685.

ON MR COTTERILL'S GONIOMETRICAL PROBLEM.

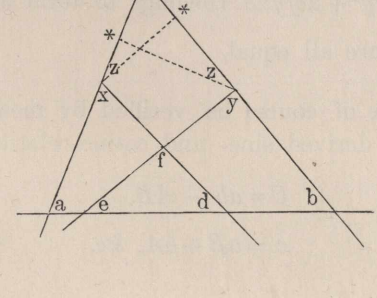
[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 196—198.]

THE very remarkable formulæ contained in Mr Cotterill's paper, "A goniometrical problem, to be solved analytically in one move, or more simply synthetically in two moves," *Quarterly Mathematical Journal*, t. VII. (1866), pp. 259—272, are presented in a form which, to say the least, is not as easily intelligible as might be; and they have not, I think, attracted the attention which they well deserve.

Using his notation, except that I write for angles small roman letters, in order to be able to have the corresponding italic small letters and capitals for the sines and cosines respectively of the same angles, we consider nine angles

$$\begin{array}{ccc} \mathrm{a}, & \mathrm{b}, & \mathrm{c}, \\ \mathrm{d}, & \mathrm{e}, & \mathrm{f}, \\ \mathrm{x}, & \mathrm{y}, & \mathrm{z}, \end{array}$$

which are such that the sum of three angles in the same line, or in the same column, is an odd multiple of π. Of course, any four angles such as a, b, d, e are



arbitrary, and each of the remaining angles is then determinate save as to an even multiple of π. And it may be remarked that these angles a, b, d, e may represent the inclinations of any four lines to a fifth line, and that the remaining angles are then at once obtained, as in the figure. The small roman letters are here used to denote as well angles as points, being so placed as to show what the angles are which they respectively denote; the points $*$, $*$ are constructed as the intersections of the lines ac, bc by the circle circumscribed about fxy, and the angle z is the angle which the points $*$, $*$ subtend at x or y. It will be observed that the sum of the three angles in a line or column is in each case $=\pi$.

But this in passing: the analytical theorem is, *first*, we can form with the sines and cosines of the angles in any two lines or columns a function S presenting itself under two distinct forms, which are in fact equal in value, or say S is a symmetrical function of the two lines or columns, viz. for the first and second lines this is

$$S\begin{pmatrix}\mathrm{a}, & \mathrm{b}, & \mathrm{c}\\ \mathrm{d}, & \mathrm{e}, & \mathrm{f}\end{pmatrix} = d^2 Abc + e^2 Bca + f^2 Cab$$
$$= a^2 Def + b^2 Efd + c^2 Fde,$$

where, as already mentioned, a, A denote sin a, cos a, and so for the other letters.

Secondly, if to the S of any two lines or columns we add twice the product of the six sines, we obtain a sum M which has the same value from whichever two lines or columns we obtain it; or, say M is a symmetrical function of the matrix of the nine angles. Thus

$$M = S\begin{pmatrix}\mathrm{a}, & \mathrm{b}, & \mathrm{c}\\ \mathrm{d}, & \mathrm{e}, & \mathrm{f}\end{pmatrix} + 2abcdef,$$

which is one of a system of six forms each of which (on account of the two forms of the S contained in it) may be regarded as a double form, and the twelve values are all of them equal. There are, moreover, 15 other forms, of M, viz. 3 line-forms, such as

$$bcdx + caey + abfz \text{ (belongs to line a, b, c)},$$

3 column-forms, such as

$$dxbc + xaef + adyz \text{ (belongs to column a, d, x)},$$

and 9 term-forms, such as

$$e^2z^2 + f^2y^2 + 2efyzA \text{ (belongs to term a)},$$

and the $12 + 15, = 27$ values are all equal.

The several identities can of course be verified by means of the relations between the nine angles, or rather the derived sine- and cosine-relations

$$C = ab - AB,$$
$$c = aB + bA, \text{ \&c.}$$

Thus, as regards the two forms of $S\begin{pmatrix}\mathrm{a}, & \mathrm{b}, & \mathrm{c}\\ \mathrm{d}, & \mathrm{e}, & \mathrm{f}\end{pmatrix}$, the identity to be verified may be written

$$c(d^2Ab+e^2Ba-cFde)=f(a^2De+b^2Ed-fCab).$$

Proceeding to reduce the factor $a^2De+b^2Ed-fCab$, if we first write herein $f=eD+dE$, it becomes

$$a^2De+b^2Ed-(eD+dE)\,Cab,$$

which is

$$=aDe(a-bC)+bEd(b-aC),$$

and then writing $C=ab-AB$, we have $a-bC=a(1-b^2)+bAB$, $=B(aB+bA)$, $=Bc$; and, similarly, $b-aC=Ac$; whence the term is $=c(aeBD+bdAE)$; or, in the equation to be verified, the right-hand side is $=cf(aeBD+bdAE)$, and by a similar reduction, the left-hand side is found to have the same value.

The paper contains various other interesting results.