938.

ON TWO CUBIC EQUATIONS.

[From the *Messenger of Mathematics*, vol. XXII. (1893), pp. 69—71.]

STARTING from the equations

$$2 + a = b^2,$$
$$2 + b = c^2,$$
$$2 + c = a^2,$$

then eliminating b, c, we find

$$(a^4 - 4a^2 + 2)^2 - (a + 2) = 0,$$

that is,

$$a^8 - 8a^6 + 20a^4 - 16a^2 - a + 2 = 0;$$

we satisfy the equations by $a = b = c$, and thence by

$$a^2 - a - 2 = (a - 2)(a + 1) = 0;$$

there remains a sextic equation breaking up into two cubic equations; the octic equation may in fact be written

$$(a - 2)(a + 1)(a^3 + a^2 - 2a - 1)(a^3 - 3a + 1) = 0,$$

and we have thus the two cubic equations

$$x^3 + x^2 - 2x - 1 = 0, \quad x^3 - 3x + 1 = 0,$$

for each of which the roots (a, b, c) taken in a proper order are such that $2 + a = b^2$, $2 + b = c^2$, $2 + c = a^2$.

It may be remarked that starting from $y^3 + y^2 - 2y - 1 = 0$, $y^2 = x + 2$, the first equation gives $(y^3 - 2y)^2 - (y^2 - 1)^2 = 0$, that is, $y^6 - 5y^4 + 6y^2 - 1 = 0$, whence

$$(x + 2)^3 - 5(x + 2)^2 + 6(x + 2) - 1 = 0,$$

that is,

$$x^3 + x^2 - 2x - 1 = 0.$$

And similarly, starting from $y^3 - 3y + 1 = 0$, $y^2 = x + 2$, the first equation gives $(y^3 - 3y)^2 - 1 = 0$, that is, $y^6 - 6y^4 + 9y^2 - 1 = 0$, whence

$$(x+2)^3 - 6(x+2)^2 + 9(x+2) - 1 = 0,$$

that is,

$$x^3 - 3x + 1 = 0.$$

To find the roots of the equation $x^3 + x^2 - 2x - 1 = 0$, taking ω an imaginary cube root of unity, and writing $\alpha = \sqrt[3]{\{7(2+3\omega)\}}$, $\beta = \sqrt[3]{\{7(2+3\omega^2)\}}$, where observe that $2 + 3\omega$, $2 + 3\omega^2$ are imaginary factors of 7, viz.

$$7 = (2+3\omega)(2+3\omega^2),$$

and therefore also $\alpha^3 + \beta^3 = 7$, $\alpha\beta = 7$, then the roots of the equation are

$$3a = -1 + \alpha + \beta,$$
$$3b = -1 + \omega\alpha + \omega^2\beta,$$
$$3c = -1 + \omega^2\alpha + \omega\beta.$$

I verify herewith the equation $a^2 = 2 + c$, viz. this gives

$$(-1 + \alpha + \beta)^2 = 18 + 3(-1 + \omega^2\alpha + \omega\beta),$$

or writing herein $2\alpha\beta = 14$, this is

$$\alpha^2 - (2+3\omega^2)\alpha + \beta^2 - (2+3\omega)\beta = 0,$$

that is,

$$\alpha^2 - \tfrac{1}{7}\beta^3\alpha + \beta^2 - \tfrac{1}{7}\alpha^3\beta = 0,$$

or finally

$$(\alpha^2 + \beta^2)(1 - \tfrac{1}{7}\alpha\beta) = 0,$$

satisfied in virtue of $\alpha\beta = 7$.

For the second equation $x^3 - 3x + 1 = 0$, ω denoting as before, the roots are

$$a = \omega^{\frac{1}{3}} + \omega^{\frac{8}{3}}, \text{ whence } a^2 = \omega^{\frac{2}{3}} + \omega^{\frac{7}{3}} + 2, = 2 + c,$$
$$b = \omega^{\frac{4}{3}} + \omega^{\frac{5}{3}}, \quad ,, \quad b^2 = \omega^{\frac{8}{3}} + \omega^{\frac{1}{3}} + 2, = 2 + a,$$
$$c = \omega^{\frac{7}{3}} + \omega^{\frac{2}{3}}, \quad ,, \quad c^2 = \omega^{\frac{5}{3}} + \omega^{\frac{4}{3}} + 2, = 2 + b.$$

The equation $x^3 - 5x^2 + 6x - 1 = 0$, which, writing therein $x + 2$ for x, gives

$$x^3 + x^2 - 2x - 1 = 0,$$

is considered in Hermite's *Cours d'Analyse,* Paris 1873, p. 12, and this suggested to me the foregoing investigation.