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NOTE ON THE THEORY OF ELLIPTIC INTEGRALS.

[From the *Mathematische Annalen*, t. XII. (1877), pp. 143—146.]

THE equation

$$\frac{Mdy}{\sqrt{1-y^2 \,.\, 1-k^2y^2}} = \frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}}$$

is integrable algebraically when M is rational: and so long as the modulus is arbitrary, then conversely, in order that the equation may be integrable algebraically, M must be rational. For particular values however of the modulus, the equation is integrable algebraically for values of the form M, or (what is the same thing) $\frac{1}{M}$, = a rational quantity $\pm$ square root of a negative rational quantity, say $= \frac{1}{p}(l + m\sqrt{-n})$, where l, m, n, p are integral and n is positive; we may for shortness call this a half-rational numerical value. The theory is considered by Abel in two Memoirs in the *Astr. Nach.* Nos. 138 & 147 (1828), being the Memoirs* XIII & XIV in the *Œuvres Complètes* (Christiania 1839). I here reproduce the investigation in a somewhat altered (and, as it appears to me, improved) form.

Putting the two differentials each $= du$, we have $x = \text{sn}(u + \alpha)$, $y = \text{sn}\left(\frac{u}{M} + \beta\right)$; and the question is whether there exists an algebraical relation between these functions, or, what is the same thing, an algebraical relation between the functions $x = \text{sn}\, u$ and $y = \text{sn}\, \frac{u}{M}$.

Suppose that A and B are independent periods of $\text{sn}\, u$; so that $\text{sn}(u + A) = \text{sn}\, u$, $\text{sn}(u + B) = \text{sn}\, u$, and that every other period is $= mA + nB$, where m and n are integers. Then if u has successively the values u, $u + A$, $u + 2A$, etc., the value of x

[* They are the Memoirs XIX. and XX. in the *Œuvres Complètes*, t. I., Christiania, 1881.]

remains always the same, and if x and y are algebraically connected, y can have only a finite number of values: there are consequently integer values p', p'' for which $\operatorname{sn}\frac{1}{M}(u+p'A)=\operatorname{sn}\frac{1}{M}(u+p''A)$: or writing $u-p'A$ for u and putting $p''-p'=p$, there is an integer value p for which $\operatorname{sn}\frac{1}{M}(u+pA)=\operatorname{sn}\frac{1}{M}u$.

Similarly there is an integer value q for which $\operatorname{sn}\frac{1}{M}(u+qB)=\operatorname{sn}\frac{1}{M}u$; and we are at liberty to assume $q=p$; for if the original values are unequal, we have only in the place of each of them to substitute their least common multiple.

We have thus an integer p, for which

$$\operatorname{sn}\frac{1}{M}(u+pA)=\operatorname{sn}\frac{1}{M}u,$$

$$\operatorname{sn}\frac{1}{M}(u+pB)=\operatorname{sn}\frac{1}{M}u.$$

There are consequently integers m, n, r, s such that

$$\frac{pA}{M}=mA+nB,$$

$$\frac{pB}{M}=rA+sB,$$

equations which will constitute a single relation $\frac{p}{M}=m$, if $m=s$, $r=n=0$; but in every other case will be two independent relations. In the case first referred to, the modulus is arbitrary and M is rational.

But excluding this case, the equations give

$$B(mA+nB)=A(rA+sB),$$

or, what is the same thing,

$$rA^2-(m-s)AB-nB^2=0,$$

an equation which implies that the modulus has some one value out of a set of given values. The ratio $A:B$ of the two periods is of necessity imaginary, and hence the integers m, n, r, s must be such that $(m-s)^2+nr$ is negative.

The foregoing equations may be written

$$\left(m-\frac{p}{M}\right)A+\qquad nB=0,$$

$$rA+\left(s-\frac{p}{M}\right)B=0,$$

whence eliminating A and B we have

$$\left(m-\frac{p}{M}\right)\left(s-\frac{p}{M}\right)-nr=0,$$

that is,

$$\left(\frac{p}{M}\right)^2 - (m+s)\frac{p}{M} + ms - nr = 0,$$

and consequently

$$\frac{p}{M} = \tfrac{1}{2}(m+s) \pm \tfrac{1}{2}\sqrt{(m-s)^2 + nr},$$

where, by what precedes, the integer under the radical sign is negative: and we have thus the above mentioned theorem.

As a very general example, consider the two rational transformations

$$z = (x, u, v); \text{ mod. eq. } Q(u, v) = 0; \quad \frac{Ndz}{\sqrt{1-z^2 \,.\, 1-v^8z^2}} = \frac{dx}{\sqrt{1-x^2 \,.\, 1-u^8x^2}},$$

$$y = (z, v, w); \text{ mod. eq. } P(v, w) = 0; \quad \frac{Mdy}{\sqrt{1-y^2 \,.\, 1-w^8y^2}} = \frac{dz}{\sqrt{1-z^2 \,.\, 1-v^8z^2}}:$$

viz. z is taken to be a rational function of x, and of the modular fourth roots u, v; and y to be a rational function of z, and of the modular fourth roots v, w; the transformations being (to fix the ideas) of different orders. We have y a rational function of x, corresponding to the differential relation

$$\frac{MNdy}{\sqrt{1-y^2 \,.\, 1-w^8y^2}} = \frac{dx}{\sqrt{1-x^2 \,.\, 1-u^8x^2}}.$$

Suppose here $w^8 = u^8$, or say $w = \theta u$, θ being an eighth root of unity: we then have $Q(u, v) = 0$, $P(v, \theta u) = 0$, equations which determine u. The differential equation is then

$$\frac{MNdy}{\sqrt{1-y^2 \,.\, 1-u^8y^2}} = \frac{dx}{\sqrt{1-x^2 \,.\, 1-u^8x^2}},$$

an equation the algebraical integral of which is $y =$ a rational function of x as above: hence, by what precedes, we have

$$\frac{1}{MN} = \frac{1}{2p}\{m + s \pm \sqrt{(m-s)^2 + nr}\},$$

a half-rational numerical value, as above.

To explain what the algebraical theorem implied herein is, observe that the equations $Q(u, v) = 0$, $P(v, \theta u) = 0$, give for u an algebraical equation. Admitting θ as an adjoint radical, suppose that an irreducible factor is $\phi(u)$, and take u to be determined by the equation $\phi u = 0$; then v, and consequently also any rational function $\frac{1}{MN}$ of u, v, can be expressed as a rational integral function of u, of a degree which is at most equal to the degree of the function ϕu less unity. The theorem is that, in virtue of the equation $\phi u = 0$, this rational function of u becomes equal to a half-rational numerical value as above. Thus in a simple case, which actually presented itself, the equation $\phi u = 0$ was $u^2 - 4u + 1 = 0$; and $\frac{1}{MN}$ had the value $u - 2$, which in virtue of this equation becomes $= \pm\sqrt{-3}$.

Thus if the second transformation be the identity $z=y$, $w=v$, $M=1$: we have $v=\theta u$; and the equations are

$$y=(x,\ u,\ \theta u),\quad Q(u,\ \theta u)=0,\quad \frac{Ndy}{\sqrt{1-y^2 \,.\, 1-u^8y^2}}=\frac{dx}{\sqrt{1-x^2 \,.\, 1-u^8x^2}}.$$

In particular, if the relation between y, x be given by the cubic transformation

$$y=\frac{\dfrac{v+2u^3}{v}x+\dfrac{u^6}{v^2}x^3}{1+vu^2(v+2u^3)x^2},$$

so that the modular equation $Q(u,\ v)=0$ is $u^4-v^4+2uv(1-u^2v^2)=0$; then, writing herein $v=\theta u$, and taking θ a prime eighth root of unity, that is, a root of $\theta^4+1=0$, we have

$$Q(u,\ \theta u)=-2\theta^3u^2(\theta u^2+\theta^2+u^4);$$

viz. disregarding the factor u^2, the equation for u is $u^4+\theta u^2+\theta^2=0$; or, if ω be an imaginary cube root of unity $(\omega^2+\omega+1=0)$, this is $(u^2-\omega\theta)(u^2-\omega^2\theta)=0$; so that a value of u^2 is $u^2=-\omega\theta$.

Assuming then $\theta^4+1=0$, $v=\theta u$ and $u^2=-\omega\theta$, we have $(v+2u^3)v=\theta^3\omega(1+2\omega)$, $=\theta^3\omega(\omega-\omega^2)$; $\dfrac{v+2u^3}{v}=\omega-\omega^2$; $\dfrac{u^6}{v^2}=\omega^2$, $(v+2u^3)vu^2=-\omega^2(\omega-\omega^2)$, $u^8=\omega^4\theta^4=-\omega$; and the formula becomes

$$y=\frac{(\omega-\omega^2)x+\omega^2x^3}{1-\omega^2(\omega-\omega^2)x^2},$$

giving

$$\frac{dy}{\sqrt{1-y^2 \,.\, 1+\omega y^2}}=\frac{(\omega-\omega^2)\,dx}{\sqrt{1-x^2 \,.\, 1+\omega x^2}},$$

where as before $\omega^2+\omega+1=0$, a result which can be at once verified. We have $(\omega-\omega^2)^2=-3$; or the coefficient $\omega-\omega^2$ in the differential equation is $=\sqrt{-3}$, which is of the form mentioned in the general theorem.

We might, instead of $z=y$, have assumed between y and z the relation corresponding to any other of the six linear transformations of an elliptic integral, and thus have obtained in each case, for a properly determined value of the modulus, a cubic transformation to the same modulus.

Cambridge, 10 *April*, 1877.