. CHERN and W. PRAGER, *Optimal design of a rotating disc for a given radical displacement of edge*, Optimization Theory and Appl., **6**, 2, 161–170, 1970.

10. J. M. CHERN and W. PRAGER, *Optimal design of beams for prescribed compliance under alternating loads*, J. of Optimization Theory and Appl., **5**, 6, 424–430, 1970.
11. M. M. VAINBERG, *Variational methods for the investigation of non-linear operators* [translated from Russian], Holden Day, San Francisco 1963.
12. S. G. MIKHLIN, *Mathematical physics* (an advanced course), translated from Russian, North Holland Publ. Co., Amsterdam 1970.

TEXAS TECH. UNIVERSITY
LUBBOCK, TEXAS
79409, U.S.A.

Received May 17, 1974.