684.

ON A RELATION BETWEEN CERTAIN PRODUCTS OF DIFFERENCES.

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CONSIDER the function

$$3\left\{\begin{array}{l} abc\,.\,de \\ +\,bcd\,.\,ea \\ +\,cde\,.\,ab \\ +\,dea\,.\,bc \\ +\,eab\,.\,cd \end{array}\right\} - \left\{\begin{array}{l} abd\,.\,ce \\ +\,bce\,.\,da \\ +\,cda\,.\,eb \\ +\,deb\,.\,ac \\ +\,eac\,.\,bd \end{array}\right\},$$

where

$$abc = (a-b)(b-c)(c-a),$$

$$ab = (a-b)(b-a), = -(a-b)^2,$$

&c.;

therefore

$$abc = bca = cab = -bac, \text{ \&c.};$$

$$ab = ba.$$

It is to be shown that the function vanishes if $e = d$. Writing $e = d$, the value is

$$\begin{array}{r} 3\,(bcd\,.\,da + dab\,.\,cd) - abd\,.\,cd \\ -\,bcd\,.\,da \\ -\,cda\,.\,db \\ -\,dac\,.\,bd, \end{array}$$

viz. this is

$$\begin{aligned} &3\ bcd\,.\,ad - \ abd\,.\,cd \\ &+3\ abd\,.\,cd - \ bcd\ .\,ad \\ &\qquad\qquad - 2acd\ .\,bd \\ &= 2\ bcd\,.\,ad - 2acd\ .\,bd + 2abd\,.\,cd \\ &= 2\,(bcd\,.\,ad + \ cad\ .\,bd + \ abd\,.\,cd), \end{aligned}$$

which is easily seen to vanish; the value is

$$\begin{aligned} &(b-c)(c-d)(d-b)(a-d)^2 = -(b-c)(a-d)^2(b-d)\ (c-d) \\ +&(c-a)(a-d)(d-c)(b-d)^2 \quad -(c-a)(a-d)\ (b-d)^2(c-d) \\ +&(a-b)(b-d)(d-a)(c-d)^2 \quad -(a-b)(a-d)\ (b-d)\ (c-d)^2: \end{aligned}$$

viz. omitting the factor $(a-d)(b-d)(c-d)$, this is

$$\begin{aligned} = &-(b-c)(a-d) \\ &-(c-a)(b-d) \\ &-(a-b)(c-d), \end{aligned}$$

which vanishes. Hence the function also vanishes if $e=a$, or $a=b$ or $b=c$, or $c=d$; and it is thus a mere numerical multiple of $(a-b)(b-c)(c-d)(d-e)(e-a)$, or say it is $= Mabcde$.

To find M write $e=c$, the equation becomes

$$\begin{aligned} &3abc\,.\,dc - cda\,.\,cb = Mabcdc, \ = Mabc\,.\,dc, \\ +\ &3bcd\,.\,ca - ac \\ +\ &3dca\,.\,bc \\ +\ &3cab\,.\,cd, \end{aligned}$$

viz. this is

$$6abc\,.\,dc + 4dbc\,.\,ac + 4adc\,.\,bc = M\,.\,abc\,.\,dc,$$

giving $M=10$. In fact, we then have

$$-4abc\,.\,dc + 4dbc\,.\,ac + 4adc\,.\,bc = 0,$$

that is,

$$-\ abc\,.\,dc - \ bdc\,.\,ac - \ dac\,.\,bc = 0,$$

which is right. And we have thus the identity

$$3\left\{\begin{array}{l} \quad abc\,.\,de \\ +\,bcd\,.\,ea \\ +\,cde\,.\,ab \\ +\,dea\,.\,bc \\ +\,eab\,.\,cd \end{array}\right\} - \left\{\begin{array}{l} \quad abd\,.\,ce \\ +\,bce\,.\,da \\ +\,cda\,.\,eb \\ +\,deb\,.\,ac \\ +\,eac\,.\,bd \end{array}\right\} = 10\,.\,abcde,$$

or say

$$3\,[abcde] - [acebd] = 10\,\{abcde\}.$$