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ON THE FLEFLECNODAL PLANES OF A SURFACE.

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If at a node (or double point) of a plane curve there is on one of the branches an inflexion, (that is, if the tangent has a 3-pointic intersection with the branch), the node is said to be a flecnode; and if there is on each of the branches an inflexion, then the node is said to be a fleflecnode. The tangent plane of a surface intersects the surface in a plane curve having at the point of contact a node; if this is a flecnode or a fleflecnode, the tangent plane is said to be a flecnodal or a fleflecnodal plane accordingly. For a quadric surface each tangent plane is fleflecnodal; this is obvious geometrically (since the section is a pair of lines), and it will presently appear that the analytical condition for such a plane is satisfied. In fact, if the origin be taken at a point of a surface, so that $z=0$ shall be the equation of the tangent plane, then in the neighbourhood of the point we have

$$z=(x,\ y)^2+(x,\ y)^3+\&\text{c.};$$

and the condition for a fleflecnodal plane is that the term $(x, y)^2$ shall be a factor of the succeeding term $(x, y)^3$. Now for a quadric surface the equation is

$$z=\tfrac{1}{2}\{ax^2+2hxy+by^2+2(fy+gx)z+cz^2\};$$

that is,

$$z(1-fy-gx-\tfrac{1}{2}cz)=\tfrac{1}{2}(ax^2+2hxy+by^2),$$

or developing as far as the third order in (x, y), we have

$$z=\tfrac{1}{2}(ax^2+2hxy+by^2)(1+fy+gx),$$

so that the condition in question is satisfied.

In what follows, I take for greater simplicity $h=0$, (viz. $x=0$, $y=0$ are here the tangents to the two curves of curvature at the point in question), and to avoid fractions write $2f$, $2g$ in place of f, g respectively; the developed equation of the quadric surface is thus

$$z=\tfrac{1}{2}(ax^2+by^2)+(ax^2+by^2)(gx+fy).$$

I consider the parallel surface, obtained by measuring off on the normal a constant length k. If, as usual, p, q denote $\frac{dz}{dx}$ and $\frac{dz}{dy}$ respectively, then, in general, (X, Y, Z) being the coordinates of the point on the parallel surface,

$$Z=z+\frac{k}{\sqrt{(1+p^2+q^2)}},$$

$$X=x-\frac{kp}{\sqrt{(1+p^2+q^2)}},$$

$$Y=y-\frac{kq}{\sqrt{(1+p^2+q^2)}}.$$

But in the present case

$$p=ax+3agx^2+2afxy+\ bgy^2,$$

$$q=by+\ afx^2+2bgxy+3bfy^2,$$

whence

$$X=x-k(ax+3agx^2+2afxy+\ bgy^2),$$

$$Y=y-k(by+\ afx^2+2bgxy+3bfy^2);$$

or, putting for convenience,

$$X=(1-ka)\xi,\quad Y=(1-kb)\eta,$$

then, for a first approximation $x=\xi$, $y=\eta$; whence, writing

$$P=3ag\xi^2+2af\xi\eta+\ bg\eta^2,$$

$$Q=\ af\xi^2+2bg\xi\eta+3bf\eta^2,$$

we find

$$x=\xi+\frac{kP}{1-ka},\quad y=\eta+\frac{kQ}{1-kb},$$

and thence

$$p=a\left(\xi+\frac{k}{1+ka}P\right)+P=a\xi+\frac{P}{1-ka},$$

$$q\qquad\qquad=b\eta+\frac{Q}{1-kb}.$$

Hence

$$Z=\tfrac{1}{2}(a\xi^2+b\eta^2)+\frac{ka}{1-ka}\xi P+\frac{kb}{1-kb}\eta Q+(a\xi^2+b\eta^2)(g\xi+f\eta)$$

$$+k\left\{1-\tfrac{1}{2}(a^2\xi^2+b^2\eta^2)-\frac{a\xi P}{1-ka}-\frac{b\eta Q}{1-kb}\right\};$$

or, finally,

$$Z-k=\tfrac{1}{2}\{a(1-ka)\xi^2+b(1-kb)\eta^2\}+(a\xi^2+b\eta^2)(g\xi+f\eta),$$

where, changing the origin to the point $x=0$, $y=0$, $z=k$ on the parallel surface, the coordinates of the consecutive point are $Z-k$, $X, =(1-ka)\,\xi$, and $Y, =(1-kb)\,\eta$.

We cannot, by any determination of the value of k, make the plane $Z-k=0$ a fleflecnodal plane of the parallel surface; but if

$$k=\frac{af^2+bg^2}{a^2f^2+b^2g^2},$$

then

$$1-ka=\frac{bg^2\,(b-a)}{a^2f^2+b^2g^2},\quad 1-kb=\frac{af^2\,(a-b)}{a^2f^2+b^2g^2},$$

and the equation becomes

$$Z-k=\tfrac{1}{2}\,\frac{ab\,(b-a)}{a^2f^2+b^2g^2}\,(g^2\xi^2-f^2\eta^2)+(a\xi^2+b\eta^2)\,(g\xi+f\eta)\,;$$

viz. the term of the second has here a factor $g\xi+f\eta$ which divides the term of the third order, and the plane $Z-k=0$ is a flecnodal plane of the parallel surface.