921.

ON SOME PROBLEMS OF ORTHOMORPHOSIS.

[From *Crelle's Journal der Mathem.*, t. CVII. (1891), pp. 262—277.]

IN the interesting Memoir, Schwarz, "Ueber einige Abbildungsaufgaben," *Crelle*, t. LXX. (1869), pp. 105—120, the author considers the orthomorphic transformation (or, as I call it, the orthomorphosis) of a square into the infinite half-plane, or into a circle, and of a rectangle into the infinite half-plane. It is of course easy to deduce the orthomorphosis of the rectangle into a circle; and then, by giving a proper value to the modulus of the elliptic function involved in the formula, we obtain the orthomorphosis of the square into a circle, this solution (although equivalent thereto) being under a different form from that previously given for the square. But as appears from my paper "On the Binodal Quartic and the Graphical Representation of the Elliptic Functions," *Camb. Phil. Trans.* vol. XIV. (1889), pp. 484—494, [891], there is for the rectangle (and consequently also for the square) an orthomorphosis wherein the boundary of the rectangle or square corresponds, not to the circumference, but to the circumference together with two twice-repeated portions of a diameter. I propose to consider in I. these several transformations: II. relates to the orthomorphosis of a circle into a circle.

I.

1. We are concerned with the elliptic function sn for which $K' = 2K$, and also with the elliptic function snl of the lemniscate form. The modulus in the former case is

$$k = \frac{\sqrt{2}-1}{\sqrt{2}+1}, = (\sqrt{2}-1)^2, = 3 - 2\sqrt{2}, \text{ or say } \sqrt{k} = \sqrt{2}-1, \ \frac{1}{\sqrt{k}} = \sqrt{2}+1.$$

For the lemniscate function snl, the modulus is $= i$, and if (with Gauss) we write $\frac{1}{2}\varpi$ for the value of the complete function K or F_1, then we have $\frac{1}{4}\varpi = K(\sqrt{2}-1)$,

$=K\sqrt{k}$, where k has the foregoing special value: I notice the numerical values $\frac{1}{2}\varpi = 1{\cdot}311028$, $k = 3 - 2\sqrt{2} = \sin 9^\circ\,52'$, $k = 1{\cdot}582548$. The relation between the lemniscate function snl, and the sn with the foregoing value of k, is easily shown to be

$$\sqrt{k}\,\mathrm{sn}\,U = \frac{(i+1)\,\mathrm{snl}\,u + i\sqrt{2}}{(i-1)\,\mathrm{snl}\,u + \sqrt{2}}, \text{ where } U - \tfrac{1}{2}iK' = \frac{1+i}{2\sqrt{k}}u;$$

and it may be added that we have

$$\mathrm{cn}\,U = \frac{\sqrt{2}}{(i-1)\,\mathrm{snl}\,u + \sqrt{2}}\sqrt{\frac{1+k}{k}}\sqrt{(1+\mathrm{snl}\,u)\,(1-i\,\mathrm{snl}\,u)},$$

$$\mathrm{dn}\,U = \frac{\sqrt{2}}{(i-1)\,\mathrm{snl}\,u + \sqrt{2}}\sqrt{1+k}\sqrt{(1-\mathrm{snl}\,u)\,(1+i\,\mathrm{snl}\,u)}.$$

2. The Schwarzian orthomorphosis of the rectangle into the infinite half-plane is given (Memoir, p. 113) by the formula $X_1 + iY_1 = \mathrm{sn}\,(X+iY)$, where the modulus is real, positive, and less than unity. Here, see the figures $1\,(XY)$ and $2\,(X_1Y_1)$,

Fig. 1 (XY).

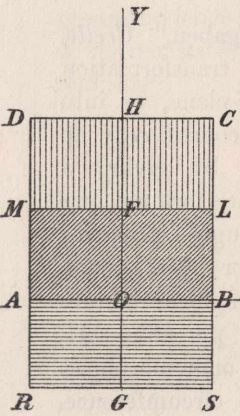
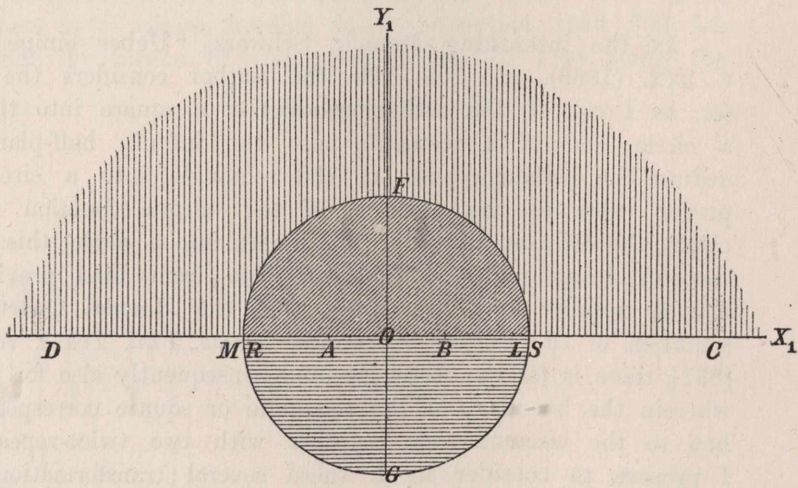


Fig. 2 (X_1Y_1).

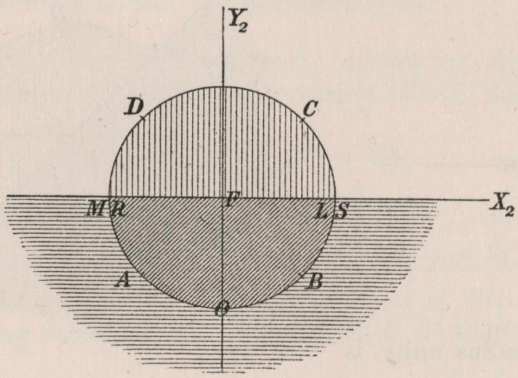


the rectangle $ABCD$, the sides of which are $AB = 2K$ and $BC = K'$, is transformed into the upper infinite half-plane ($Y_1 = +$), the four corners of the rectangle corresponding to the points A, B, C, D on the axis of X, where $OB\,(=OA) = 1$, $OC\,(=OD) = \dfrac{1}{k}$.

3. We can, by a properly determined quasi-inversion (as will be explained), transform the X_1Y_1-figure into a new figure see figure $3\,(X_2Y_2)$, the infinite X_1-axis being transformed into the circumference of a circle (the radius of which may be taken to be $=1$) and the infinite half-plane into the area within the circle. The four points A, B, C, D are thus transformed into points on the circle, which if the quasi-inversion be a symmetrical one, will be situate, A and B symmetrically, and

also C and D symmetrically, in regard to the axis OY_2: and in the case of the before-mentioned modulus $k = 3 - 2\sqrt{2}$, for which the rectangle $ABCD$ becomes a

Fig. 3* (X_2Y_2).



square, the quasi-inversion may be so determined that the points A, B, C, D shall be situate midway (that is, at inclinations $\pm 45°$, $\pm 135°$) in the four quadrants of the circle.

4. But if in figure 1 we take $BL = \frac{1}{2}K'$ and draw FL parallel to OX, then as shown in my memoir above referred to, the foregoing transformation $X_1 + iY_1 = \operatorname{sn}(X + iY)$ changes the rectangle $OBLF$, the sides of which are $OB = K$ and $BL = \frac{1}{2}K'$, into the quadrant $OBLF$ of figure 2, $OB = 1$ (as already mentioned) and radius $OL = \dfrac{1}{\sqrt{k}}$. Hence completing the rectangle $RSLM$, the sides of which are $RS = 2K$ and $SL = K'$, (viz. this rectangle differs only in position from the first-mentioned rectangle $ABCD$), we have an orthomorphosis of the rectangle $RSLM$ into the circle of figure 2: but so nevertheless that to the boundary of the rectangle there corresponds not the circumference of the circle but the boundary $RGSBLFMAR$ composed of the circumference and the portions SB, BL and MA, AR (that is, BL, AM each twice) of a diameter of the circle. See *post* No. 12.

5. The Schwarzian orthomorphosis of the square into a circle is given (Memoir, pp. 111—113) by the formula $x_1 + iy_1 = \operatorname{snl}(x + iy)$; viz. here—see figures 4 (xy) and 5 (x_1y_1)—the square $ABCD$, the half-diagonals whereof are each $= \frac{1}{2}\varpi$, corresponds to the circle radius $= 1$ of figure 5, the four points A, B, C, D of the circle being the quadrantal points as shown in the figure. Figure 5 is, in fact, figure 3 turned through an angle of $45°$; and it thus appears that Schwarz's lemniscate solution for the square into a circle is the quasi-inversion of his solution for the rectangle into the infinite half-plane, when by putting $k = 3 - 2\sqrt{2}$ the rectangle is made to be a square. See *post* Nos. 13 and 14.

[* The scale of this figure is double that of figures 1, 2, 4, 6, 7 in this paper.]

6. The general formula of quasi-inversion whereby the infinite X_1-axis is changed

Fig. 4 (xy).

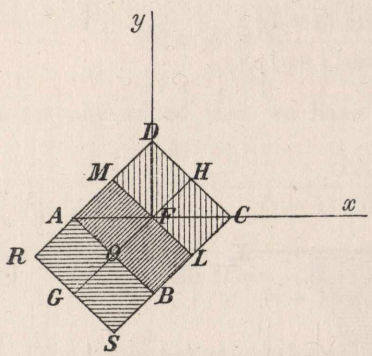
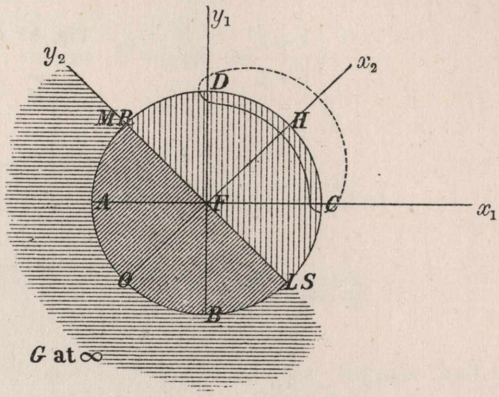


Fig. 5* (x_1y_1).



into a circumference† radius unity, is

$$X_2 + iY_2 = \frac{1 + Mi(X_1 + iY_1)}{M(X_1 + iY_1) + i},$$

where M is real. In fact, writing this equation in the form

$$X_2 + iY_2 = \frac{1 + Mi(X_1 + iY_1)}{i\{1 - Mi(X_1 + iY_1)\}},$$

we have

$$X_2 - iY_2 = \frac{1 - Mi(X_1 - iY_1)}{-i\{1 + Mi(X_1 - iY_1)\}},$$

and thence $X_2^2 + Y_2^2 - 1 = 0$, if only $Y_1 = 0$; that is, the infinite X_1-axis is transformed into the circumference $X_2^2 + Y_2^2 - 1 = 0$.

Writing $Y_1 = 0$, we have

$$X_2 + iY_2 = \frac{1 + MiX_1}{MX_1 + i} = \frac{2MX_1 + i(M^2X_1^2 - 1)}{M^2X_1^2 + 1},$$

so that to a pair of points $(\pm X_1, 0)$ there corresponds a pair of points $(\pm X_2, Y_2)$ situate symmetrically in regard to the axis OY_2.

The equation gives

$$M(X_1 + iY_1) = \frac{1 - i(X_2 + iY_2)}{(X_2 + iY_2) - i}.$$

Hence, writing this in the form

$$M(X_1 + iY_1) = \frac{1 - i(X_2 + iY_2)}{-i\{1 + i(X_2 + iY_2)\}},$$

[* The scale of this figure is double that of figures 1, 2, 4, 6, 7 in this paper.]

† I use here and elsewhere the term circumference rather than circle, to mark more clearly the distinction between the curve and the included area.

we have

$$M(X_1 - iY_1) = \frac{1 + i(X_2 - iY_2)}{i\{1 - i(X_2 - iY_2)\}},$$

and consequently $M^2(X_1^2 + Y_1^2) - 1 = 0$, if only $Y_2 = 0$; that is, the circumference $X_1^2 + Y_1^2 - \frac{1}{M^2} = 0$ is transformed into the infinite X_2-axis.

Although for the purposes of the present memoir we require the coefficient M, yet there is no real loss of generality in assuming $M = 1$, and the transformation is best studied under the form

$$X_2 + iY_2 = \frac{1 + i(X_1 + iY_1)}{X_1 + iY_1 + i};$$

see *post* No. 11.

7. As already mentioned, the coordinates (X_1, Y_1) and (X_2, Y_2) are connected by an equation of the foregoing form, and in the case of the square ($k = 3 - 2\sqrt{2}$ as before), the corresponding values for the points A, B, C, D should be

$$A, \quad X_1 + iY_1 = -1, \quad X_2 + iY_2 = \frac{-1 - i}{\sqrt{2}},$$

$$B, \quad X_1 + iY_1 = \ \ 1, \quad X_2 + iY_2 = \frac{1 - i}{\sqrt{2}},$$

$$C, \quad X_1 + iY_1 = \ \ \frac{1}{k}, \quad X_2 + iY_2 = \frac{1 + i}{\sqrt{2}},$$

$$D, \quad X_1 + iY_1 = -\frac{1}{k}, \quad X_2 + iY_2 = \frac{-1 + i}{\sqrt{2}}.$$

The proper value of M is $M = \sqrt{2} - 1, = \sqrt{k}$; the required formula thus is

$$X_2 + iY_2 = \frac{1 + i\sqrt{k}(X_1 + iY_1)}{\sqrt{k}(X_1 + iY_1) + i}$$

or say

$$\sqrt{k}(X_1 + iY_1) = \frac{1 - i(X_2 + iY_2)}{X_2 + iY_2 - i}.$$

Thus for the point B, we should have

$$\sqrt{k} = \frac{1 - i\dfrac{1 - i}{\sqrt{2}}}{\dfrac{1 - i}{\sqrt{2}} - i}, \text{ that is, } \sqrt{k} = \frac{\sqrt{2} - i - 1}{1 - i - i\sqrt{2}} = \frac{\sqrt{k} - i}{1 - \dfrac{i}{\sqrt{k}}},$$

which is right. And similarly for the point C, we should have

$$\frac{1}{\sqrt{k}} = \frac{1 - i\dfrac{1 + i}{\sqrt{2}}}{\dfrac{1 + i}{\sqrt{2}} - i}, \text{ that is, } \frac{1}{\sqrt{k}} = \frac{\sqrt{2} - i + 1}{1 + i - i\sqrt{2}} = \frac{\dfrac{1}{\sqrt{k}} - i}{1 - i\sqrt{k}},$$

which is right. And similarly for the points A and D.

8. We connect the square $ABCD$ of figure 1 with that of figure 4 by the equation

$$X + iY - \tfrac{1}{2}iK' = \frac{1+i}{2\sqrt{k}}(x+iy);$$

in fact, recollecting that $\frac{1}{2}K' = K = \dfrac{\varpi}{4\sqrt{k}}$, we have for the four points respectively

$$\begin{array}{lll} A, & X+iY=-K, & x+iy=-\tfrac{1}{2}\varpi\,, \\ B, & X+iY=K, & x+iy=-\tfrac{1}{2}\varpi i, \\ C, & X+iY=K+iK', & x+iy=\tfrac{1}{2}\varpi\,, \\ D, & X+iY=-K+iK', & x+iy=\tfrac{1}{2}\varpi i, \end{array}$$

values which satisfy the relation in question.

9. Hence writing

$$X + iY = U, \quad x + iy = u,$$

we have

$$U - \tfrac{1}{2}iK' = \frac{1+i}{2\sqrt{k}}u,$$

which is the relation in No. 1 between the arguments U, u of the elliptic functions sn, snl. We have $X_1 + iY_1 = \operatorname{sn} U$, $x_1 + iy_1 = \operatorname{snl} u$; and consequently

$$\sqrt{k}\,(X_1 + iY_1) = \frac{(i+1)(x_1+iy_1) + i\sqrt{2}}{(i-1)(x_1+iy_1)+\sqrt{2}}.$$

Hence, substituting for $\sqrt{k}\,(X_1 + iY_1)$ its value in terms of $X_2 + iY_2$, we find

$$\frac{(i+1)(x_1+iy_1) + i\sqrt{2}}{(i-1)(x_1+iy_1)+\sqrt{2}} = \frac{1 - i(X_2+iY_2)}{X_2+iY_2-i},$$

an equation which $\left(\text{multiplying on the left-hand side the numerator and the denominator each by } \dfrac{-i}{\sqrt{2}}\right)$ may be written

$$\frac{1 - i\,\dfrac{1+i}{\sqrt{2}}(x_1+iy_1)}{\dfrac{1+i}{\sqrt{2}}(x_1+iy_1) - 1} = \frac{1 - i(X_2+iY_2)}{X_2+iY_2-i},$$

that is, we have

$$X_2 + iY_2 = \frac{1+i}{\sqrt{2}}(x_1+iy_1),$$

an equation which shows that the figure 5 is in fact the figure 3 turned through an angle of 45°. We have thus proved the conclusion stated in No. 5 as to the connexion between the lemniscate solution for the square and the solution for the rectangle.

10. It is convenient to collect here the several equations relating to the orthomorphosis of the square. We have

$$X_1 + iY_1 = \text{sn}\,(X + iY), \quad k = 3 - 2\sqrt{2};$$

$$x_1 + iy_1 = \text{snl}\,(x + iy);$$

$$\sqrt{k}\,(X_1 + iY_1) = \frac{1 - i\,(X_2 + iY_2)}{X_2 + iY_2 - i},$$

$$X + iY - \tfrac{1}{2}iK' = \frac{1+i}{2\sqrt{k}}(x + iy),$$

$$\sqrt{k}\,(X_1 + iY_1) = \frac{(i+1)(x_1 + iy_1) + i\sqrt{2}}{(i-1)(x_1 + iy_1) + \sqrt{2}},$$

$$X_2 + iY_2 = \frac{1+i}{\sqrt{2}}(x_1 + iy_1),$$

which are the equations connecting together the coordinates of the five figures.

11. I examine more in detail the above-mentioned transformation

$$X_2 + iY_2 = \frac{1 + i\,(X_1 + iY_1)}{X_1 + iY_1 + i};$$

see the foregoing figures 1 (XY) and 2 ($X_1 Y_1$), in which we now regard the two circles as having each of them the radius unity; changing the sign of i, the equation gives

$$X_2 - iY_2 = \frac{1 - i\,(X_1 - iY_1)}{X_1 - iY_1 - i},$$

and we hence find

$$X_2^2 + \ Y_2^2 = \frac{X_1^2 + Y_1^2 - 2Y_1 + 1}{X_1^2 + Y_1^2 + 2Y_1 + 1};$$

consequently if $X_1^2 + Y_1^2 + 1 = 0$, then also $X_2^2 + Y_2^2 + 1 = 0$, or the transformation changes the first of these imaginary circles into the second of them: or say it changes the concentric orthotomic of the circle $X_1^2 + Y_1^2 - 1 = 0$ into the concentric orthotomic of the circle $X_2^2 + Y_2^2 - 1 = 0$.

We have moreover

$$X_2 = \frac{2X_1}{X_1^2 + Y_1^2 + 2Y_1 + 1}, \quad Y_2 = \frac{X_1^2 + Y_1^2 - 1}{X_1^2 + Y_1^2 + 2Y_1 + 1};$$

values which give the foregoing expression for $X_2^2 + Y_2^2$. But we further obtain

$$X_2^2 + Y_2^2 - 1 - \frac{2}{\mu}\,Y_2 = \frac{\frac{2}{\mu}\,(X_1^2 + Y_1^2 - 1 + 2\mu Y_1)}{X_1^2 + Y_1^2 + 2Y_1 + 1},$$

and it thus appears that the circumferences

$$X_1^2 + Y_1^2 - 1 + 2\mu Y_1 = 0, \quad X_2^2 + Y_2^2 - 1 - \frac{2}{\mu}\,Y_2 = 0,$$

correspond to each other. These are circles passing through the pairs of points $(Y_1 = 0,\ X_1 = \pm 1)$, $(Y_2 = 0,\ X_2 = \pm 1)$ respectively; or imagining them in the same figure, say they are circles which belong each to the series of circles $x^2 + y^2 - 1 + 2\beta y = 0$, and which moreover cut at right angles at the points $y = 0,\ x = \pm 1$. But attending more carefully to the nature of the correspondence, it is to be observed that taking Y positive, and giving to X any positive value from 0 to ∞, we have in figure 2 an arc LJM lying wholly within the upper semicircle LFM; and that, corresponding hereto in figure 3, we have an arc LJM lying wholly within the lower semicircle LOM; and that as in figure 2, the arc LJM lies nearer to the semicircumference LFM or to the diameter LOM, so in figure 3 the arc LJM lies nearer to the diameter LFM or to the semicircumference LOM. Thus the upper semicircle LFM of figure 2 corresponds to the lower semicircle LOM of figure 3; but so that the semicircumference LFM of the first figure corresponds to the diameter LFM of the second figure; and the diameter LOM of the first figure to the semicircumference LOM of the second figure. And further supposing that Y_1 is still positive, but that μ has any negative value from $-\infty$ to 0, we have in figure 2 an arc in the upper half-plane lying wholly outside the semicircle; and corresponding thereto in figure 3, an arc lying wholly inside the upper semicircle LHM; that is, in figure 2 the infinite space in the upper half-plane outside the semicircle corresponds in figure 3 to the space within the upper half-circle LHM; the infinity of figure 2 corresponding to the semicircumference LHM of figure 3, and the semicircumference LFM of figure 2 to the diameter LFM of figure 3. And thus in figure 2, the upper half-plane inside and outside the semicircle corresponds in figure 3 to the lower and upper half-circles, that is, to the whole circular area $OLHM$ of figure 3.

It is to be observed, moreover, that we have

$$X_2^2 + Y_2^2 + 1 + 2\lambda X_2 = \frac{2\,(X_1^2 + Y_1^2 + 1 + 2\lambda X_1)}{X_1^2 + Y_1^2 + 2Y_1 + 1},$$

that is, the circles $X_1^2 + Y_1^2 + 1 + 2\lambda X_1 = 0$ and $X_2^2 + Y_2^2 + 1 + 2\lambda X_2 = 0$ correspond to each other. Imagining the circles as belonging to the same figure, these are one and the same circle of the series $x^2 + y^2 + 1 - 2\alpha x = 0$ each passing through the pair of points $(x = 0,\ y = \pm i)$ which are the antipoints of the pair $(y = 0,\ x = \pm 1)$; these circles thus cut at right angles those of the series $x^2 + y^2 - 1 + 2\beta y = 0$. We have, by means of the two series of circles, an easy construction for the correspondence between the figures 2 and 3.

12. In explanation of No. 4, observe that, starting from the equation

$$X_1 + iY_1 = \operatorname{sn}(X + iY),$$

and writing $\operatorname{sn} X = P$, $\operatorname{sn} iY = iQ$, we have

$$X_1 + iY_1 = \frac{P\sqrt{1 + Q^2 \,.\, 1 + k^2Q^2} + iQ\sqrt{1 - P^2 \,.\, 1 - k^2P^2}}{1 + k^2P^2Q^2},$$

that is,

$$X_1 = \frac{P\sqrt{1 + Q^2 \,.\, 1 + k^2Q^2}}{1 + k^2P^2Q^2},\quad Y_1 = \frac{Q\sqrt{1 - P^2 \,.\, 1 - k^2P^2}}{1 + k^2P^2Q^2},$$

and thence

$$X_1^2+Y_1^2=\frac{P^2+Q^2}{1+k^2P^2Q^2}=r^2 \text{ (if } X_1^2+Y_1^2 \text{ be put } =r^2);$$

hence

$$P^2(1-k^2Q^2r^2)=r^2-Q^2,\quad Q^2(1-k^2P^2r^2)=r^2-P^2.$$

Now considering in figure 1 any line in the rectangle parallel to the axis of X, that is, taking Y constant and therefore also Q constant, and proceeding to eliminate P, we have

$$P^2=\frac{r^2-Q^2}{1-k^2Q^2r^2},\quad 1-P^2=\frac{1+Q^2-(1+k^2Q^2)\,r^2}{1-k^2Q^2r^2},$$

$$1-k^2P^2=\frac{1+k^2Q^2-k^2(1+Q^2)\,r^2}{1-k^2Q^2r^2},\quad 1+k^2P^2Q^2=\frac{1-k^2Q^4}{1-k^2Q^2r^2},$$

and consequently

$$X_1=\frac{\sqrt{1+Q^2\,.\,1+k^2Q^2}}{1-k^2Q^4}\sqrt{r^2-Q^2\,.\,1-k^2Q^2r^2},$$

$$Y_1=\frac{Q\sqrt{1+Q^2-(1+k^2Q^2)\,r^2}\,\sqrt{1+k^2Q^2-k^2(1+Q^2)\,r^2}}{1-k^2Q^4},$$

giving X_1, Y_1 each of them in terms of Q^2 and r^2, $=X_1^2+Y_1^2$. The former of these equations, replacing therein r^2 by its value, gives easily

$$(X_1^2+Y_1^2)^2-2AX_1^2-2BY_1^2+\frac{1}{k^2}=0,$$

where

$$2A=\frac{1+Q^2}{1+k^2Q^2}+\frac{1}{k^2}\,\frac{1+k^2Q^2}{1+Q^2},\quad 2B=Q^2+\frac{1}{k^2Q^2};$$

viz. we have thus the equation of the curve, a bicircular quartic which in figure 2 corresponds to the line parallel to the axis of X_1 in figure 1.

In particular, for the line LM of figure 1 we have $Y=\frac{1}{2}K'$ and thence $iQ=\operatorname{sn}\frac{1}{2}iK'=\frac{i}{\sqrt{k}}$, that is, $Q=\frac{1}{\sqrt{k}}$, and thence $A=B=\frac{1}{k}$; the equation of the bicircular quartic is

$$(X_1^2+Y_1^2)^2-\frac{2}{k}(X_1^2+Y_1^2)+\frac{1}{k^2}=0;$$

viz. this is the circle $X_1^2+Y_1^2-\frac{1}{k}=0$ twice repeated, and we have thus this circle, or rather the half-circumference LFM of figure 2, corresponding to the line LM of figure 1. More simply,

$$X_1+iY_1=\operatorname{sn}(X+\tfrac{1}{2}iK')=\frac{\frac{1+k}{\sqrt{k}}\operatorname{sn}X+\frac{i}{\sqrt{k}}\operatorname{cn}X\operatorname{dn}X}{1+k^2\,.\,\frac{1}{k}\operatorname{sn}^2X}=\frac{(1+k)\,P+i\sqrt{1-P^2\,.\,1-k^2P^2}}{\sqrt{k}\,(1+kP^2)},$$

and thence

$$X_1^2 + Y_1^2 = \frac{(1+k)^2 P^2 + 1 - (1+k^2) P^2 + k^2 P^4}{k(1 + 2kP^2 + k^2P^4)} = \frac{1}{k},$$

that is, $X_1^2 + Y_1^2 - \frac{1}{k} = 0$ as before. It is easy to see that the points A, O, B of figure 1 correspond to the points A, O, B of figure 2, and hence that the area of the rectangle $AOBLFMA$ of figure 1 corresponds to that of the semicircle $AOBLFMA$ of figure 2.

Returning to the equation $X_1 + iY_1 = \operatorname{sn}(X + iY)$, if we write herein successively

$$Y_1 = \tfrac{1}{2}K' - \beta, \quad \operatorname{sn} iY_1 = iQ_1 = \operatorname{sn} i(\tfrac{1}{2}K' - \beta),$$

and

$$Y_1 = \tfrac{1}{2}K' + \beta, \quad \operatorname{sn} iY_1 = iQ_2 = \operatorname{sn} i(\tfrac{1}{2}K' + \beta),$$

then we have

$$iQ_1 . iQ_2 = \operatorname{sn} i(\tfrac{1}{2}K' - \beta) \operatorname{sn} i(\tfrac{1}{2}K' + \beta) = -\frac{1}{k},$$

that is, $Q_1Q_2 = \frac{1}{k}$: hence for q writing Q_1 or Q_2, we have in each case the same values of A and B, that is, we have the same bicircular quartic for two lines parallel to and equidistant from the line LM, but to one of these (viz. the line between LM and BA) there corresponds the half-perimeter lying within the semicircumference LFM, and to the other of them (viz. the line between LM and CD) there corresponds the half-perimeter lying without the semicircumference LFM.

It may be shown in like manner that, to any line in figure 1 parallel to the axis OY, there corresponds in figure 2 a bicircular quartic of the like form

$$(X_1^2 + Y_1^2)^2 - 2AX_1^2 - 2BY_1^2 + \frac{1}{k^2} = 0.$$

13. Similarly in explanation of No. 5, observe that, starting from the equation $x_1 + iy_1 = \operatorname{snl}(x + iy)$ and writing $\operatorname{snl} x = p$, $\operatorname{snl} iy = i \operatorname{snl} y = iq$, we have

$$x_1 + iy_1 = \frac{p\sqrt{1-q^4} + iq\sqrt{1-p^4}}{1 - p^2q^2},$$

that is,

$$x_1 = \frac{p\sqrt{1-q^4}}{1-p^2q^2}, \quad y_1 = \frac{q\sqrt{1-p^4}}{1-p^2q^2},$$

and thence

$$x_1^2 + y_1^2 = \frac{p^2+q^2}{1-p^2q^2}:$$

writing $x + y = \tfrac{1}{2}\varpi$, we have

$$\operatorname{sn} y = \frac{\operatorname{cn} x}{\operatorname{dn} x}, \text{ that is, } q = \frac{\sqrt{1-p^2}}{\sqrt{1+p^2}}, \text{ or } q^2 = \frac{1-p^2}{1+p^2}, \text{ that is, } \frac{p^2+q^2}{1-p^2q^2} = 1;$$

and thus to the line $x + y = \tfrac{1}{2}\varpi$, there corresponds the circle $x_1^2 + y_1^2 - 1 = 0$. More precisely, to the line CD of figure 4, there corresponds the quarter-circumference CD

of figure 5: and similarly to DA, AB and BC of figure 4 the remaining quarter-circumferences DA, AB, BC of figure 5; that is, to the whole boundary of the square in figure 4 there corresponds the whole circumference of the circle in figure 5.

14. To any line in figure 4, parallel to the axis Ox or to the axis Oy, there corresponds in figure 5 a bicircular quartic of the form $x_1^2+y_1^2-2Ax_1^2-2By_1^2-1=0$. The investigation is substantially the same as that contained in No. 12, and need not be here given. But it is remarkable that also, to any line of figure 4 parallel to a side of the square (that is, to any line $x \pm y = c$), there corresponds in figure 5 a bicircular quartic of the like form (for the sides of the square, or lines $x \pm y = \pm \tfrac{1}{2}\varpi$ of figure 4, this bicircular quartic becomes the twice repeated circle $x_1^2+y_1^2-1=0$ of figure 5, which is the result just obtained in No. 13). I investigate this result as follows. Writing, as in No. 13,

$$\operatorname{snl} x = p, \quad \operatorname{snl} iy = i \operatorname{snl} y = iq,$$

we have, as above,

$$x_1 = \frac{p\sqrt{1-q^4}}{1-p^2q^2}, \quad y_1 = \frac{q\sqrt{1-p^4}}{1-p^2q^2}, \text{ and thence } x_1^2+y_1^2 = \frac{p^2+q^2}{1-p^2q^2}.$$

Now assuming between x and y the relation $x+y=C$, and writing $\operatorname{snl} C = c$, this gives

$$c = \frac{p\sqrt{1-q^4}+q\sqrt{1-p^4}}{1+p^2q^2};$$

to obtain the required curve, we must between these equations (three independent equations) eliminate p and q. We have

$$x_1+y_1 = \frac{p\sqrt{1-q^4}+q\sqrt{1-p^4}}{1-p^2q^2},$$

and consequently

$$(1+p^2q^2)\,c = (1-p^2q^2)(x_1+y_1),$$

or, writing for convenience $\Omega = 1-p^2q^2$, this equation gives

$$\Omega = \frac{2c}{x_1+y_1+c}.$$

Hence $\Omega x_1 = p\sqrt{1-q^4}$, $\Omega y_1 = q\sqrt{1-p^4}$; these equations may be written

$$\begin{aligned} \Omega^2 x_1^2 &= \qquad\qquad p^2 - (1-\Omega)\,q^2, \\ \Omega^2 y_1^2 &= -(1-\Omega)\,p^2 + \qquad\qquad q^2, \end{aligned}$$

and from these equations obtaining the expressions for p^2 and q^2, and thence the expression for p^2q^2, $=1-\Omega$, we find, after some easy reductions,

$$\{(x_1^2+y_1^2)^2-1\} + \frac{\Omega^2}{1-\Omega}\,x_1^2y_1^2 = \frac{4(1-\Omega)}{\Omega^2}.$$

But we have

$$\frac{\Omega^2}{1-\Omega} = \frac{4c^2}{(x_1+y_1)^2-c^2},$$

or substituting this value, the equation becomes

$$(x_1^2+y_1^2)^2 - 1 + \frac{4c^2x_1^2y_1^2}{(x_1+y_1)^2-c^2} = \frac{(x_1+y_1)^2}{c^2} - 1,$$

that is,

$$\left\{(x_1^2+y_1^2)^2-\frac{(x_1+y_1)^2}{c^2}\right\}\{(x_1+y_1)^2-c^2\}+4c^2x_1^2y_1^2=0.$$

Writing for a moment $x_1^2+y_1^2=P$, $x_1y_1=Q$, this is

$$\left\{P^2-\frac{1}{c^2}(P+2Q)\right\}\{P+2Q-c^2\}+4c^2Q^2=0,$$

an equation which contains the factor $P+2Q$; throwing this out, the equation becomes, after an easy reduction,

$$(P-1)^2+(1-c^2)\left\{P-2Q-\frac{1}{c^2}(P+2Q)\right\}=0,$$

that is,

$$(x_1^2+y_1^2-1)^2+(1-c^2)\left\{(x_1-y_1)^2-\frac{1}{c^2}(x_1+y_1)^2\right\}=0,$$

the required equation. Transforming through an angle of 45° by writing

$$x_1=\frac{x_2+y_2}{\sqrt{2}},\quad y_1=\frac{x_2-y_2}{\sqrt{2}},$$

(where observe that the axis of x_2 is the line FH of figure 5), the equation becomes

$$(x_2^2+y_2^2-1)^2+(1-c^2)\left(2y_2^2-\frac{1}{c^2}2x_2^2\right)=0,$$

or writing $c=\cos\gamma$ and therefore $1-c^2=\sin^2\gamma$, this equation becomes

$$(x_2^2+y_2^2)^2-\frac{2}{\cos^2\gamma}x_2^2-2\cos^2\gamma\,.\,y_2^2+1=0,$$

a curve consisting of two indented ovals situate symmetrically in regard to the axis Fy_2 of figure 5. In fact, writing in the equation $y_2=0$, we have for x_2^2 two real positive values; but, writing $x_2=0$, we have for y_2^2 two imaginary values. For $\gamma=0$, the equation becomes

$$(x_2^2+y_2^2)^2-2(x_2^2+y_2^2)+1=0,$$

that is, we have the circle $x_2^2+y_2^2-1=0$ twice repeated. One of the ovals is shown in figure 5; the portion of it lying within the circle agrees with Schwarz's figure, p. 113, turning this round through an angle of 45°.

For the lines $x-y=C$ of figure 4, we have in figure 5 the same system of bicircular quartics turned round through an angle of 90°.

II.

15. I consider the general problem of the orthomorphosis of a circle into a circle: we can, for the transformation of the circumference of the circle $x^2+y^2-1=0$ into that of the circle $x_1^2+y_1^2-1=0$, find a formula involving an arbitrary function

or (what is the same thing) an indefinite number of arbitrary constants. In fact, writing for shortness $z=x+iy$, $z_1=x_1+iy_1$, and $\bar{z}$, $\bar{z}_1$ for the conjugate functions $x-iy$, x_1-iy_1; also $\phi(z)$ for a function of z involving in general imaginary coefficients $a+ib$, &c., and $\bar{\phi}(z)$ for the like function with the conjugate coefficients $a-ib$, &c.; then if we assume

$$z_1=\frac{\phi(z)}{z^m\bar{\phi}\left(\frac{1}{z}\right)},$$

where m is any positive or negative integer, this implies

$$\bar{z}_1=\frac{\phi(\bar{z})}{\bar{z}^m\phi\left(\frac{1}{\bar{z}}\right)};$$

consequently, if $x^2+y^2-1=0$, that is, $z\bar{z}=1$, or $\bar{z}=\frac{1}{z}$, we have

$$\bar{z}_1=\frac{\bar{\phi}\left(\frac{1}{z}\right)}{\left(\frac{1}{z}\right)^m\phi(z)}=\frac{z^m\bar{\phi}\left(\frac{1}{z}\right)}{\phi(z)},\ =\frac{1}{z_1},$$

or $z_1\bar{z}_1=1$, that is, $x_1{}^2+y_1{}^2-1=0$.

In a slightly different form, taking α, β, &c., to denote any imaginary quantities, and $\bar{\alpha}$, $\bar{\beta}$, ... the conjugate quantities; assuming

$$\phi(z)=(z-\alpha)(z-\beta)\ldots,$$

and taking m for the number of factors, we have

$$z_1=\frac{(z-\alpha)(z-\beta)\ldots}{(1-\bar{\alpha}z)(1-\bar{\beta}z)\ldots},$$

and then (repeating the demonstration) we have

$$\bar{z}_1=\frac{(\bar{z}-\bar{\alpha})(\bar{z}-\bar{\beta})\ldots}{(1-\alpha\bar{z})(1-\beta\bar{z})\ldots},$$

which, writing therein $\bar{z}=\frac{1}{z}$, becomes

$$\bar{z}_1=\frac{\left(\frac{1}{z}-\bar{\alpha}\right)\left(\frac{1}{z}-\bar{\beta}\right)\ldots}{\left(1-\frac{\alpha}{z}\right)\left(1-\frac{\beta}{z}\right)\ldots},\ =\frac{(1-\bar{\alpha}z)(1-\bar{\beta}z)\ldots}{(z-\alpha)(z-\beta)\ldots}=\frac{1}{z},$$

and consequently, if $\bar{z}=\frac{1}{z}$, then also $\bar{z}_1=\frac{1}{z_1}$ as before.

We may in the expression for z_1 introduce a factor $\frac{A}{\bar{A}}$, or, what is the same thing, a factor A which is such that $A\bar{A}=1$. In particular, we thus have the solution

$$z_1=\frac{A(z-\alpha)}{(1-\bar{\alpha}z)},$$

giving

$$\bar{z}_1 = \frac{\bar{A}\,(\bar{z} - \bar{\alpha})}{1 - \alpha \bar{z}},$$

which, for $z\bar{z} = 1$, becomes

$$\bar{z}_1 = \frac{\bar{A}\,(1 - \bar{\alpha} z)}{z - \alpha},$$

so that ($A\bar{A}$ being $= 1$) this gives $z_1 \bar{z}_1 = 1$.

16. This is a solution with three arbitrary constants, viz. A (which may be put $= \cos\lambda + i \sin\lambda$) is a single arbitrary constant, and $\alpha, = a + ib$, is two arbitrary constants; and these constants may be so determined that a given point in the interior of the one circle, and a given point on the circumference thereof, shall correspond respectively to a given point in the interior of the other circle and to a given point on the circumference thereof. According to a well-known theorem of Riemann's, any two simply connected areas included within given closed curves respectively may be made to correspond to each other, and that in one way only, under the foregoing conditions as to a pair of interior points and a pair of boundary points: and we have, in what just precedes, the solution of the problem in the case of two equal circles.

17. In the case of any other solution, we thus know that the correspondence between the two circumferences cannot and does not imply a (1, 1) correspondence between the areas of the two circles: but it is interesting to enquire what happens. I take a very particular case

$$z_1 = \frac{z\,(z - 2)}{1 - 2z},$$

and therefore

$$\bar{z}_1 = \frac{\bar{z}\,(\bar{z} - 2)}{1 - 2\bar{z}},$$

and consequently

$$z_1 \bar{z}_1 = \frac{z\bar{z}\,\{z\bar{z} - 2\,(z + \bar{z}) + 4\}}{1 - 2\,(z + \bar{z}) + 4z\bar{z}},$$

that is,

$$x_1^2 + y_1^2 = \frac{(x^2 + y^2)\,(x^2 + y^2 - 4x + 4)}{4\,(x^2 + y^2) - 4x + 1},$$

so that, writing $x^2 + y^2 - 1 = 0$, we have $x_1^2 + y_1^2 - 1 = 0$, a correspondence of the two circumferences. But to the circumference $x_1^2 + y_1^2 - 1 = 0$ there corresponds not only the circumference $x^2 + y^2 - 1 = 0$, but another circumference. In fact, writing $x_1^2 + y_1^2 = 1$, we have

$$4\,(x^2 + y^2) - 4x + 1 = (x^2 + y^2)\,(x^2 + y^2 - 4x + 4),$$

that is,

$$(x^2 + y^2)^2 - 4x\,(x^2 + y^2) + 4x - 1 = 0,$$

or

$$(x^2 + y^2 - 1)\,(x^2 + y^2 - 4x + 1) = 0,$$

and there is thus the other circle

$$x^2 + y^2 - 4x + 1 = 0, \text{ or say } (x - 2)^2 + y^2 - 3 = 0,$$

viz. this is a circle, coordinates of centre (2, 0) and radius $=\sqrt{3}$, cutting the circle $x^2+y^2-1=0$ in two real points. Referring to the figures 6 (xy) and 7 (x_1y_1), and observing that $x_1=0$, $y_1=0$, that is, $z_1=0$, gives $z=0$, or $z=2$, that is, the points

Fig. 6 (xy).

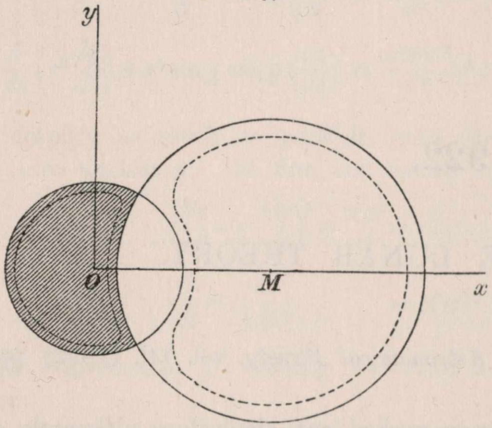
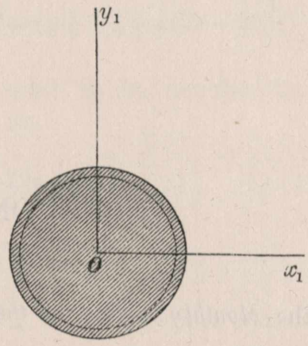


Fig. 7 (x_1y_1).



$x=0$, $y=0$, and $x=2$, $y=0$, we see that to the centre O in figure 7, there correspond in figure 6 the points O, M which are the centres of the two circles. To any small closed curve, or say any small circle surrounding the point O of figure 7, there correspond in figure 6 small closed curves surrounding the points O, M respectively; and if in figure 7 the radius of the circle continually increases and becomes nearly equal to unity, the closed curves of figure 6 continually increase, changing at the same time their forms, and assume the forms shown by the dotted lines of figure 6. It thus appears that, to the whole area of the circle $x_1^2+y_1^2-1=0$ of figure 7, there correspond the two lunes ACB and ABD of figure 6; or if we attend only to the area included within the circle $x^2+y^2-1=0$ of this figure, then there corresponds not the whole area of this circle, but only the area of the lune ACB: and thus that the assumed relation $z_1=\dfrac{z(z-2)}{1-2z}$ establishes, in fact, an orthomorphosis of the circle $x_1^2+y_1^2-1=0$ into the lune ACB which lies inside the circle $x^2+y^2-1=0$ and outside the circle $(x-2)^2+y^2-3=0$. It may be added that, to the infinite area outside the circle $x_1^2+y_1^2-1=0$ of figure 7, there correspond in figure 6 first the area of the lens AB common to the two circles, and secondly the area outside the two circles: we have thus an orthomorphosis of the area outside the circle $x_1^2+y_1^2-1=0$ into these two areas respectively.

A somewhat more elegant example would have been that of the correspondence

$$z_1=\frac{z(z-\sqrt{2})}{1-\sqrt{2}z};$$

here, corresponding to the circumference $x_1^2+y_1^2-1=0$, we have the two equal circumferences $x^2+y^2-1=0$, and $(x-\sqrt{2})^2+y^2-1=0$: and to the whole area of the circle $x_1^2+y_1^2-1=0$, there correspond two equal lunes ACB and ABD.