651.

ON A SPECIAL SURFACE OF MINIMUM AREA.

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A VERY remarkable form of the surface of minimum area was obtained by Prof. Schwarz in his memoir "Bestimmung einer speciellen Minimal-fläche," Berlin, 1871, [*Ges. Werke*, t. I., pp. 6—125], crowned by the Academy of Sciences at Berlin. The equation of the surface is

$$1 + \mu\nu + \nu\lambda + \lambda\mu = 0,$$

where λ, μ, ν are functions of x, y, z respectively, viz.

$$x = -\int_\lambda^\infty \frac{d\theta}{\sqrt{(\frac{3}{4}\theta^4 + \frac{5}{2}\theta^2 + \frac{3}{4})}},$$

and y, z are the same functions of μ, ν respectively. A direct verification of the theorem that this is a surface of minimum area, satisfying, that is, the differential equation

$$r(1 + q^2) - 2pqs + t(1 + p^2) = 0,$$

is given in the memoir; but the investigation may be conducted in quite a different manner, so as to be at once symmetrical and somewhat more general, viz. we may enquire whether there exists a surface of minimum area

$$1 + \mu\nu + \nu\lambda + \lambda\mu = 0,$$

where the determining equations are

$$\lambda'^2 = a\lambda^4 + b\lambda^2 + c,$$

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$\left(\lambda' = \dfrac{d\lambda}{dx}, \text{ \&c.}\right)$. I find that the coefficients a, b, c must satisfy four homogeneous quadric equations, which, in fact, admit of simultaneous solution, and that in three distinct ways; viz. assuming $a = 1$, the solutions are

$$\begin{aligned} a &= 1, \quad b = \tfrac{10}{3}, \quad c = 1, \\ a &= 1, \quad b = -2, \quad c = 1, \\ a &= 1, \quad b = -\tfrac{3}{2}, \quad c = -\tfrac{1}{3}; \end{aligned}$$

that is,

$$\lambda'^2 = \lambda^4 + \tfrac{10}{3}\lambda^2 + 1 \; \{= \tfrac{4}{3}(\tfrac{3}{4}\lambda^4 + \tfrac{5}{2}\lambda^2 + \tfrac{3}{4})\},$$

which gives Schwarz's surface:

$$\lambda'^2 = \lambda^4 - 2\lambda^2 + 1 \text{ or } \lambda' = \pm(\lambda^2 - 1),$$

which, it is easy to see, gives only $x + y + z = \text{const.}$; and

$$\lambda'^2 = \lambda^4 - \tfrac{2}{3}\lambda^2 - \tfrac{1}{3}, \quad = (\lambda^2 - 1)(\lambda^2 + \tfrac{1}{3}),$$

which is a surface similar in its nature to Schwarz's surface.

The investigation is as follows: the condition to be satisfied by a surface of minimum area $U = 0$ is

$$(\mathrm{a} + \mathrm{b} + \mathrm{c})(X^2 + Y^2 + Z^2) - (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h} \between X, Y, Z)^2 = 0,$$

where (X, Y, Z) are the first derived coefficients and (a, b, c, f, g, h) the second derived coefficients of U in regard to the coordinates. Considering U as a function of λ, μ, ν, which are functions of x, y, z respectively, and writing (L, M, N) and (a, b, c, f, g, h) for the first and second derived functions of U in regard to λ, μ, ν, also λ', λ'' for the first and second derived functions of λ in regard to x, and so for μ', μ'' and ν', ν'': we have

$$(X, Y, Z) = (L\lambda', M\mu', N\nu'),$$

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h}) = (a\lambda'^2 + L\lambda'', b\mu'^2 + M\mu'', c\nu'^2 + N\nu'', f\mu'\nu', g\nu'\lambda', h\lambda'\mu'),$$

and for the particular surface $U = 1 + \mu\nu + \nu\lambda + \lambda\mu = 0$, the values are

$$(L, M, N, a, b, c, f, g, h) = (\mu + \nu, \nu + \lambda, \lambda + \mu, 0, 0, 0, 1, 1, 1).$$

Hence the condition is found to be

$$\begin{aligned} &2\mu'^2\nu'^2(\lambda + \mu)(\lambda + \nu) \\ &+ 2\nu'^2\lambda'^2(\mu + \nu)(\mu + \lambda) \\ &+ 2\lambda'^2\mu'^2(\nu + \lambda)(\nu + \mu) \\ &- \lambda''(\mu + \nu)\{(\lambda + \nu)^2\mu'^2 + (\lambda + \mu)^2\nu'^2\} \\ &- \mu''(\nu + \lambda)\{(\mu + \lambda)^2\nu'^2 + (\mu + \nu)^2\lambda'^2\} \\ &- \nu''(\lambda + \mu)\{(\nu + \mu)^2\lambda'^2 + (\nu + \lambda)^2\mu'^2\} = 0, \end{aligned}$$

or say this is

$$2\Sigma\mu'^2\nu'^2(\lambda+\mu)(\lambda+\nu)$$
$$-\Sigma\lambda''(\mu+\nu)\{(\lambda+\nu)^2\mu'^2+(\lambda+\mu)^2\nu'^2\}=0.$$

We have to write in this equation $\lambda'^2=a\lambda^4+b\lambda^2+c$, and therefore $\lambda''=2a\lambda^3+b\lambda$, &c.; the left-hand side, call it Ω, is a symmetrical function of λ, μ, ν, and is consequently expressible as a rational function of

$$p, =\lambda+\mu+\nu,$$
$$q, =\mu\nu+\nu\lambda+\lambda\mu,$$
$$r, =\lambda\mu\nu.$$

We ought to have $\Omega=0$, not identically, but in virtue of the equation $1+q=0$, that is, Ω should divide by $1+q$; or, what is the same thing, Ω should vanish on writing therein $q=-1$.

To effect the reduction as easily as possible, observe that we have $(\lambda+\mu)(\lambda+\nu)=\lambda^2+q$; and therefore

$$\Sigma\mu'^2\nu'^2(\lambda+\mu)(\lambda+\nu)=\Sigma\lambda^2\mu'^2\nu'^2+q\Sigma\mu'^2\nu'^2.$$

Similarly, in the second term,

$$(\mu+\nu)(\lambda+\nu)^2=(\nu+\lambda)(\nu^2+q) \text{ and } (\mu+\nu)(\lambda+\mu)^2=(\mu+\lambda)(\mu^2+q).$$

The complete value of Ω thus is

$$\Omega=2(Aq+B)-[(C+D)q+E+F],$$

where

$$A=\Sigma\lambda^2\mu'^2\nu'^2, \qquad B=\Sigma\mu'^2\nu'^2,$$
$$C=\Sigma\lambda\lambda''(\nu^2\mu'^2+\mu^2\nu'^2), \quad D=\Sigma\lambda''(\nu^2\mu'^2+\mu^2\nu'^2),$$
$$E=\Sigma\lambda\lambda''(\mu'^2+\nu'^2), \qquad F=\Sigma\lambda''(\nu\mu'^2+\mu\nu'^2).$$

We find without difficulty

$$\begin{aligned}A=\ & a^2\,(\quad q^4-4q^2pr+4qr^2+2p^2r^2)\\ &+ab\,(-2q^3+q^2p^2+4qpr-3r^2-2p^3r)\\ &+ac\,(\quad 4q^2-8qp^2+8pr+2p^4)\\ &+b^2\,(\quad q^2-2pr)\\ &+bc\,(-4q+2p^2)\\ &+c^2\,(\quad 3),\end{aligned}$$

$$\begin{aligned}B=\ & a^2\,(\quad q^2r^2+2pr^3)\\ &+ab\,(-4qr^2+2p^2r^2)\\ &+ac\,(-2q^3+q^2p^2+4qpr-3r^2-2p^3r)\\ &+b^2\,(\quad 3r^2)\\ &+bc\,(\quad 2q^2-4pr)\\ &+c^2\,(-2q+p^2),\end{aligned}$$

$$
\begin{aligned}
C = {} & a^2\,(\ 4q^4 - 16q^2pr + 16qr^2 + 8p^2r^2) \\
& + ab\,(-6q^3 + 3q^2p^2 + 12qpr - 9r^2 - 6p^3r) \\
& + ac\,(\ 8q^2 - 16qp^2 + 16pr + 4p^4) \\
& + b^2\,(\ 2q^2 - 4pr) \\
& + bc\,(-4q + 2p^2),
\end{aligned}
$$

$$
\begin{aligned}
D = {} & a^2\,(\ 2q^2pr - 2qr^2 - 4p^2r^2) \\
& + ab\,(-4qpr + 2p^3r) \\
& + ac\,(-4q^2 + 2qp^2 - 2pr) \\
& + b^2\,(\ 2pr) \\
& + bc\,(\ 2q),
\end{aligned}
$$

$$
\begin{aligned}
E = {} & + a^2\,(\ 4q^2r^2 - 8pr^3) \\
& + ab\,(-12qr^2 + 6p^2r^2) \\
& + ac\,(-4q^3 + 2q^2p^2 + 8qpr - 6r^2 - 4p^3r) \\
& + b^2\,(\ 6r^2) \\
& + bc\,(\ 2q^2 - 4pr),
\end{aligned}
$$

$$
\begin{aligned}
F = {} & a^2\,(\ 4pr^3) \\
& + ab\,(\ q^2pr + 3qr^2 - 2p^2r^2) \\
& + ac\,(\ 4q^3 - 12qpr + 12r^2) \\
& + b^2\,(\ qpr - 3r^2) \\
& + bc\,(-2q^2 + qp^2 - pr),
\end{aligned}
$$

where in each line the terms are arranged according to their order in p, r.

Substituting, we find

$$
\begin{aligned}
\Omega = {} & a^2\,(-2q^5 + 6q^3pr - 8q^2r^2) \\
& + ab\,(\ 2q^4 - q^3p^2 - q^2pr + 4qr^2) \\
& + ac\,(\ -2q^2p^2 + 14qpr - 12r^2) \\
& + b^2\,(\ -3qpr + 3r^2) \\
& + bc\,(-2q^2 + qp^2 - 3pr) \\
& + c^2\,(\ 2q + 2p^2);
\end{aligned}
$$

viz. writing $q = -1$, this is

$$
\begin{aligned}
\Omega = {} & a^2\,(\ 2 - 6pr - 8r^2) \\
& + ab\,(\ 2 + p^2 - pr - 4r^2) \\
& + ac\,(\ -2p^2 - 14pr - 12r^2) \\
& + b^2\,(\ 3pr + 3r^2) \\
& + bc\,(-2 - p^2 - 3pr) \\
& + c^2\,(-2 - 2p^2);
\end{aligned}
$$

or, what is the same thing, it is

$$\begin{aligned}
= &\ \ \ \ \ \ \ (\ 2a^2 + 2ab \qquad\qquad\qquad\quad - 2bc - 2c^2)\\
&+ p^2\ (\qquad\quad ab - \ 2ac \qquad\ \ - \ bc + 2c^2)\\
&+ pr\ (- 6a^2 - \ ab - 14ac + 3b^2 - 3bc \qquad\ )\\
&+ r^2\ \ (- 8a^2 - 4ab - 12ac + 3b^2 \qquad\qquad\ );
\end{aligned}$$

so that, writing for convenience $a = 1$, the equations to be satisfied are

$$\begin{aligned}
2 - 2c^2 + 2\,(1 - \ c)\ \ b &= 0,\\
-\ 2c + 2c^2 + \ (1 - \ c)\ \ b &= 0,\\
-6 - 14c + 3b^2 - \ (1 + 3c)\ \ b &= 0,\\
-8 - 12c + 3b^2 - \qquad\quad 4b &= 0.
\end{aligned}$$

The first and second are $(1-c)(2+2c+2b) = 0$ and $(1-c)(-2c+b) = 0$; viz. they give $c = 1$, or else $b = -\frac{2}{3}$, $c = \frac{1}{3}$. In the former case, the third and fourth equations each become $3b^2 - 4b - 20 = 0$, that is $(3b-10)(b-2) = 0$; in the latter case, they are satisfied identically; hence we have for a, b, c the three systems of values mentioned at the beginning.

This completes the investigation; but it is interesting to find the values assumed by the other factor of Ω on substituting therein for a, b, c the foregoing several systems of values. We have in general

$$\begin{aligned}
\Omega = &\qquad - 2a^2q^3 + 2abq^4 - \ 2bcq^2 + 2c^2q\\
&+ p^2\ (- \ abq^3 - 2acq^2 + \quad bcq \ + 2c^2 \qquad\quad)\\
&+ pr\ (\ \ 6a^2q^3 - \ abq^2 + 14acq \ - 3b^2q - 3bc)\\
&+ r^2\ (- 8a^2q^2 + 4abq - 12ac \ \ + 3b^2 \qquad\quad)\\
= &- 2a^2\,(q^5+1) \qquad\quad + 2ab\,(q^4-1) - 2bc\,(q^2-1) + 2c^2\,(q+1)\\
&+ p^2\ \ \{-ab\ \ (q^3+1) - 2ac\,(q^2-1) + bc\,(q+1)\}\\
&+ pr\ \ \{\ \ 6a^2\,(q^3+1) - \ ab\,(q^2-1) + (14ac - 3b^2)\,(q+1)\}\\
&+ r^2\ \ \{\qquad\qquad\qquad - 8a^2\ \ (q^2-1) + 4ab\,(q+1)\}\\
= &\ (q+1)\left\{\begin{array}{l}
-2a^2\,(q^4 - q^3 + q^2 - q + 1) + 2ab\,(q^3 - q^2 + q - 1) - 2bc\,(q-1) + 2c^2\\
+p^2\ \ \{-\ \ ab\,(q^2 - q + 1) - 2ac\,(q-1) + bc\}\\
+pr\ \ \{\ \ 6a^2\,(q^2 + q + 1) - \ ab\,(q-1) + (14ac - 3b^2)\}\\
+r^2\ \ \{\qquad\qquad\qquad\qquad - 8a^2\,(q-1) + \ \ 4ab\}
\end{array}\right\}.
\end{aligned}$$

Hence writing, first, $a = c = 1$, $b = \frac{10}{3}$, we obtain, after some reductions,

$$\Omega = (q+1)\,\{-2q\,(q-1)\,(q^2 - \tfrac{10}{3}q + 1) + p^2\,(q-1)(-\tfrac{10}{3}q - 2) + pr\,(6q^2 - \tfrac{28}{3}q - 10) + r^2 - 8q + \tfrac{64}{3}\};$$

secondly, writing $a = c = 1$, $b = -2$, we obtain

$$\Omega = (q+1)\,\{-2\,(q+1)^2\,(q^2+1) + p^2\,.\,2\,(q-1)^2 + 2pr\,(3q^2 - 2q + 6) - 8r^2q\};$$

and, thirdly, writing $a = 1$, $b = -\frac{1}{3}$, $c = -\frac{2}{3}$, we obtain

$$\Omega = (q+1)\,\{(-2q^4 + \tfrac{4}{3}q^3 - \tfrac{4}{3}q^2 + \tfrac{8}{9}q) + p^2\,(-\tfrac{1}{3}q^2 + \tfrac{5}{3}q - \tfrac{13}{9}) + pr\,(6q^2 - \tfrac{17}{3}q - \tfrac{10}{3}) + r^2\,(-8q + \tfrac{20}{3})\}.$$