687.

NOTE ON THE FUNCTION $\vartheta x = a^2(c-x) \div \{c(c-x)-b^2\}$.

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STARTING from the general form

$$\vartheta x = \frac{\alpha x + \beta}{\gamma x + \delta},$$

we have

$$\vartheta^n x = \frac{(\lambda^{n+1}-1)(\alpha x+\beta)+(\lambda^n-\lambda)(-\delta x+\beta)}{(\lambda^{n+1}-1)(\gamma x+\delta)+(\lambda^n-\lambda)(\gamma x-\alpha)},$$

where

$$\lambda + \frac{1}{\lambda} = \frac{\alpha^2+\delta^2+2\beta\gamma}{\alpha\delta-\beta\gamma}.$$

For the function in question

$$\vartheta x = \frac{a^2(c-x)}{c(c-x)-b^2},$$

(a form which presents itself in the problem of the distribution of electricity upon two spheres), the values of $\alpha, \beta, \gamma, \delta$ are

$$\alpha = -a^2,\quad \beta = a^2c,\quad \gamma = -c,\quad \delta = c^2-b^2;$$

the equation for λ therefore is

$$\lambda + \frac{1}{\lambda} = \frac{a^4+(c^2-b^2)^2-2a^2c^2}{a^2b^2};$$

or, what is the same thing,

$$\frac{(\lambda+1)^2}{\lambda} = \frac{(a^2+b^2-c^2)^2}{a^2b^2}.$$

Suppose that a, b, c are the sides of a triangle the angles whereof are A, B, C; then $c^2 = a^2 + b^2 - 2ab\cos C$, or we have

$$\frac{(\lambda+1)^2}{\lambda} = 4\cos^2 C;$$

or, writing this under the form

$$\sqrt{(\lambda)} + \frac{1}{\sqrt{(\lambda)}} = 2\cos C,$$

the value of λ is at once seen to be $= e^{2iC}$; and it is interesting to obtain the expression of the nth function in terms of the sides and angles of the triangle.

The numerator and the denominator are

$$\lambda^n P + Q,$$
$$\lambda^n R + S,$$

where

$$P = \lambda(\alpha x + \beta) + (-\delta x + \beta), \quad R = \lambda(\gamma x + \delta) + \gamma x - \alpha,$$
$$Q = -(\alpha x + \beta) - \lambda(-\delta x + \beta), \quad S = -(\gamma x + \delta) - \lambda(\gamma x - \alpha).$$

Hence, writing the numerator and the denominator in the forms

$$\lambda^{\frac{1}{2}n} P + \lambda^{-\frac{1}{2}n} Q,$$
$$\lambda^{\frac{1}{2}n} R + \lambda^{-\frac{1}{2}n} S,$$

these are

$$(P+Q)\cos nC + (P-Q)\, i\sin nC,$$
$$(R+S)\cos nC + (R-S)\, i\sin nC;$$

viz. they are

$$(\lambda-1)(\alpha+\delta)\, x\cos nC + (\lambda+1)\{(\alpha-\delta)\, x + 2\beta\}\, i\sin nC,$$
$$(\lambda-1)(\alpha+\delta)\, .\cos nC + (\lambda+1)\{2\gamma x - (\alpha-\delta)\}\, i\sin nC,$$

or, observing that $\dfrac{\lambda-1}{\lambda+1} = i\tan C$ and removing the common factor $i(\lambda+1)$, they may be written

$$\tan C(\alpha+\delta)\, x\cos nC + \{(\alpha-\delta)\, x + 2\beta\}\sin nC,$$
$$\tan C(\alpha+\delta)\, .\cos nC + \{2\gamma x - (\alpha-\delta)\}\sin nC.$$

Substituting for α, β, γ, δ their values, these are

$$\tan C\{(c^2-a^2-b^2)\, x\cos nC\} + \{(b^2-a^2-c^2)\, x + 2a^2c\}\sin nC,$$
$$\tan C\{(c^2-a^2-b^2)\, .\cos nC\} + \{-2cx - (b^2-a^2-c^2)\}\sin nC,$$
$$= \tan C\{-ab\cos Cx\cos nC\} + \{-ac\cos B\, .\, x + a^2c\}\sin nC,$$
$$\tan C\{-ab\cos Cx\cos nC\} + \{-cx + ac\cos B\}\sin nC,$$
$$= x\{-ab\sin C\cos nC - ac\cos B\sin nC\} + a^2c\sin nC,$$
$$-cx\sin nC + \{ac\cos B\sin nC - ab\sin C\cos nC\};$$

or, writing herein $b \sin C = c \sin B$, these are

$$\begin{array}{ll} -acx\{\sin B \cos nC + \cos B \sin nC\} & + a^2 c \sin nC, \\ -cx \sin nC & + ac\{\cos B \sin nC - \sin B \cos nC\}, \end{array}$$

whence finally

$$\vartheta^n x = \frac{a^2 \sin nC - ax \sin(nC + B)}{a \sin(nC - B) - x \sin nC}.$$

As a verification, writing $n = 1$, we have

$$\vartheta x = \frac{a^2 \sin C - ax \sin A}{a \sin(C - B) - x \sin C}$$

$$= \frac{a^2 c - acx \dfrac{\sin A}{\sin C}}{ac \dfrac{\sin(C-B)}{\sin C} - cx},$$

or observing that

$$ac \frac{\sin(C-B)}{\sin C} = c^2 - b^2,$$

(for this is $\sin A \sin(C - B) = \sin^2 C - \sin^2 B$), we have

$$\vartheta x = \frac{a^2(c-x)}{c^2 - b^2 - cx}$$

as it should be. If in the formula for $\vartheta^n x$ we write $x = 0$, we have a formula given in the Senate-House Problems, January 14, 1878: it was thus that I was led to investigate the general expression.