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Thermoelastic Rolling Contact Problems for a Multi-Layer Structure

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ABSTRACT: This paper is concerned with the numerical solution of the thermoelastic wheel-rail rolling contact problems including friction, frictional heat generation and transport across contact surface as well as wear. Three-layer model of rail material is assumed. Materials in upper and lower layers are characterized by distinct constant mechanical and thermal parameters. Middle layer material parameters are dependent on the depth of this layer according to the exponential law. The displacement and the temperature of the rail are governed by the coupled hyperbolic variational inequality of the second order and the parabolic equation, respectively. Using the special features of the rolling contact problem the original time dependent problem is transformed into the quasistatic elliptic problem and numerically solved. Distributions of stress and temperature fields in the contact zone are provided and discussed.

1 INTRODUCTION

Contact phenomena appear in different fields of engineering sciences and are subject of intensive research (Choi et al. 2008, Giannakopoulos et al. 2000, Guler 2009, Sextro 2007, Suresh 2001, Wriggers 2006). These phenomena may include among others friction, frictional heat generation as well as heat transfer across the contact surface and wear. Contact phenomena may generate high stresses between contacting surfaces. Repeated overstressing of the surface or sub-surface material by intensive wheel-rail contact cycles may lead to rolling contact fatigue, noise generation and reducing of journey comfort for passengers. The control and/or reduction of wheel-rail contact stress is subject of great interest of the railway engineering community.

This paper deals with the numerical solution of the thermoelastic rolling contact problems. The contact of a rigid wheel with an elastic rail lying on a rigid foundation is considered. The friction between the bodies is assumed to be governed by Coulomb law (Chudzikiewicz et al. 2011, Chudzikiewicz et al. 2012, Shillor et al. 2004, Wriggers 2006). The heat generated due to the friction is transported across the contact surface and increases the temperature of the contacting bodies. We employ Archard's law of wear

(Meng et al. 1995, Paczelt et al. 2007). On a macro-scale the existence of the wear process can be identified as wear debris. This debris is assumed to disappear immediately at the point where it is formed. In the model the wear is identified as an increase in the gap between bodies. Moreover the dissipation energy is being changed due to wear. The displacement and temperature of the contacting bodies are governed by the weakly coupled system of hyperbolic variational inequality and parabolic heat equation, respectively (Chudzikiewicz et al. 2011, Ertz et al. 2002, Jang et al. 2007).

The elastic or thermoelastic rolling contact problems were considered by many authors. For details see the references in monographs (Guler 2009, Han et al. 2002, Sextro 2007, Shillor et al. 2004, Wriggers 2006). Among others, in (Ertz et al. 2002) thermoelastic wheel-rail contact problem was solved numerically using Hertz contact model as well as Green function approach to solve the heat equation. It was also pointed out in (Ertz et al. 2002) that not only heat conduction but also heat convection should be taken into account in wheel-rail heat flow. Thermoelastic instability in two dimensional contact problem has been considered in (Jang et al. 2007) where the heat flow in friction material components was described by conduction convection term. Numerical experiments in

(Jang et al. 2007, Guler et al. 2013) indicate that the use of a graded layer or a coating material attached to the conventional steel body of the rail reduces the magnitude of residual and/or thermal stresses. Similar conclusions based on results of experiments are reported in (Hiensh et al. 2005, Suresh 2001).

In this paper we solve numerically the thermoelastic rolling contact problem assuming the existence of a small elastic multi-layer on the rail surface rather than a single graded layer as in (Chudzikiewicz et al. 2011). It consists from two sublayers having constant distinct material parameters and a functionally graded material layer between them having material properties dependent on its depth according to the exponential law. Following (Chudzikiewicz et al. 2011, Chudzikiewicz et al. 2012, Chudzikiewicz et al. 1992, Ertz et al. 2002) special features of this rolling contact problem are taken into account and so-called quasistatic approach to solve this problem is used. In this approach the inertial terms in displacement and temperature governing equations are replaced by the stationary terms reflecting the dynamics of the body rather than completely neglecting it as in classical quasistatic formulation (Han et al. 2002, Shillor et al. 2004). Therefore, after brief introduction of the thermoelastic model of the rolling contact problem with friction and wear in the framework of two-dimensional linear elasticity theory (Han et al. 2002, Sextro 2007) the general coupled time dependent system describing this physical phenomenon is formulated. This system is transformed into equivalent stationary system in so-called quasistatic formulation (Chudzikiewicz et al. 1992). To solve numerically this stationary system we will decouple it into mechanical and thermal parts (Chudzikiewicz et al. 1992). Finite element method is used as a discretization method (Efendiev et al. 2009, Hou et al. 1999, Kim et al. 2002). The numerical results are provided and discussed.

2 FORMULATION OF THE PROBLEM

Consider deformations of an elastic strip lying on a rigid foundation (see Fig. 1). The strip has constant height h and occupies domain $\Omega \in \mathbb{R}^2$ with the boundary Γ . A wheel rolls along the upper surface Γ_C of the strip. The wheel has radius r_0 , rotating speed ω and linear velocity V . The axis of the wheel is moving along a straight line at a constant altitude h_0 where $h_0 < h + r_0$, i.e., the wheel is pressed in the elastic strip. It is assumed, that the head and tail ends of the strip are clamped, i.e., we assume that the length of the strip is much bigger than the radius of the wheel. Moreover it is assumed, that there is no mass forces in the strip. The body is clamped along a portion Γ_0 of the boundary Γ of the domain Ω . The contact conditions are prescribed on a portion Γ_C of the boundary Γ . Moreover, $\Gamma_0 \cap \Gamma_C = \emptyset$, $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_C$.

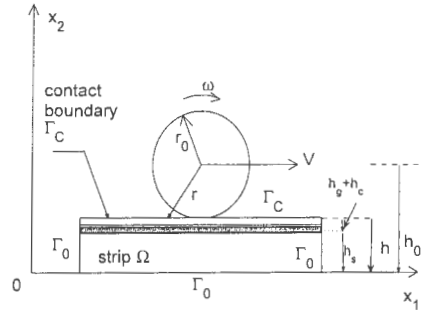


Figure 1: The wheel rolling over the rail strip.

2.1 Material properties of the functionally graded material

Assume $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ where Ω_1 (Ω_3) denotes upper (lower) part of the rail strip having thickness h_c (h_s). Material properties of layers Ω_1 and Ω_3 are assumed to be constant and homogeneous throughout their whole volumes. The middle layer Ω_2 has thickness h_g and constitutes a graded coating layer composed of steel and ceramic (see Fig. 2). The coating layer material has a variation property along its height in the occupied subdomain. The graded material coating of rail is assumed to be processed in such a way that the property grading is smooth. It implies that the discontinuities in the material property distribution are eliminated and stresses through the composite layer are bounded.

In literature (Yang et al. 2008) are used power, exponential or sigmoidal models of the graded layers. In the paper we use the exponential model of the graded layer rather than the power one as in (Chudzikiewicz et al. 2011). Material properties in this layer are assumed to be a function of the layer height and are described by the following equation (Yang et al. 2008):

$$P(x) = P_c e^{n(x_2/h_g)}, \quad n = \ln\left(\frac{P_s}{P_c}\right),$$

$$x_2 \in [0, h_g] \quad (1)$$

where $x = (x_1, x_2) \in \Omega_2$, $P = P(x)$ denotes the material property dependent on spatial variable x_2 and P_c , P_s are the ceramic property and the steel property, respectively. Constant n denotes the non-homogeneity parameter of the graded material. P may be used for the elastic modulus, the thermal expansion, the thermal conductivity or the thermal diffusivity coefficients. Distribution model (1) indicates that the composition would vary continuously from homogeneous steel material near the lower interface to homogeneous ceramic near the surface. Thus the material is purely steel at the core part and gradually move and approaches the ceramic properties at the upper surface of coating layer. The inner material distribution

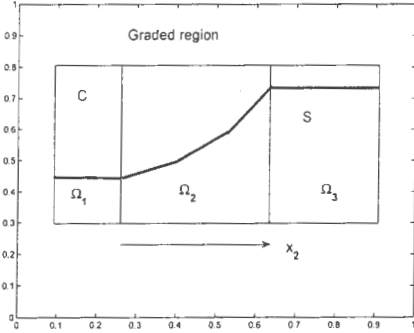


Figure 2: Graded joint.

of graded layer is determined by the parameter n . For $n < 0$ the coating layer is more ceramic rich while for $n > 0$ is more steel rich. Remark $n = 0$ denotes the homogeneous rail material, i.e., there is no graded layer. Along the boundaries of the subdomains Ω_1 and Ω_2 as well as Ω_2 and Ω_3 the material properties are assumed to be continuous. (see Fig. 2).

2.2 Thermoelastic model

The thermoelastic rolling contact problem is described by the coupled thermoelastic system of equations. Let us denote by $u = (u_1, u_2)$, $u = u(x, t)$, $x \in \Omega$, $t \in (0, T)$, $T > 0$, a displacement of the strip and by $\theta = \theta(x, t)$ the absolute temperature of the strip. Since the rail consists from the graded layer Ω_2 and the layers Ω_1 and Ω_3 we denote by u_g , u_c and u_s as well as by θ_g , θ_c and θ_s the displacement and temperature of these layers, respectively. Therefore $u = u_c, u_g, u_s$ in Ω_1, Ω_2 and Ω_3 , respectively. The similar relation holds for the temperature θ .

Assume the wheel and the rail are brought into contact under the action of the static wheel load. The contact area and the contact pressure distribution are usually calculated using Hertz's theory (Hiensch et al. 2005). Here we use two dimensional elastic linear model and Coulomb friction model to determine contact area and stress distribution. The displacement u and the temperature θ of the rail strip satisfy the system of evolution equations in $\Omega \times (0, T)$ (Chudzikiewicz et al. 2011, Meng et al. 1995):

$$\rho \frac{\partial^2 u}{\partial t^2} = A^* D A u - \alpha(3\lambda + 2\gamma) \nabla \theta, \quad (2)$$

$$\rho c_p \frac{\partial \theta}{\partial t} = \bar{\kappa} \Delta \theta, \quad (3)$$

with the initial and the boundary conditions:

$$u = 0 \quad \text{on } \Gamma_0 \times (0, T), \quad (4)$$

$$B^* D A u = F \quad \text{on } \Gamma_C \times (0, T), \quad (5)$$

$$u(0) = \bar{u}_0, \quad u'(0) = \bar{u}_1 \quad \text{in } \Omega, \quad (6)$$

$$\theta(0) = \bar{\theta} \quad \text{in } \Omega, \quad (7)$$

$$\frac{\partial \theta}{\partial n}(x, t) = q(t, x) \quad \text{on } \Gamma \times (0, T), \quad (8)$$

where $u(0) = u(x, 0)$, $u' = du/dt$, $\bar{u}_0, \bar{u}_1, \bar{\theta}, q(t, x)$ are given functions, ρ is a mass density of the strip material, α is a coefficient of thermal expansion, $\bar{\kappa}$ is a thermal conductivity coefficient, c_p is a heat capacity coefficient, $\Gamma_0 = \Gamma \setminus \Gamma_C$. The operators A, B and D are defined as follows (Chudzikiewicz et al. 2011, Chudzikiewicz et al. 1992)

$$A = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2^2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}, \quad B = \begin{bmatrix} n_1 & 0 \\ 0 & n_2 \\ n_2 & n_1 \end{bmatrix}, \quad (9)$$

$$D = \begin{bmatrix} \lambda + 2\gamma & \lambda & 0 \\ \lambda & \lambda + 2\gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad (10)$$

where $n = (n_1, n_2)$ is the outward normal vector to the boundary Γ of the domain Ω , λ and γ are Lamé coefficients (Chudzikiewicz et al. 2011) related with Young modulus E and Poisson's ratio ν by relations

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \gamma = \frac{E}{2(1+\nu)}.$$

Lamé coefficients are constant in Ω_1 and Ω_3 . In Ω_2 they are dependent on x_2 in Ω_2 according to (1). Therefore operator D is also assumed to be constant in $\Omega_1 \cup \Omega_3$ and to depend on the depth of the rail (Hiensch et al. 2005, Jang et al. 2007) according to the exponential law (1) in Ω_2 . Coefficients $\bar{\kappa}$ and c_p in (3) are assumed to be constant in Ω_1 and Ω_3 as well as to depend on spatial variable x_2 in Ω_2 according to (1). The matrix A^* (B^*) denotes a transpose of A (B).

Due to the dependence of stress also on temperature in the equation (2) appears the gradient $\nabla \theta = [\frac{\partial \theta}{\partial x_1}, \frac{\partial \theta}{\partial x_2}]^T$ of temperature θ . Therefore the operator equations (2) and (3) are mildly coupled.

By $\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})$ and F we denote the stress tensor in domain Ω and surface traction vector on the boundary Γ , respectively. The surface traction vector $F = (F_1, F_2)$ on the boundary Γ_C is a priori unknown and is given by the conditions of contact and friction. Under the assumptions that the strip displacement is suitable small the contact conditions on $\Gamma_C \times (0, T)$ take the form (Chudzikiewicz et al. 2011):

$$u_2 + g_r + w \leq 0, \quad F_2 \leq 0, \quad (11)$$

$$(u_2 + g_r + w)F_2 = 0,$$

$$|F_1| \leq \mu |F_2|, \quad F_1 \frac{du_1}{dt} \leq 0, \quad (12)$$

$$(|F_1| - \mu |F_2|) \frac{du_1}{dt} = 0, \quad (13)$$

where μ is a friction coefficient and u_2 denotes vertical component of u . For the sake of simplicity we assume here this friction coefficient is constant. In general it may be dependent on temperature θ or sliding velocity $\frac{du_1}{dt}$. Under suitable assumptions the gap between the bodies is equal to $g_r = h - h_0 + \sqrt{r_0^2 - (u_1 + x_1)^2}$ (Chudzikiewicz et al. 2012). Conditions (11)-(13) describe the contact phenomenon. Nonpenetration condition (11) implies that either the bodies are in contact characterized by the displacement satisfying first condition in (11) as equality and nonzero normal contact stress or the contact does not appear, i.e., the displacement condition is satisfied as strong inequality and the normal contact stress is zero. Conditions (12)-(13) describe Coulomb law of friction.

Function $w = w(x, t)$ in (11) denotes the additional distance between the bodies due to wear (Chudzikiewicz et al. 2011) of the contacting surfaces. For details concerning wear phenomenon and models see (Meng et al. 1995, Paczelt et al. 2007). We assume wear on the boundary $\Gamma_C \times (0, T)$ is governed by Archard's law (Meng et al. 1995),

$$\frac{dw}{dt} = kV F_2, \quad (14)$$

$w = w(x, t)$ is an internal state variable to model the wear process taking place at the contact interface (Meng et al. 1995). $k > 0$ denotes a given dimensional wear coefficient. In the considered model the wear is described as an increase in the gap in the normal direction between the contacting bodies. For generalizations of wear models see (Paczelt et al. 2007).

Displacement continuity condition is also assumed on the interfaces between the layers, i.e.,

$$u_c = u_g \quad \text{on } \Omega_1 \cap \Omega_2, \quad (15)$$

$$u_g = u_s \quad \text{on } \Omega_2 \cap \Omega_3.$$

It is well known (Hiensh et al. 2005) that for the observer fixed to the wheel the contact patch moves with respect to the wheel surface. The frictional heating within this patch is generally a time dependent heat source. We confine to consider the simpler case when the heat source term is stationary, i.e., we assume that in (8) function $q(t, x) = q(x) = \bar{q}$ is not dependent on time. The boundary conditions associated with the heat equation are dependent on contact between the bodies and surrounding environment. Consider the boundary condition (8). The first case is when the surfaces of the wheel and the rail are in contact. It causes heat generation due to friction. This heat flow is expressed as the boundary condition

$$-\bar{\kappa} \frac{\partial \theta_c}{\partial \bar{n}}(x, t) = \bar{q}, \quad \text{on } \Gamma_C \times (0, T), \quad (16)$$

where \bar{q} is a given function which may depend on space variables. We use \bar{q} given by (31). When the contact surfaces are separated the boundary condition (8) takes the form

$$-\bar{\kappa} \frac{\partial \theta_c}{\partial \bar{n}}(x, t) = 0. \quad (17)$$

Temperature continuity condition is also assumed on the interfaces between the layers, i.e.,

$$\theta_c = \theta_g \quad \text{on } \Omega_1 \cap \Omega_2. \quad (18)$$

$$\theta_g = \theta_s \quad \text{on } \Omega_2 \cap \Omega_3.$$

Along the boundary Γ_0 temperature $\theta_c = \theta_g = \theta_s = \theta_a$ where θ_a denotes the given temperature of the surrounding air. At the initial moment $t = 0$ temperature of rail is assumed to be the same as the given temperature θ_a of the surrounding air, i.e., $\theta_c(0, x) = \theta_s(0, x) = \theta_g(0, x) = \theta_a$ in Ω .

3 QUASISTATIC FORMULATION

The system of thermoelastic equations (2)-(3) with the boundary and initial conditions (4)-(18) may be solved using general or specialized methods. Laplace transform method belongs to the first group. To solve this system usually Green function approach is used (Hiensh et al. 2005). Assuming that heat source is constant in time and space surface temperature integrals are expressed in terms of the error function and the exponential integral for which series expansions are known. In this approach, as in (Hiensh et al. 2005), time t elapsed since entering the contact area is substituted with the current position x in a coordinate system fixed to the contact patch. It results in wheel or rail temperature integrals depending only on spatial variables. We propose slightly different way to eliminate time t from this system.

Taking into account the special features of the contact problem (2)-(18) one can reformulate it in the framework of the quasistatic approach (Chudzikiewicz et al. 2012). This approach is based on the main assumption that for the observer moving with a wheel with the constant linear velocity V its displacement does not depend on time. Moreover let us assume:

- (i) the length of the strip is much bigger than the radius of the wheel,
- (ii) for the observer moving with the wheel the displacement of the strip does not depend on time,
- (iii) the velocity of the wheel is small enough,
- (iv) the temperatures very soon approach steady-state values,
- (v) in the contact area the heat is generated due to friction and the heat flow rate is transformed completely into heat,

- (vi) the wear debris disappear immediately at the point where it is formed influencing the contact conditions by increasing the gap between the contacting bodies only.

Since we consider the rail which has finite length rather than infinite length assumption (i) is the minimal requirement to formulate displacement equation (2) as well as thermal equation (3) with initial and boundary conditions (4)-(18). Assumption (ii) is essential to transform the original contact problem into quasistatic one. The observer does not distinguish between points of the upper surface of the rail. Assumption (iii) is introduced to ensure the positive definiteness of the stiffness matrix, i.e., the existence of solutions to the contact problem. Remark this assumption imposes an upper bound on the admissible wheel velocity. However due to the application of the scaling technique, numerically wide range of wheel velocities appearing in the operational systems can be covered. Assumption (iv) means that the considered heat flow rate is not strongly dependent on time, i.e., is stationary or close to this state. Such assumption appear also in (Hiensh et al. 2005). However let us remark that in (Dohrman 2001), where strongly non steady heat flow is considered this assumption does not hold. Taking into account heat convection it leads also to more complicated and difficult finite element model to be solved. These elements motivates the assumption (iv). Assumption (v) states that all friction energy without dissipation is transformed into heat energy and is used to increase the temperature of rail only. Recall frictional heat energy may increase temperature of wheel as well as of rail (Hiensh et al. 2005, Chudzikiewicz et al. 2012). The last assumption simplifies the wear phenomenon. Contact models assuming the existence of intermediate layer of wear particles between contacting surfaces are still under development and it is not known whether they possess a solution.

To transform equations (2)-(18) into quasistatic form let us introduce the new Cartesian coordinate system $O'x'_1x'_2$ hooked in the middle of the wheel. The new system $O'x'_1x'_2$ and the original one Ox_1x_2 are related by:

$$x'_1 = x_1 - Vt, \quad x'_2 = x_2. \quad (19)$$

Since by assumption (ii) $u(x'_1, x'_2)$ does not depend on time we obtain:

$$\frac{\partial u}{\partial t}(x'_1, x'_2) = \frac{\partial u}{\partial t}(x_1 - Vt, x_2, t) = 0. \quad (20)$$

By elementary differentiation (19)-(20) imply

$$\frac{\partial u}{\partial t} = -V \frac{\partial u}{\partial x_1} \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = V^2 \frac{\partial^2 u}{\partial x_1^2}. \quad (21)$$

Using the same argument we obtain:

$$\frac{\partial \theta}{\partial t} = -V \frac{\partial \theta}{\partial x_1}, \quad \frac{\partial w}{\partial t} = -V \frac{\partial w}{\partial x_1}. \quad (22)$$

Let Ω denotes now the moving part of the strip seen by the observer. Taking into account (19) and using (21)-(22) we can transform system (2)-(18) into the quasistatic form. This problem has the following form: find $u = u_c, u_g, u_s$ and $\theta = \theta_c, \theta_g, \theta_s$ depending on spatial variables only satisfying displacement governing equations

$$A^*DAu - \rho V^2 \frac{\partial^2 u}{\partial x_1 \partial x_2} - \alpha(3\lambda + 2\gamma)\nabla\theta = 0 \quad \text{in } \Omega, \quad (23)$$

and temperature governing equations

$$-V \frac{\partial \theta}{\partial x_1} = \kappa \frac{\partial^2 \theta}{\partial x_2^2} \quad \text{in } \Omega, \quad (24)$$

where $\kappa(x_2) = \frac{\bar{\kappa}(x_2)}{\rho(x_2)c(x_2)}$ is thermal diffusivity coefficient depending in Ω_2 on x_2 according to (1). In Ω_1 and Ω_3 thermal diffusivity coefficient has distinct constant values. Let us transform the boundary conditions (4)-(18) into equivalent ones using (19)-(22). Strip clamped condition (4) takes the form

$$u_c = u_g = u_s = 0 \quad \text{and} \quad \theta_c = \theta_g = \theta_s = \theta_a \quad \text{on } \Gamma_0. \quad (25)$$

Displacement continuity conditions (15) and (18) become

$$u_c = u_g, \quad \theta_c = \theta_g \quad \text{on } \Omega_1 \cap \Omega_2. \quad (26)$$

$$u_g = u_s, \quad \theta_g = \theta_s \quad \text{on } \Omega_2 \cap \Omega_3.$$

In the contact zone the surface traction vector F in condition (5) is determined by

$$B^*D(x_2)Au_c = F \quad \text{on } \Gamma_C. \quad (27)$$

Contact and friction conditions (11)-(13) take the following equivalent form on Γ_C ,

$$u_{c2} + g_r + w \leq 0, \quad F_2 \leq 0, \quad (u_{c2} + g_r + w)F_2 = 0, \quad (28)$$

$$|F_1| \leq \mu |F_2|, \quad -F_1 \frac{\partial u_1}{\partial x_1} \leq 0, \quad (|F_1| - \mu |F_2|) \frac{\partial u_1}{\partial x_1} = 0. \quad (29)$$

The boundary heat flow condition (8) becomes

$$-\kappa \frac{\partial \theta}{\partial x_2} = \bar{q} \quad \text{on } \Gamma_C, \quad (30)$$

where

$$\bar{q} = \bar{\alpha} \left[\frac{\theta}{r} F_2 + (1 - \frac{k\rho c \theta}{\mu}) \mu V F_2 \right]. \quad (31)$$

In (31) $\bar{\alpha}$ represents the fraction of frictional heat flow rate entering the rail, τ is thermal resistance constant (Chudzikiewicz et al. 1992). Remark, the form of \bar{q} follows from the balance of heat energy along the contact interface. Under assumption (vi) the formed wear debris disappear immediately without interfering with contact conditions apart from changing the gap between the wheel and the rail. The disappearing wear debris are warm due to wear process as well as due to conduction from surrounding heated material. Term $\mu V F_2$ in (31) is the power of frictional forces. Power generated due to these forces is decreased by the power carried away as heat in loose wear particles. The dissipated wear debris power is assumed to be proportional to the wear rate and the temperature of the contacting surfaces. The first term in (31) describes the transfer of heat from the wear debris into rail by conduction. Here it is assumed that the contact resistance to heat flow is inversely proportional to the contact pressure. For detailed derivation of \bar{q} see references in (Chudzikiewicz et al. 2012).

Moreover wear condition (14) takes the form

$$\frac{\partial w}{\partial x_1} = -kF_2. \quad (32)$$

There are also given initial conditions (5), (6) where the given functions are assumed to be zero. The initial temperature of the rail is equal to the given temperature θ_a .

In order to solve numerically quasistatic system (23)-(32) and ensure the existence (Shillor et al. 2004) of solutions to this system we have to consider it as a problem with the prescribed friction. It means this problem has to be replaced by the regularized one. Let $\varepsilon > 0$ denotes a regularization parameter. We use in numerical algorithm the following formula relating tangential F_1 and normal F_2 tractions on the contact boundary Γ_C (Chudzikiewicz et al. 2011)

$$F_1 = F_1(\varepsilon, F_2, u_1) =$$

$$-\mu |F_2| \arctan\left[\frac{1}{\varepsilon} \left(V \frac{\partial u_1}{\partial x_1}\right)\right]. \quad (33)$$

Remark the proposed quasistatic approach based on the assumptions (i)-(vi) consists in replacing the time derivatives terms in equations (2)-(3) by the stationary terms depending on the wheel velocity and spatial derivatives of displacement or temperature. These terms still reflect the dynamics of the moving body rather than completely neglect it as in the classical quasistatic formulation (Sextro 2007). Therefore the original nonstationary system (2)-(14) is transformed into the stationary one (23)-(33).

4 NUMERICAL ALGORITHM

Problem (23)-(33) is a coupled thermoelastic problem. Remark, the contact traction depends on the thermal distortion of the bodies and wear process. On

the other hand, the amount of heat generated due to friction depends on the contact traction. The main solution strategies for coupled problems are global solution algorithms where the differential systems for the different variables are solved together or operator splitting methods. In this paper we employ operator split algorithm. The conceptual algorithm for solving quasistatic system (23)-(33) is as follows

- Step 1 : Choose $\theta = \theta^0$ and $w = w^0$. Choose $\eta \in (0, 1)$. Set $k = 0$.
- Step 2 : For given θ^k and w^k find u^k and F_2^k satisfying equation (23) with boundary conditions (25)-(29).
- Step 3 : For calculated u^k and F_2^k find θ^{k+1} as well as w^{k+1} satisfying equations (24) and (32), respectively, with boundary conditions (25), (26), (31).
- Step 4 : If $\|\theta^{k+1} - \theta^k\| \leq \eta$, Stop. Otherwise: set $k = k + 1$, go to Step 2.

The conceptual algorithm consists first in calculating for a given temperature field and wear the corresponding displacement and stress fields, i.e., in solving the mechanical subproblem. Next for the calculated displacement and stress fields we solve the thermal part of the system and calculate wear. The algorithm is terminated when the calculated temperature becomes steady, i.e., the temperature changes from iteration to iteration are less than the prescribed tolerance. The convergence of the operator split algorithm is shown using Fixed Point Theorem (see references in (Chudzikiewicz et al. 2012)).

Let us briefly present the algorithms for solving discrete mechanical and thermal subproblems. In order to solve the mechanical part of this system we introduce regularization of the friction conditions. The mechanical subproblem of the discretized contact problem is reformulated as a quadratic optimization problem in terms of tangent and normal contact tractions. In order to solve this auxiliary optimization problem one has to approximate inverse stiffness matrix of the discretized system. This matrix is calculated using collocation approach. Newton method is employed to calculate regularized tangent traction. Linearization based optimization method (Chudzikiewicz et al. 1992) is used to solve auxiliary quadratic optimization problem and to find tangent and normal tractions. Having calculated these tractions one can calculate by back substitution displacement and stresses in the whole strip as well as the wear. Next the thermal problem is solved using Choleski algorithm. For algorithm details concerning solving discrete mechanical and thermal subproblems see references (Han et al. 2002).

5 NUMERICAL RESULTS AND DISCUSSION

A series of simulations are conducted to calculate and to investigate, governed by system (23)-(33), the influence of the elastic grading on the stress and temperature distributions in the contact area. Polygonal domain Ω occupied by the rail has a form

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (-2, 2), x_2 \in (0, 1)\}. \quad (34)$$

The contact boundary is modeled by 39 segments. The coating layer, in zone, where the wheel - rail contact may occur has been covered with fine mesh while the coarse mesh has been used to divide the steel layer (Efendiev et al. 2009, Hou et al. 1999, Kim et al. 2002). The thickness of surface layer is made large enough to ensure that stresses in wheel-rail contact zone are not affected by the boundary between the fine and coarse meshes. Multi - point constraints are applied on the boundary between layers and coarse and fine meshes. The ratio $h_c/h_g = .25$ is chosen. The layer material properties are assembled in Table 1. The computations were performed for the non-homogeneity index n equal to 0.28, 0, -0.28 corresponding to ratio P_s/P_c equal to 1.32, 1, 0.75, respectively. $n < 0$ ($n > 0$) indicates that the surface coating layer is stiffer (softer) than the lower layer. $n = 0$ corresponds to the homogeneous case, i.e., all three layers have the same material parameters. Other data are as follows: the velocity $V = 10$ m/s, radius of the wheel $r_0 = 0.46$ m, the thermal resistance coefficient $r = 1000$ KNs/J, the wear constant $k = 0.5 \cdot 10^{-6}$ MPa $^{-1}$. The friction coefficient μ is equal to 0.45. The penetration of the wheel is taken as $\delta = 0.1 \cdot 10^{-3}$ m. The regularization parameter ε in (33) is set to 0.001. Functions \tilde{u}_{0j} and \tilde{u}_{1j} , for $j = "c", "g"$ or $j = "s"$, in (6) are selected as equal to 0. θ_0 is equal to 20° C and $\bar{\alpha}$ in (31) is equal to 1. The obtained distributions of stresses and temperatures in the contact area for different values of non-homogeneity index n are displayed on Fig. 3–11.

Normal contact pressure distribution is shown on Fig. 3. It attains maximal value in the middle of the contact zone. It is sensitive with respect to the change of the non-homogeneity index n . The decrease of the normal contact pressure maximal value is accompanied by the extension of the contact zone.

Fig. 4–6 display through-thickness stresses $\sigma_{22}, \sigma_{12}, \sigma_{11}$, respectively. These stresses attain maximum at a surface contact point and rapidly decrease inside the strip.

Fig. 7–9 display longitudinal stresses $\sigma_{22}, \sigma_{12}, \sigma_{11}$, respectively. These stresses attain maximum at a surface point inside the contact zone.

Fig 10–11 display the distribution of the temperature in the contact zone. The maximal temperature is observed in the contact area. The temperature is rapidly decreasing inside the strip and in front of the

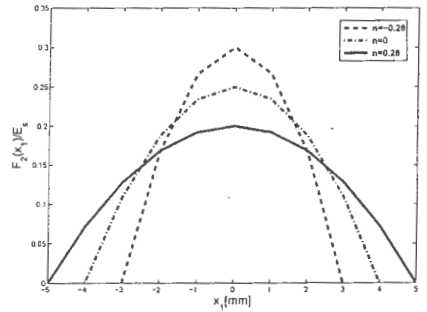


Figure 3: Normal contact pressure distribution as a function of n .

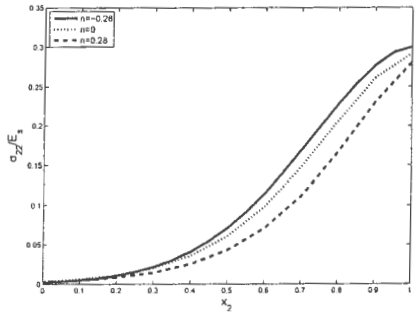


Figure 4: Through - thickness stress σ_{22} at $x_1 = 0$.

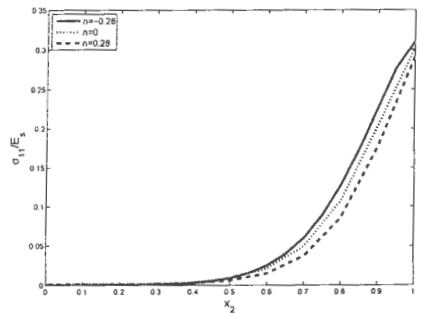


Figure 5: Through - thickness stress σ_{11} at $x_1 = 0$.

Table 1: Material properties of coating and steel layers.

Properties	Ω_1	Ω_3
Elastic modulus E (GPa)	151	200
Poisson ratio ν	0,24	0,30
density ρ (kg/m^3)	3260	7800
thermal conductivity $\bar{\kappa}$ ($W/(m \text{ deg}(C))$)	6,1	50
thermal diffusivity κ (m^2/s)	$0,06 \cdot 10^{-5}$	$1,44 \cdot 10^{-5}$
thermal expansion α ($\text{deg } C^{-1}$)	$10 \cdot 10^{-6}$	$12 \cdot 10^{-6}$

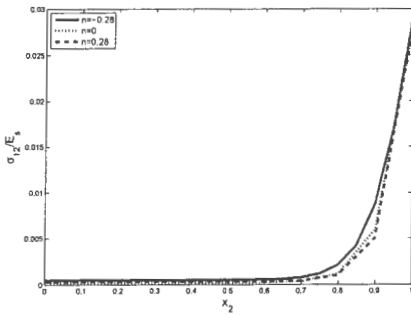


Figure 6: Through - thickness stress σ_{12} at $x_1 = 0$.

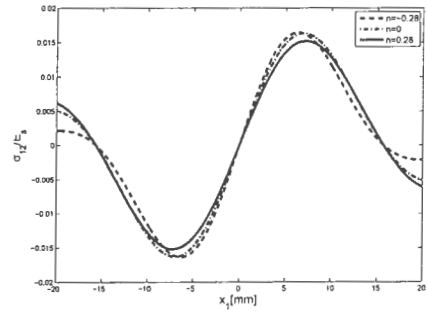


Figure 9: Longitudinal stress σ_{12} on $x_2 = 1$.

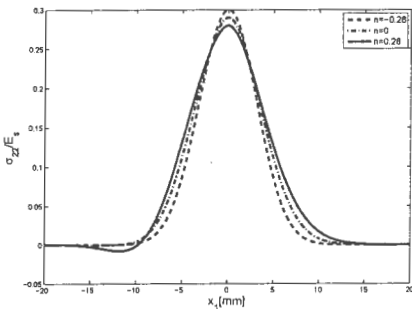


Figure 7: Longitudinal stress σ_{22} on $x_2 = 1$.

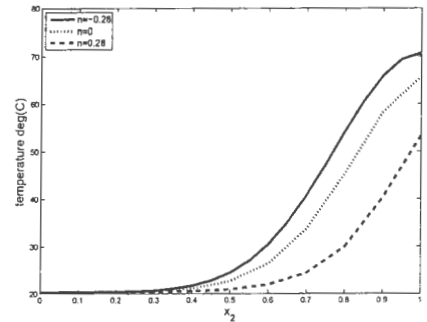


Figure 10: Rail temperature distribution at $x_1 = 0$ (along x_2 direction).

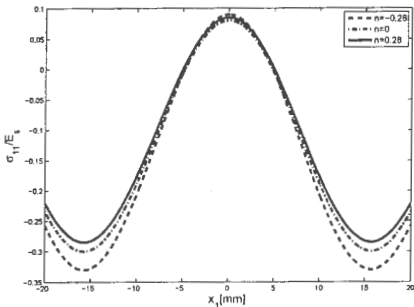


Figure 8: Longitudinal stress σ_{11} on $x_2 = 1$.

wheel. Behind the wheel the decrease of temperature is mild.

6 CONCLUSIONS

The applied exponential model of the graded material allows to control the normal contact pressure, temperature and the size of the contact area. The obtained numerical results seems to be in accordance with physical reasoning. The normal traction F_2 has its peak in the middle of the contact area. The quasi-static approach allows also to observe dynamic phenomena associated with the rolling wheel (for details see (Chudzikiewicz et al. 1992))). The decrease in the non-homogeneity index n reduces the maximum

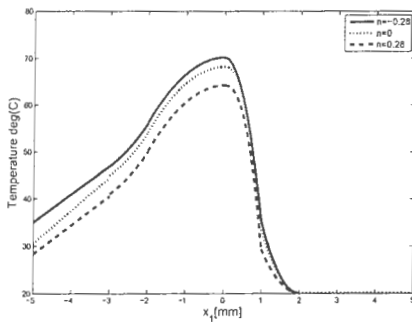


Figure 11: Temperature distribution along the contact interface at $x_2 = 1$ (along x_1 direction).

normal contact pressure and temperature at a cost of the widening of the contact area. The relationship between the applied normal load and the size of the contact zone is nonlinear. The designing and the analysis of functionally graded materials is also considered in the framework of topology and/or material optimization (see (Paulino et al. 2009)). These methods allow to investigate the sensitivity of the displacement or the temperature fields with respect to the perturbation of the material parameters.

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the 1990s, the number of people in the world who are illiterate has increased from 1.2 billion to 1.5 billion (UNESCO, 2003).

There are many reasons for the increase in illiteracy. One of the reasons is that the population of the world is increasing rapidly. Another reason is that the number of people who are unable to read and write is increasing. This is because many people do not have access to education. In many countries, especially in the developing world, there are many children who do not go to school. This is because their parents cannot afford to send them to school. In addition, many children are forced to work to help support their families. This means that they do not have time to go to school.

Another reason for the increase in illiteracy is that many people do not have access to books and other educational materials. In many countries, there are very few libraries and very few books. This means that people do not have the opportunity to read and learn. In addition, many people do not have access to the internet. This means that they do not have access to online educational resources.

There are many ways to reduce the number of illiterate people in the world. One way is to increase access to education. This can be done by building more schools and hiring more teachers. Another way is to provide more books and other educational materials. This can be done by building more libraries and providing more books. In addition, it is important to provide access to the internet. This can be done by building more internet cafes and providing more internet access.

It is important to note that reducing the number of illiterate people in the world is not just a matter of providing education. It is also a matter of providing access to other resources that are necessary for learning. For example, many people do not have access to electricity. This means that they do not have access to many educational resources that require electricity. In addition, many people do not have access to clean water. This means that they do not have access to many educational resources that require clean water.

There are many other ways to reduce the number of illiterate people in the world. For example, it is important to provide access to health care. This is because many people are unable to learn because they are sick. In addition, it is important to provide access to food. This is because many people are unable to learn because they are hungry. In addition, it is important to provide access to housing. This is because many people are unable to learn because they do not have a place to live.

There are many ways to reduce the number of illiterate people in the world. It is important to provide access to education, books, and other educational materials. It is also important to provide access to other resources that are necessary for learning, such as electricity, clean water, health care, food, and housing. By providing access to these resources, we can help to reduce the number of illiterate people in the world and improve the lives of many people.