# 682.

## FORMULÆ RELATING TO THE RIGHT LINE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 169—171.]

1. LET $\lambda$, $\mu$, $\nu$ be the direction-angles of a line; $\alpha$, $\beta$, $\gamma$ the coordinates of a point on the line; and write

$$a = \cos\lambda,\quad f = \beta\cos\nu - \gamma\cos\mu,$$
$$b = \cos\mu,\quad g = \gamma\cos\lambda - \alpha\cos\nu,$$
$$c = \cos\nu,\quad h = \alpha\cos\mu - \beta\cos\lambda,$$

whence

$$a^2 + b^2 + c^2 = 1,$$
$$af + bg + ch = 0,$$

or the six quantities $(a, b, c, f, g, h)$, termed the coordinates of the line, depend upon four arbitrary parameters.

2. It is at once shown that the condition for the intersection of any two lines $(a, b, c, f, g, h)$, $(a', b', c', f', g', h')$, is $af' + bg' + ch' + a'f + b'g + c'h = 0$.

3. Given two lines $(a, b, c, f, g, h)$, $(a', b', c', f', g', h')$, it is required to find their shortest distance, and the coordinates of their line of shortest distance.

Let

$$Ax + By + Cz + D = 0,$$
$$Ax + By + Cz + D' = 0,$$

be parallel planes containing the two lines respectively; then the first plane contains the point $\alpha + r\cos\lambda$, $\beta + r\cos\mu$, $\gamma + r\cos\nu$, and the second contains the point $\alpha' + r'\cos\lambda'$, $\beta' + r'\cos\mu'$, $\gamma' + r'\cos\nu'$; that is, we have

$$A\alpha + B\beta + C\gamma + D = 0,$$
$$A\alpha' + B\beta' + C\gamma' + D' = 0,$$
$$A\cos\lambda + B\cos\mu + C\cos\nu = 0,$$
$$A\cos\lambda' + B\cos\mu' + C\cos\nu' = 0,$$

which last equations may be written

$$Aa + Bb + Cc = 0,$$
$$Aa' + Bb' + Cc' = 0,$$

giving

$$A : B : C = bc' - b'c : ca' - c'a : ab' - a'b,$$

or, if we write

$$\theta = aa' + bb' + cc',$$

and assume, as is convenient,

$$A^2 + B^2 + C^2 = 1,$$

then

$$A,\ B,\ C = \frac{bc' - b'c}{\sqrt{(1-\theta^2)}},\ \frac{ca' - c'a}{\sqrt{(1-\theta^2)}},\ \frac{ab' - a'b}{\sqrt{(1-\theta^2)}},$$

where $\theta$, = cosine-inclination, $= aa' + bb' + cc'$.

Hence, shortest distance $= D - D'$

$$\begin{aligned} &= A(\alpha - \alpha') + B(\beta - \beta') + C(\gamma - \gamma') \\ &= \frac{1}{\sqrt{(1-\theta^2)}}\{(bc' - b'c)(\alpha - \alpha') + (ca' - c'a)(\beta - \beta') + (ab' - a'b)\} \\ &= \frac{1}{\sqrt{(1-\theta^2)}}\{a'(c\beta - b\gamma) + b'(a\gamma - c\alpha) + c'(b\alpha - a\beta) \\ &\qquad + a(c'\beta' - b'\gamma') + b(a'\gamma' - c'\alpha') + c(b'\alpha' - a'\beta')\} \\ &= \frac{1}{\sqrt{(1-\theta^2)}}(af' + bg' + ch' + a'f + b'g + c'h),\ = \delta \text{ suppose.} \end{aligned}$$

The six coordinates of the line of shortest distance are $A, B, C, F, G, H$, where $A, B, C$ denote as before, and $F, G, H$ are to be determined.

Since the line meets each of the given lines, we have

$$Af + Bg + Ch + Fa + Gb + Hc = 0,$$
$$Af' + Bg' + Ch' + Fa' + Gb' + Hc' = 0,$$

and we have also

$$FA + GB + HC = 0,$$

which equations give $F, G, H$. Multiplying the first equation by $b'C - c'B$, the second by $Bc - Cb$, and the third by $bc' - b'c$, we find

$$(b'C - c'B)(Af + Bg + Ch) + (Bc - Cb)(Af' + Bg' + Ch') + F\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ A, & B, & C \end{vmatrix} = 0.$$

Here

$$\begin{aligned} b'C - c'B &= \frac{1}{\sqrt{(1-\theta^2)}}\{b'(ab' - a'b) - c'(ca' - c'a)\} \\ &= \frac{1}{\sqrt{(1-\theta^2)}}\{a(a'^2 + b'^2 + c'^2) - a'(aa' + bb' + cc')\} \\ &= \frac{1}{\sqrt{(1-\theta^2)}}(a - a'\theta), \end{aligned}$$

and similarly

$$cB - bC = \frac{1}{\sqrt{(1-\theta^2)}}(a' - a\theta).$$

Also, putting for shortness

$$\Omega = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ f, & g, & h \end{vmatrix}, \quad \Omega' = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ f', & g', & h' \end{vmatrix},$$

we have

$$Af + Bg + Ch = \frac{1}{\sqrt{(1-\theta^2)}}\,\Omega, \quad Af' + Bg' + Ch' = \frac{1}{\sqrt{(1-\theta^2)}}\,\Omega',$$

and finally, the determinant which multiplies $F$ is

$$\frac{1}{\sqrt{(1-\theta^2)}}\{(bc' - b'c)^2 + (ca' - c'a)^2 + (ab' - a'b)^2\} = \frac{1}{\sqrt{(1-\theta^2)}}(1-\theta^2), \; = \sqrt{(1-\theta^2)}.$$

We have thus the value of $F$; forming in the same way those of $G$ and $H$, we find

$$F = \frac{-1}{(1-\theta^2)^{\frac{3}{2}}}\{(a - a'\theta)\,\Omega + (a' - a\theta)\,\Omega'\},$$

$$G = \frac{-1}{(1-\theta^2)^{\frac{3}{2}}}\{(b - b'\theta)\,\Omega + (b' - b\theta)\,\Omega'\},$$

$$H = \frac{-1}{(1-\theta^2)^{\frac{3}{2}}}\{(c - c'\theta)\,\Omega + (c' - c\theta)\,\Omega'\},$$

which, with the foregoing equations for $A$, $B$, $C$, give the six coordinates of the line of shortest distance.