905.

ON THE EQUATION $x^{17}-1=0$.

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WRITING $\rho=\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{17}$, I carry the solution up to the determination of the periods each of two roots, $\rho+\rho^{16}$, $=2\cos\frac{2\pi}{17}$, &c. The expressions contain the radicals

$$a=\sqrt{(17)},\quad b=\sqrt{\{2(17-a)\}},\quad c=\sqrt{\{4(17+3a)-2(3+a)b\}},$$

where a, b, c are taken to be positive ($a=4{\cdot}12$, $b=5{\cdot}07$, $c=6{\cdot}72$). Taking for a moment r to be any imaginary seventeenth root, $r=\rho^\theta$, then the algebraical expression for the period P_1 of eight roots is $P_1=\frac{1}{2}(-1\pm a)$, but I assume the value to be $P_1=\frac{1}{2}(-1+a)$, and thus determine θ to denote some one of the values 1, 2, 4, 8, 9, 13, 15, 16; similarly, I assume the value of Q_1 to be $=\frac{1}{4}(-1+a+b)$; and thus further determine θ to denote some one of the values 1, 4, 13, 16: and, again, I assume the value of R_1 to be $=\frac{1}{8}(-1+a+b+c)$, and thus further determine θ to denote one of the values 1 and 16. As regards the values of the periods R, it is obviously indifferent which value is taken, and I assume therefore $\theta=1$. This comes to saying that the signs of the radicals are determined in suchwise that r shall denote the root $\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{17}$; and it is to be understood that r has this value.

I write now

$$\begin{aligned}
P_1&=r+r^9+r^{13}+r^{15}+r^{16}+r^8+r^4+r^2,\\
P_2&=r^3+r^{10}+r^5+r^{11}+r^{14}+r^7+r^{12}+r^6,\\
Q_1&=r+r^{13}+r^{16}+r^4,\\
Q_2&=r^9+r^{15}+r^8+r^2,\\
Q_3&=r^3+r^5+r^{14}+r^{12},\\
Q_4&=r^{10}+r^{11}+r^7+r^6,\\
R_1&=r+r^{16},\\
R_2&=r^{13}+r^4,\\
R_3&=r^9+r^8,
\end{aligned}$$

$$R_4 = r^{15} + r^2,$$
$$R_5 = r^3 + r^{14},$$
$$R_6 = r^5 + r^{12},$$
$$R_7 = r^{10} + r^7,$$
$$R_8 = r^{11} + r^6,$$

and moreover

$$a = P_1 - P_2,$$
$$b = 2(Q_1 - Q_2),$$
$$+b_1 = 2(Q_3 - Q_4),$$
$$c = 4(R_1 - R_2),$$
$$-c_1 = 4(R_3 - R_4),$$
$$+c_2 = 4(R_5 - R_6),$$
$$-c_3 = 4(R_7 - R_8).$$

It will appear by what follows that a is determined by the quadric equation $a^2=17$, but I have assumed that a denotes the positive root $a=\sqrt{(17)}$; similarly, b is determined by the quadric equation $b^2=2(17-a)$, but it is assumed that b denotes the positive root, $b=\sqrt{\{2(17-a)\}}$; and c is determined by the quadric equation $c^2=4(17+3a)-2(3+a)b$, but it is assumed that c denotes the positive root,

$$c=\sqrt{\{4(17+3a)-2(3+a)b\}}.$$

If in the equations I had written b_1, c_1, c_2, c_3, instead of $+b_1$, $-c_1$, $+c_2$, $-c_3$, then b_1 comes out rationally in terms of a, b; and c_1, c_2, c_3 come out rationally in terms of a, b, c; the signs were attached to them *à posteriori*, in suchwise that the values of b_1, c_1, c_2, c_3 might be each of them positive; for their independent determination, we have, in fact, for b_1^2 an expression such as that for b^2; and for c_1^2, c_2^2, c_3^2 expressions such as that for c^2; and taking as above for each of them the positive value of the square root, we have

$$a = \sqrt{(17)}, \qquad (=4{\cdot}12),$$
$$b = \sqrt{\{2(17-a)\}}, \qquad (=5{\cdot}07),$$
$$b_1 = \sqrt{\{2(17+a)\}}, \qquad (=6{\cdot}49),$$
$$c = \sqrt{\{4(17+3a)-2(3+a)b\}}, \qquad (=6{\cdot}72),$$
$$c_1 = \sqrt{\{4(17+3a)+2(3+a)b\}}, \qquad (=13{\cdot}77),$$
$$c_2 = \sqrt{\{4(17-3a)+2(-3+a)b_1\}}, \qquad (=5{\cdot}75),$$
$$c_3 = \sqrt{\{4(17-3a)-2(-3+a)b_1\}}, \qquad (=2{\cdot}02).$$

The relations between the periods P are

$$P_1 + P_2 = -1,$$

	P_1	P_2
P_1	$-P_1+4$	-4
P_2		$-P_2+4$

	a
a	17

that is,

$$P_1^2 = -P_1 + 4, \text{ \&c.}; \quad a^2 = 17.$$

Here for the second square, we have

$$a^2 = (P_1 - P_2)^2 = P_1^2 + P_2^2 - 2P_1P_2 = (-P_1 + 4) + (-P_2 + 4) + 8, \; = -P_1 - P_2 + 16, \; = 17;$$

and similarly in the other cases which follow.

For the periods Q, we have

$$Q_1 + Q_2 = P_1,$$
$$Q_3 + Q_4 = P_2,$$

	Q_1	Q_2	Q_3	Q_4
Q_1	$Q_2 + 2Q_3 + 4$	-1	$2Q_1 + Q_2 + Q_4$	$Q_2 + Q_3 + 2Q_4$
Q_2		$Q_1 + 2Q_4 + 4$	$Q_1 + 2Q_3 + 4$	$Q_1 + 2Q_2 + Q_3$
Q_3			$2Q_2 + Q_4 + 4$	-1
Q_4				$2Q_1 + Q_3 + 4$

	b	b_1
b	$34 - 2a$	$8a$
b_1		$34 + 2a$

so that $bb_1 = 8a$, or b_1 is given rationally in terms of a, b.

And for the periods R, we have

$$R_1 + R_2 = Q_1,$$
$$R_3 + R_4 = Q_2,$$
$$R_5 + R_6 = Q_3,$$
$$R_7 + R_8 = Q_4,$$

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
R_1	$R_4 + 2$	$\overline{R_5 + R_6}$	$R_3 + R_7$	$R_1 + R_5$	$R_2 + R_4$	$R_2 + R_8$	$R_3 + R_8$	$R_6 + R_7$
R_2		$R_3 + 2$	$R_2 + R_6$	$R_4 + R_8$	$R_1 + R_7$	$R_1 + R_3$	$R_5 + R_8$	$R_4 + R_7$
R_3			$R_1 + 2$	$\overline{R_7 + R_8}$	$R_6 + R_8$	$R_2 + R_5$	$R_1 + R_4$	$R_4 + R_5$
R_4				$R_2 + 2$	$R_1 + R_6$	$R_5 + R_7$	$R_3 + R_6$	$R_2 + R_3$
R_5					$R_8 + 2$	$\overline{R_3 + R_4}$	$R_2 + R_7$	$R_3 + R_5$
R_6						$R_7 + 2$	$R_4 + R_6$	$R_1 + R_8$
R_7							$R_5 + 2$	$\overline{R_1 + R_2}$
R_8								$R_3 + 2$

	c	c_1	c_2	c_3
c	$\overline{4(17+3a)-2(3+a)b}$	$8(b+b_1)$	$\overline{4(2a-b+b_1)}$	$\overline{4(-2a+b+b_1)}$
c_1		$\overline{4(17+3a)+2(3+a)b}$	$4(2a+b+b_1)$	$\overline{4(2a+b-b_1)}$
c_2			$\overline{4(17-3a)+2(-3+a)b_1}$	$8(-b+b_1)$
c_3				$4(17-3a)-2(-3+a)b_1$

where observe that the overlined terms R_5+R_6, R_7+R_8, R_3+R_4, and R_1+R_2, have the values Q_4, Q_3, Q_2, Q_1, respectively, and in the last table b_1 may be considered as denoting its value $=\frac{8a}{b}$, so that c_1, c_2, c_3 are each given rationally in terms of a, b, c.

And from the foregoing results, we have

$$\begin{aligned}
P_1 &= \tfrac{1}{2}(-1+a), & &= 1{\cdot}56,\\
P_2 &= \tfrac{1}{2}(-1-a), & &= -2{\cdot}56,\\
Q_1 &= \tfrac{1}{4}(-1+a+b), & &= 2{\cdot}05,\\
Q_2 &= \tfrac{1}{4}(-1+a-b), & &= -0{\cdot}49,\\
Q_3 &= \tfrac{1}{4}(-1-a+b_1), & &= 0{\cdot}34,\\
Q_4 &= \tfrac{1}{4}(-1-a-b_1), & &= -2{\cdot}90,\\
R_1 &= \tfrac{1}{8}(-1+a+b+c), & &= 1{\cdot}87,\\
R_2 &= \tfrac{1}{8}(-1+a+b-c), & &= 0{\cdot}18,\\
R_3 &= \tfrac{1}{8}(-1+a-b-c_1), & &= -1{\cdot}96,\\
R_4 &= \tfrac{1}{8}(-1+a-b+c_1), & &= 1{\cdot}47,\\
R_5 &= \tfrac{1}{8}(-1-a+b_1+c_2), & &= 0{\cdot}89,\\
R_6 &= \tfrac{1}{8}(-1-a+b_1-c_2), & &= -0{\cdot}55,\\
R_7 &= \tfrac{1}{8}(-1-a-b_1-c_3), & &= -1{\cdot}70,\\
R_8 &= \tfrac{1}{8}(-1-a-b_1+c_3), & &= -1{\cdot}20.
\end{aligned}$$

The approximate numerical values have been given throughout only for the purpose of showing that the signs of the square roots have been rightly determined.