913.

ON THE EPITROCHOID.

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If we have a curve C_1 rolling on a fixed curve C, and consider the epitrochoid described by a point P, attached to and carried along with the curve C_1, then there is a known construction for the radius of curvature at any point of the epitrochoid; the construction is probably much older, but I refer to Mannheim, *Géométrie Descriptive*, Paris, 1886, pp. 177 and 194. In fact, if Q be a position of the point of contact, P_1 the corresponding position of the describing point, R_1 and R the radii of curvature at Q of the two curves respectively, ρ the distance QP_1, and ϕ the inclination of this distance to the common normal at Q, then the radius of curvature x at the point P_1 of the epitrochoid is given by the formula

$$\frac{x}{x-\rho} = \frac{\rho}{\cos\phi}\left(\frac{1}{R} + \frac{1}{R_1}\right).$$

I prove this as follows: take Q, Q' consecutive positions of the point of contact, P_1, P_1' consecutive positions of P_1; the centre of curvature is the intersection of the lines P_1Q, $P_1'Q'$; hence, if for a moment M and N are the perpendicular distances between these two lines at the points P_1 and Q respectively, we have $\dfrac{x}{x-\rho} = \dfrac{M}{N}$. Let ds be the element QQ' considered as belonging to each of the two curves respectively, θ and θ_1 the angles which this element subtends at the two centres of curvature respectively; we have $ds = R\theta = R_1\theta_1$, whence

$$\theta + \theta_1 = ds\left(\frac{1}{R} + \frac{1}{R_1}\right).$$

The instantaneous centre is P_1, and hence

$$M = P_1P_1' = \rho(\theta + \theta_1);\ \text{also}\ N = ds\cos\phi;$$

we thus have

$$\frac{x}{x-\rho}=\frac{\rho\,(\theta+\theta_1)}{ds\cos\phi};$$

or, substituting for $\theta+\theta_1$ the foregoing value, we have the required expression

$$\frac{x}{x-\rho}=\frac{\rho}{\cos\phi}\left(\frac{1}{R}+\frac{1}{R_1}\right).$$

If the curve C_1 roll on a straight line, $R=\infty$ and therefore

$$\frac{x}{x-\rho}=\frac{\rho}{R_1\cos\phi}.$$

Suppose that at any instant we have on the epitrochoid an inflexion, then $x=\infty$, and we have $\rho=R_1\cos\phi$; this denotes that P_1 is a point of the circle described on the radius of curvature at Q as its diameter, and we have thus the theorem:

If at any instant we consider the circle, which has for its diameter the radius of curvature of the rolling curve at its point of contact with the right line, then each point on this circle is an inflexion on the epitrochoid described by the same point as the curve rolls on the straight line.

The circle in question for any point Q of the rolling curve C_1 is of course a circle, having its centre on the normal at Q and its radius equal to one-half of the radius of curvature; we may call it the circle of double curvature. Regarding Q as a given point on the curve C_1, and supposing that the circle of double curvature intercepts the curve C_1 in a point X, then taking P_1 at X, we have the theorem:

If the curve C_1 roll on a straight line, then the epitrochoid described by the point X has an inflexion corresponding to the position of the rolling curve for which the point Q comes into contact with the straight line.

Consider the curve C_1 referred to axes fixed in the plane of the curve and moveable with it; take x, y as the coordinates of the point Q; α, β as the coordinates of the corresponding centre of curvature, and X, Y as current coordinates; the equation of the circle of double curvature is

$$(X-x)(X-\alpha)+(Y-y)(Y-\beta)=0,$$

viz. this is the equation of a circle having for a diameter the two points (x, y) and (α, β).

Suppose that C_1 is a conic; the circle of double curvature at any point Q meets it in the point Q counting twice, and in two other points, say P_1, P_2, which are determined as the intersections of the conic by the line P_1P_2 which joins them. I propose to determine the equation of this line. The investigation would be more symmetric for the ellipse, but I nevertheless prefer to consider for the curve C_1 the hyperbola $b^2X^2-a^2Y^2-a^2b^2=0$.

Representing for a moment the equation of the circle by $U=0$, and that of the hyperbola by $V=0$, it must be possible to determine the ratio of the coefficients p, q, in such wise that $pU+qV=0$ shall break up into a pair of lines, one of which is the tangent at Q of the hyperbola; and then the other of them will be the required line P_1P_2.

The equation of the tangent is $b^2xX-a^2yY-a^2b^2=0$; we then see that the equation of the line P_1P_2 must be of the form $b^2xX+a^2yY+\Omega=0$; and we have to find p, q, Ω so as to verify the identity

$$(p+qb^2)X^2+(p-qa^2)Y^2-p(\alpha+x)X-p(\beta+y)Y+p(\alpha x+\beta y)-q\alpha^2\beta^2$$
$$=(b^2xX-a^2yY-a^2b^2)(b^2xX+a^2yY+\Omega);$$

that is, we ought to have

$$p+qb^2=b^4x^2,$$
$$p-qa^2=a^4y^2,$$
$$p(x+\alpha)=(\Omega-a^2b^2)b^2x,$$
$$p(y+\beta)=(\Omega+a^2b^2)a^2y,$$
$$qa^2b^2-p(\alpha x+\beta y)=\Omega a^2b^2.$$

It will be recollected that α, β denote the coordinates of the centre of curvature corresponding to the point (x, y) of the hyperbola; their values thus are

$$\alpha=\quad(a^2+b^2)\frac{x^3}{a^4},$$
$$\beta=-(a^2+b^2)\frac{y^3}{b^4}.$$

There is no difficulty in finding the values

$$p=\frac{a^4b^4}{a^2+b^2},\quad q=\frac{b^4x^2+a^4y^2}{a^2+b^2},$$

whence also

$$q-a^2b^2=\frac{a^4b^2}{a^2+b^2}\left\{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)y^2-1\right\},$$
$$q+a^2b^2=\frac{a^2b^4}{a^2+b^2}\left\{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)x^2+1\right\},$$
$$\Omega=-a^2b^2\frac{x^2+y^2}{a^2+b^2};$$

these satisfy the identities in question.

The equation of the line P_1P_2 is thus

$$b^2xX+a^2yY-\frac{a^2b^2}{a^2+b^2}(x^2+y^2)=0,$$

and combining this with the equation

$$b^2X^2-a^2Y^2-a^2b^2=0$$

of the hyperbola, we find

$$\left\{xX - \frac{a^2}{a^2+b^2}(x^2+y^2)\right\}^2 = \frac{a^4y^2Y^2}{b^4}, = \frac{a^2y^2}{b^2}(X^2-a^2),$$

that is,

$$X^2\left(x^2 - \frac{a^2y^2}{b^2}\right) - \frac{2a^2x}{a^2+b^2}(x^2+y^2)X + \frac{a^4}{(a^2+b^2)^2}(x^2+y^2)^2 + \frac{a^4y^2}{b^2} = 0,$$

viz.

$$X^2 - 2\frac{x^2+y^2}{a^2+b^2}xX + a^2\left\{\left(\frac{x^2+y^2}{a^2+b^2}\right)^2 + \frac{y^2}{b^2}\right\} = 0,$$

and we thence obtain

$$X = \frac{x^2+y^2}{a^2+b^2}x \pm \frac{ay}{b}\sqrt{\left\{\left(\frac{x^2+y^2}{a^2+b^2}\right)^2 - 1\right\}},$$

$$Y = -\frac{x^2+y^2}{a^2+b^2}y \pm \frac{bx}{a}\sqrt{\left\{\left(\frac{x^2+y^2}{a^2+b^2}\right)^2 - 1\right\}},$$

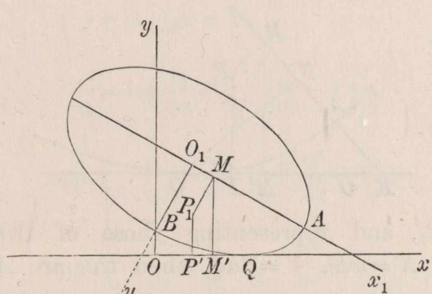
values belonging to the points P_1, P_2 respectively: these points are thus real or imaginary, according as x^2+y^2 is greater or less than a^2+b^2. In the limiting case $x^2+y^2=a^2+b^2$, we have $X=x$, $Y=-y$, viz. here the circle of double curvature has its centre on the axis of x, and has double contact with the hyperbola at the points (x, y) and $(x, -y)$ respectively.

Suppose, to fix the ideas, x and y are both positive, i.e. that Q is on the right-hand upper half branch of the hyperbola; and that $x^2+y^2=$ or $>a^2+b^2$, so that P_1, P_2 are both real. The two values of X are both positive, those of Y are originally both negative, and as Q moves along the half branch of the hyperbola one of these values increases negatively, the other increases positively until it becomes $=0$, after which it becomes positive and continually increases: say the points P_1, P_2 are originally on the right-hand lower half branch, but one of them, say P_1, moves up to the vertex, and we afterwards have P_1 on the upper half branch and P_2 on the lower half branch. The limiting case is when P_1 is at the vertex; here $\frac{x^2+y^2}{a^2+b^2}=\frac{x}{a}$, giving for (x, y) a real point, which is easily constructed on the upper half branch. There is thus on the half branch a real position of Q, such that the corresponding circle of double curvature passes through the vertex of the same half branch.

Hence in the epitrochoid described by the vertex of a hyperbola rolling on a right line there is always an inflexion.

Imagine the branch $B'AB$ of a hyperbola rolling on the line $y=0$, where A is the vertex, B', B the points at infinity, and so that as x increases positively (that is, towards the right) the point of contact Q passes from A towards B; and consider the epitrochoid described by the vertex A. Supposing that the point of contact is at first at A; the point A is a cusp on the epitrochoid, the motion of A is at first vertically upwards, and towards the right, the curve being concave to the axis; we then come to a position of the point of contact for which there is an inflexion; the motion of A is still upwards and towards the right, but the curve has now become convex to the axis, and the motion continues in this manner as the point of contact moves along the hyperbola towards B; when the point of contact is at B, i.e. when the asymptote has come to coincide with the axis, then the motion of A is again vertically upwards; the other half branch of the hyperbola now comes into contact with the axis; the point A still moves upwards but towards the left, the curve being concave to the axis, and this continues until the point of contact arrives at the other vertex (i.e. the vertex on the other branch), when the motion of A is horizontal and towards the left, viz. the coordinate y has here a maximum value, equal transverse axis of the hyperbola; the motion of A is thenceforth towards the left and downwards, until the other asymptote comes to coincide with the curve, after which it is downwards and towards the right (the point of contact being now on the half $B'A$ of the original branch $B'AB$), and ultimately the point of contact is again at A, and we have $y=0$, a cusp; the curve is symmetrical in regard to the before-mentioned maximum ordinate: it is to be noticed that the distance between the two cusps is equal to four times the difference between the length of the half asymptote and the half branch of the hyperbola (this difference being a finite value expressible by elliptic functions), and that, as already mentioned, the maximum ordinate is equal to the transverse axis. The above described portion of the curve is of course repeated continually right and left *ad infinitum*; two consecutive portions intersect each other, and there is thus a real node vertically above each cusp of the infinite curve. We have, in what precedes, a complete explanation of the motion of a hyperbola rolling on a straight line, and in particular our explanation of the form of the epitrochoid described by a vertex; the epitrochoid described by any other point of the curve is of a similar character but must be non-symmetrical in regard to the maximum ordinate; and it would not be difficult to explain the form of the curve for other positions of the describing point; but as regards the analytical theory, I prefer to consider the case of the ellipse.

In the case of the ellipse $b^2X^2+a^2Y^2-a^2b^2=0$ rolling on the line $y=0$ (see figure), take as origin the point O which comes in contact with the extremity B of



the minor axis; let x_1, y_1 be the coordinates of the describing point P_1, and for the point Q, taking ϕ as the complement of the eccentric angle, let the coordinates referred to the axes of the ellipse be

$$x_1 = a\sin\phi, \quad y_1 = b\cos\phi;$$

the arc BQ is then

$$= \int_0^\phi \sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}\, d\phi,$$

which is $= aE\phi$, the modulus being the eccentricity e of the ellipse; we then have for the coordinates of P_1

$$x = OP' = OQ - QP', \quad = BQ - QP',$$
$$y \quad = \quad P_1P',$$

where QP' and P_1P' are the projections of P_1Q in the direction of, and at right angles to, the tangent QO at Q; we find the values without difficulty, and we thus have

$$x = \int_0^\phi \sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}\, d\phi - \frac{(a^2 - b^2) \sin \phi \cos \phi - ax_1 \cos \phi + by_1 \sin \phi}{\sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}},$$

$$y = \frac{ab - bx_1 \sin \phi - ay_1 \cos \phi}{\sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}};$$

which are the expressions for x, y in terms of a variable parameter ϕ.

In particular, for the point B at the extremity of the minor axis, $x_1 = 0$, $y_1 = b$, and the equations become

$$x = \int_0^\phi \sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}\, d\phi - \frac{(a^2 - b^2) \sin \phi \cos \phi + b^2 \sin \phi}{\sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}},$$

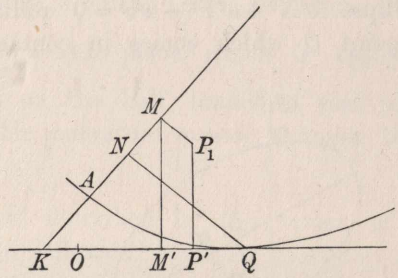
$$y = \frac{ab\,(1 - \cos \phi)}{\sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}},$$

which, for $a = b$, become

$$x = a\,(\phi - \sin \phi), \quad y = a\,(1 - \cos \phi),$$

the ordinary formulæ for the cycloid.

In the case of the parabola $Y^2 = 4aX$ (see figure), taking x_1, y_1 for the coordinates



of the describing point P_1, and representing those of the point of contact Q by means of the parameter ϕ, $X = a\phi^2$, $Y = 2a\phi$, then the arc AQ is

$$= a\,[\phi \sqrt{(\phi^2 + 1)} + \log \{\phi + \sqrt{(\phi^2 + 1)}\}],$$

so that, taking the origin at the point O, which comes into contact with the vertex A, this is also the value of OQ, and we then have

$$x = OQ - QP',$$
$$y = P_1P'.$$

Producing the axis of the parabola to meet the line $y=0$ in k, we have

$$AK = AN = a\phi^2,$$

and thence $KM = a\phi^2 + x_1$; and then, if θ be the inclination AKQ of the axis of the parabola,

$$KM' = (a\phi^2 + x_1)\cos\theta, \quad MM' = (a\phi^2 + x_1)\sin\theta;$$

and thence also

$$KP' = (a\phi^2 + x_1)\cos\theta + y_1\sin\theta, \quad P_1P' = (a\phi^2 + x_1)\sin\theta - y_1\cos\theta;$$

also

$$KQ = 2a\phi^2\sec\theta,$$

and thence

$$QP' = 2a\phi^2\sec\theta - (a\phi^2 + x_1)\cos\theta - y_1\sin\theta.$$

We have $\tan\theta = \dfrac{1}{\phi}$, and thence

$$\sin\theta = \frac{1}{\sqrt{(\phi^2+1)}}, \quad \cos\theta = \frac{\phi}{\sqrt{(\phi^2+1)}}, \quad \sec\theta = \frac{\sqrt{(\phi^2+1)}}{\phi};$$

hence

$$QP' = 2a\phi\sqrt{(\phi^2+1)} - \frac{(a\phi^2 + x_1)\phi + y_1}{\sqrt{(\phi^2+1)}} = \frac{a(\phi^3 + 2\phi) - x_1\phi - y_1}{\sqrt{(\phi^2+1)}},$$

$$P_1P' = \frac{a\phi^2 + x_1 - y_1\phi}{\sqrt{(\phi^2+1)}},$$

and the values of x, y thus become

$$x = a\left[\phi\sqrt{(\phi^2+1)} + \log\{\phi + \sqrt{(\phi^2+1)}\}\right] - \frac{a(\phi^3 + 2\phi) - x_1\phi - y_1}{\sqrt{(\phi^2+1)}},$$

that is,

$$x = a\log\{\phi + \sqrt{(\phi^2+1)}\} + \frac{(x_1 - a)\phi + y_1}{\sqrt{(\phi^2+1)}},$$

$$y = \frac{a\phi^2 + x_1 - \phi y_1}{\sqrt{(\phi^2+1)}}.$$

In particular, for the focus of the parabola, $x_1 = a$, $y_1 = 0$, and the equations become

$$x = a\log\{\phi + \sqrt{(\phi^2+1)}\},$$

$$y = a\sqrt{(\phi^2+1)},$$

that is,

$$\frac{x}{a} = \log\left\{\frac{y}{a} + \sqrt{\left(\frac{y^2}{a^2} - 1\right)}\right\},$$

or say

$$\exp.\left(\frac{x}{a}\right) = \frac{y}{a} + \sqrt{\left(\frac{y^2}{a^2} - 1\right)};$$

this gives

$$\exp.\left(-\frac{x}{a}\right) = \frac{y}{a} - \sqrt{\left(\frac{y^2}{a^2} - 1\right)};$$

and we have therefore

$$y = \tfrac{1}{2}a\left\{\exp.\left(\frac{x}{a}\right) + \exp.\left(-\frac{x}{a}\right)\right\},$$

viz. as is well known, the epitrochoid described by the focus of the parabola is the catenary.