910.

NOTE ON THE INVOLUTANT OF TWO BINARY MATRICES.

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CONSIDER the two matrices

$$M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}, \quad M' = \begin{pmatrix} a', & b' \\ c', & d' \end{pmatrix},$$

and their product in one or the other order

$$MM' = \begin{pmatrix} A, & B \\ C, & D \end{pmatrix}, \quad M'M = \begin{pmatrix} A_1, & B_1 \\ C_1, & D_1 \end{pmatrix}.$$

Then the Involutant is by definition = either of the determinants

$$I = \begin{vmatrix} 1, & a, & a', & A \\ 0, & b, & b', & B \\ 0, & c, & c', & C \\ 1, & d, & d', & D \end{vmatrix}, \qquad I_1 = \begin{vmatrix} 1, & a', & a, & A_1 \\ 0, & b', & b, & B_1 \\ 0, & c', & c, & C_1 \\ 1, & d', & d, & D_1 \end{vmatrix};$$

viz. it is to be shown that these two values are in fact equal.

We have

$$MM' = \begin{array}{c|cc|} & (a',\ c') & (b',\ d') \\ \hline (a,\ b) & ,, & ,, \\ (c,\ d) & ,, & ,, \end{array} = \begin{pmatrix} A, & B \\ C, & D \end{pmatrix},$$

that is,

$$A = aa' + bc', \quad B = ab' + bd',$$
$$C = ca' + dc', \quad D = cb' + dd',$$

and similarly

$$M'M = \begin{matrix} \\ (a',\ b') \\ (c',\ d') \end{matrix} \left| \begin{matrix} (a,\ c)\ (b,\ d) \\ \text{,,} \quad \text{,,} \\ \text{,,} \quad \text{,,} \end{matrix} \right. = \begin{pmatrix} A_1, & B_1 \\ C_1, & D_1 \end{pmatrix},$$

that is,

$$A_1 = aa' + cb', \quad B_1 = ba' + db',$$
$$C_1 = ac' + cd', \quad D_1 = bc' + dd',$$

viz. A_1, B_1, C_1, D_1 are obtained from A, B, C, D by the interchange of the accented and unaccented letters.

We have then, from the first expression for the Involutant,

$$I = A \begin{vmatrix} 0, & b, & b' \\ 0, & c, & c' \\ 1, & d, & d' \end{vmatrix} - B \begin{vmatrix} 0, & c, & c' \\ 1, & d, & d' \\ 0, & b, & b' \end{vmatrix} + C \begin{vmatrix} 1, & d, & d' \\ 1, & a, & a' \\ 0, & b, & b' \end{vmatrix} - D \begin{vmatrix} 1, & a, & a' \\ 0, & b, & b' \\ 0, & c, & c' \end{vmatrix},$$

$$= A\,(bc' - b'c) - B\,(ac' - a'c + cd' - c'd) + C\,(ab' - a'b + bd' - b'd) - D\,(bc' - b'c),$$

or substituting for $A - D$, B and C their values, this is

$$(aa' - dd' + bc' - b'c)(bc' - b'c) - (ab' + bd')(ac' - a'c + cd' - c'd)$$
$$+ (ca' + dc')(ab' - a'b + bd' - b'd);$$

and multiplying out and grouping together the terms in bc, $b'c'$, bc' and $b'c$, this is found to be

$$= -(a' - d')^2\, bc + (a - d)(a' - d')(bc' + b'c) - (a - d)^2\, b'c' + (bc' - b'c)^2,$$

which is

$$= -\{(a - d)\,b' - (a' - d')\,b\}\,\{(a - d)\,c' - (a' - d')\,c\} + (bc' - b'c)^2.$$

Hence, writing

$$\mathrm{a} = bc' - b'c, \quad \mathrm{f} = ad' - a'd,$$
$$\mathrm{b} = ca' - c'a, \quad \mathrm{g} = bd' - b'd,$$
$$\mathrm{c} = ab' - a'b, \quad \mathrm{h} = cd' - c'd,$$

we have

$$I = -(\mathrm{c} + \mathrm{g})(-\mathrm{b} + \mathrm{h}) + \mathrm{a}^2,$$

that is,

$$I = \mathrm{bc} - \mathrm{ch} + \mathrm{bg} - \mathrm{gh} + \mathrm{a}^2.$$

To obtain the value of I_1, we must interchange the accented and unaccented letters, that is, change the signs of the several quantities a, b, c, f, g, h; but I, being a quadric function of the six quantities, is not altered by the change; that is, we have $I = I_1$.