Archives of Mechanics • Archiwum Mechaniki Stosowanej • 27, 2, pp. 231–243, Warszawa 1975

On the global behaviour of one-dimensional acceleration waves in a material with internal variables

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AMPLITUDE of one-dimensional acceleration waves propagating through a material which is described by the strain and the internal variable vector is investigated. It is shown that the amplitude of the wave obeys the Bernoulli equation. The propositions on the local and global in time behaviour of the amplitude are formulated. A wave entering a homogeneous state of a material of grade two is precisely discussed.

Zbadano amplitudę jednowymiarowych fal przyśpieszenia, rozprzestrzeniających się w materiale, opisywanym przez odkształcenie i wektor parametrów wewnętrznych. Wykazano, że amplituda fali spełnia równanie Bernoulliego. Sformułowano wnioski dotyczące lokalnego i globalnego w czasie zachowania się amplitudy. Szczegółowo omówiono falę poruszającą się w jednorodnym stanie materiału drugiego rzędu.

Исследована амплитуда одномерных волн ускорения распространяющихся в материале, описыванном деформацией и вектором внутренних параметров. Доказано, что амплитуда волны удовлетворяет уравнению Бернулли. Сформулированы выводы, касающиеся локального и глобального во времени поведения амплитуды. Подробно обсуждена волна движущаяся в однородном состоянии материала второго порядка.

1. Introduction

IN THE PREVIOUS paper [4] an analysis of acceleration waves in a material with internal parameters was given. There it was assumed that the internal dissipation of a rheological material could be described by n internal scalar variables (parameters). Here the same assumption is done. But now we neglect the thermal effects.

The object of this paper is to investigate the behaviour of the acceleration wave amplitude in the material under consideration in the case of one-dimensional theory.

The governing differential equation obtained in Sec. 2 is of Bernoulli type. In Sec. 3 the propositions on the local behaviour of the amplitude are formulated. The theorem on global behaviour of amplitude given in this section enables us to state the existence of the critical initial amplitude.

In Sec. 4, a wave entering a homogeneous state of a material of grade two is precisely discussed. Furthermore, the solution of the general initial value problem in terms of homogeneous strain functions is obtained.

2. The amplitude and the velocity of acceleration waves

We shall identify a body with an open region $\mathscr{B}$ which is its image in the fixed reference configuration $\varkappa$. The motion of a body is defined by a function $\chi\colon \mathscr{B}\times R \to R$; the value $x = \chi(X, t)$ determines the location x at time t of the material point X. By R we denote a real axis.

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In the paper, acceleration waves will be discussed. It means that we shall consider the motion χ of the body $\mathcal{B}$, which is a twice continuously differentiable function on $\mathcal{B}\times R$, except for jump discontinuity across some curve $\sum$. This curve may be interpreted as a material trajectory(¹) of the acceleration wave. In other words the curve $\sum$ has the property: the second derivatives of χ have jump discontinuities across $\sum$ only and are continuous in both variables X and t everywhere else.

If $(Y(t), t) \in \sum$, then $Y(t)$ is the particle at which the wave is to be found at time t and $U(t) = \frac{d}{dt}Y(t)$ is the intrinsic velocity of the wave. We assume that $U(t) \neq 0$.

In the motion χ with an acceleration wave, the derivatives

$$E(X, t) = \frac{\partial}{\partial X}\chi(X, t) - 1, \quad \dot{x}(X, t) = \frac{\partial}{\partial t}\chi(X, t) \tag{2.1}$$

are continuous but

$$\partial_X E(X, t) = \frac{\partial^2}{\partial X^2}\chi(X, t), \ddot{x}(X, t) = \frac{\partial^2}{\partial t^2}\chi(X, t) \quad \text{and} \quad \dot{E}(X, t) = \frac{\partial^2}{\partial X \partial t}\chi(X, t), \tag{2.2}$$

as well as the higher derivatives of χ have jump discontinuities.

Let(²)

$$\mathbf{a}(t) \equiv [\ddot{x}](t) \neq 0$$

be the amplitude of the acceleration wave.

The compatibility condition [6]

$$\frac{d}{dt}[f] = [\dot{f}] + U[\partial_X f] \tag{2.3}$$

with $f = \dot{x}$ and E implies that

$$-U[\dot{E}] = U^2[\partial_X E] = \mathbf{a}. \tag{2.4}$$

In the present paper a material in particle X of the body $\mathcal{B}$ is defined by the constitutive equation for the stress ([4, 5])

$$T(X, t) = \mathcal{T}\big(E(X, t), \alpha(X, t)\big) \quad \text{for} \quad t \in [0, \infty), \tag{2.5}$$

which is supplemented by the evolution equation for the internal variables α

$$\dot{\alpha}(X, t) = A\big(E(X, t), \alpha(X, t)\big) \quad \text{for} \quad t \in [0, \infty) \tag{2.6}$$

with the initial data $\alpha(X, 0) = \alpha_0(X)$.

Here α represents n scalar internal state variables (parameters) which are introduced to describe the internal dissipation of a rheological material.

If $A\colon (-1, \infty)\times R^n \to R^n$ is a continuous function of its variables and Lipschitz continuous with respect to the second variable then, for each vector α_0 and a continuous

(¹) Cf. [4] and the literature cited there.

(²) We use the well-known notation: for a function $f(X, t)$, $[f] = f^- - f^+$ with $f^\pm \equiv \lim_{X\to Y(t)\pm} f(X, t)$.

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strain function $E(\cdot)$ on $[0, \infty)$ there exists a unique continuously differentiable solution of the problem (2.6)(3).

Hence in a motion with an acceleration wave we have (cf. [2–4])

$$[\alpha] = [\dot{\alpha}] = 0 \tag{2.7}$$

and by (2.3) with $f = \alpha$

$$[\partial_X \alpha] = 0. \tag{2.8}$$

In following considerations we assume that the constitutive function $\mathcal{T}$ is C^2-class and the function A is C^1-class in their domains.

Let ϱ denote the mass density in the reference configuration $\varkappa$. Then the law of motion in the presence of a body force b is of the form

$$\partial_X T + \varrho b = \varrho \ddot{x} \tag{2.9}$$

on either side of the wave and across the wave

$$[\partial_X T] = [\varrho \ddot{x}] \tag{2.10}$$

after the assumption that the force b is C^1-class.

Using (2.4) and (2.10) we have the expression for the velocity of the wave (cf. [2–4])

$$\varrho U^2(t) = \partial_E \mathcal{T}\big(E(Y(t), t), \alpha(Y(t), t)\big), \tag{2.11}$$

where the smoothness property for the constitutive function $\mathcal{T}$ was used.

We shall attempt to find the equation which describes changes of the amplitude $\mathrm{a}(t) = [\ddot{x}](t)$.

LEMMA 1. *The amplitude of an acceleration wave propagating into o material with internal state variables satisfies the equation*

$$\frac{d\mathrm{a}}{dt} = -\mu \mathrm{a} + \beta \mathrm{a}^2, \tag{2.12}$$

where

$$\mu(t) = -\frac{1}{2\varrho U}\Big\{\varrho \frac{dU(t)}{dt} + \frac{1}{U(t)} \partial_E^2 \mathcal{T} \dot{E}^+ - \partial_E^2 \mathcal{T} (\partial_X E)^+ + \frac{1}{U(t)} \partial_\alpha \mathcal{T} \cdot \partial_E A + \frac{1}{U(t)} \partial_\alpha \partial_E \mathcal{T} \cdot \dot{\alpha} - \partial_\alpha \partial_E \mathcal{T} \cdot \partial_X \alpha\Big\}, \tag{2.13}$$

$$\beta(t) = -\frac{\partial_E^2 \mathcal{T}}{2U(t)\partial_E \mathcal{T}}.$$

The derivatives $\partial_E^2 \mathcal{T}$, $\partial_\alpha \mathcal{T}$, $\partial_E A$ and $\partial_\alpha \partial_E \mathcal{T}$ are evaluated at $\big(E(Y(t), t), \alpha(Y(t), t)\big)$ and $\dot{\alpha}$ and $\partial_X \alpha$ at $(Y(t), t)$.

P r o o f. Equations (2.3) and (2.4) yield

$$2\sqrt{U} \frac{d}{dt}\left(\frac{\mathrm{a}}{\sqrt{U}}\right) = [\dddot{x}] - U^2 [\partial_X \dot{E}]. \tag{2.14}$$

(3) In [5] the theorem of existence and uniqueness of (2.6) was formulated.

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Differentiation (2.9) with respect to time t gives on $\sum$

$$[\dddot{x}] = \frac{1}{\varrho}[\partial_X \dot{T}]. \tag{2.15}$$

After using (2.5) and (2.6) and the smoothness assumptions about $\mathscr{T}$ and A we have

$$[\partial_X \dot{T}] = \partial_E \mathscr{T}[\partial_X \dot{E}] + \partial_E^2 \mathscr{T}[\dot{E}\partial_X E] + \partial_\alpha \mathscr{T}\cdot \partial_E A[\partial_X E] + \partial_\alpha \partial_E \mathscr{T}\cdot(\dot{\alpha}[\partial_X E] + \partial_X \alpha[\dot{E}]), \tag{2.16}$$

where (2.7) was used.

If the following equality for two arbitrary functions having jumps across $\sum$ is used in (2.16)

$$[fh] = [f]\,[h] + f^+[h] + h^+[f], \tag{2.17}$$

then substituting the result into (2.14) and after using (2.4) we have (2.12) with (2.13).

Notice that the coefficient μ, defined by $(2.13)_1$, depends on the rheological (i.e. elastic and inelastic) properties of the material; it also depends on the strain rate, strain gradient, rate of the internal variables and on the gradient of the internal variables just ahead of the waves. On the other hand, the coefficient β, defined by $(2.13)_2$, depends on the elastic properties of the material alone.

In the equation (2.13) the derivative dU/dt takes place. By an additional calculation, this derivative may be determined.

LEMMA 2. *At an acceleration wave*

$$\frac{dU}{dt} = \frac{1}{2U\varrho}(\partial_E^2 \mathscr{T}\dot{E}^+ + \partial_\alpha \partial_E \mathscr{T}\cdot\dot{\alpha}) + \frac{1}{2\varrho}(\partial_E^2 \mathscr{T}(\partial_X E)^+ + \partial_\alpha \partial_X \mathscr{T}\cdot \partial_X \alpha), \tag{2.18}$$

where $\partial_E^2 \mathscr{T}$ and $\partial_\alpha \partial_X \mathscr{T}$ are evaluated at $(E(Y(t),t), \alpha(Y(t),t))$.

Proof. Let us notice that the operator of differentiation d/dt has the form

$$\frac{d}{dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial X}. \tag{2.19}$$

If we apply (2.19) to (2.11), we obtain(⁴)

$$2\varrho U\frac{dU}{dt} = \partial_E^2 \mathscr{T}(\dot{E}^+ + U(\partial_X E)^+) + \partial_\alpha \partial_E \mathscr{T}\cdot(\dot{\alpha} + U\partial_X \alpha), \tag{2.20}$$

where the homogeneous mass distribution was assumed.

Finally we have

THEOREM 1. *The amplitude of an acceleration wave propagating into a material with internal state variables satisfies the equation* (2.12) *with μ defined by*

$$\mu = -\frac{1}{2U\varrho}\left\{\frac{3}{2U}(\partial_E^2 \mathscr{T}\dot{E}^+ + \partial_\alpha \partial_E \mathscr{T}\cdot\dot{\alpha}) - \frac{1}{2}(\partial_E^2 \mathscr{T}(\partial_X E)^+ + \partial_\alpha \partial_E \mathscr{T}\cdot \partial_X \alpha) + \frac{1}{U}\partial_\alpha \mathscr{T}\cdot \partial_E A\right\} \tag{2.21}$$

and β defined by $(2.13)_2$. The velocity U is defined by (2.11).

(⁴) Let us notice that in (2.20) the values in $-$ region can be taken, because $\dot{f}^+ + U(\partial_X f)^+ = \dot{f}^- + U(\partial_X f)^-$ if $[f] = 0$.

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3. Local and global behaviour of the amplitude

In what follows we assumed that the velocity of the wave is positive $U(t) > 0$. It follows from([5]) (2.11) and (2.13) that

$$\operatorname{sign}\beta(t) = -\operatorname{sign}\partial_E^2\mathcal{T}(E(Y(t),t),\alpha(Y(t),t)). \tag{3.1}$$

Hence we can formulate the following propositions for the local (in time) behaviour of the amplitude $a(t)$. Before proceeding to state these results let us define $\lambda(t)$ and $\tilde{E}(t)$ by the relations

$$\lambda(t) \equiv \frac{\mu(t)}{\beta(t)}, \quad \tilde{E}(t) \equiv \partial_E^2\mathcal{T}(E(Y(t),t))\,\alpha(Y(t),t)). \tag{3.2}$$

PROPOSITIONS. 1. *At any instant t, if either $\tilde{E}(t) < 0$ and $a(t) < \lambda(t)$ or $\tilde{E}(t) > 0$ and $a(t) > \lambda(t)$, then $\frac{d}{dt}|a(t)| < 0$. In particular, if either* $\operatorname{sign} a(t) = \operatorname{sign}\lambda(t) = \operatorname{sign}\tilde{E}(t)$ *and $|a(t)| > |\lambda(t)|$ or* $\operatorname{sign} a(t) = \operatorname{sign}\lambda(t) = -\operatorname{sign}\tilde{E}(t)$ *and $|a(t)| < |\lambda(t)|$, then*

$$\frac{d}{dt}|a(t)| < 0. \tag{3.3}$$

2. *At any instant t, if either $\tilde{E}(t) < 0$ and $a(t) > \lambda(t)$ or $\tilde{E}(t) > 0$ and $a(t) < \lambda(t)$, then $\frac{d}{dt}|a(t)| > 0$. In particular, if either*

$$\operatorname{sign} a(t) = \operatorname{sign}\lambda(t) = \operatorname{sign}\tilde{E}(t) \quad \textit{and} \quad |a(t)| < |\lambda(t)| \tag{3.4}$$

or

$$\operatorname{sign} a(t) = \operatorname{sign}\lambda(t) = -\operatorname{sign}\tilde{E}(t) \quad \textit{and} \quad |a(t)| > |\lambda(t)|,$$

then

$$\frac{d}{dt}|a(t)| > 0. \tag{3.5}$$

3. *At any instant t*

$$a(t) = \lambda(t) \quad \textit{if, and only if,} \quad \frac{da(t)}{dt} = 0. \tag{3.6}$$

The proofs of the above results are simple consequences of (3.1) and of the properties of Bernoulli equation (2.12) written in the form

$$\frac{da}{dt} = \beta a(a-\lambda). \tag{3.7}$$

Let us notice that the statements 1–3 are true under the assumption $\beta(t) \neq 0$ for $t \geqslant 0$. It follows that $\tilde{E}(t)$ must be nonvanishing; in other words]

$$\tilde{E}(t) = \partial_E^2\mathcal{T} \neq 0. \tag{3.8}$$

([5]) In physical applications of the present theory we expect to have $\partial_E\mathcal{T} > 0$ because $\partial_E\mathcal{T} < 0$ leads to a purely imaginary value of U.

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Physically, the condition (3.8) states that the constitutive function $\mathcal{T}$ is non-linear in the strain (i.e. the elastic response of the material is non-linear).

Further, the statement 3 does not imply that the wave is steady, i.e. has a constant amplitude over all the period of time, for $\lambda(t)$ may increase or decrease during the period. The wave can have a constant amplitude over some period of time only if $d\lambda(t)/dt = 0$ during this period.

To investigate the global behaviour of the amplitude we apply the results of Bailey and Chen [1].

Now we assume that μ and β as the functions of t from the interval $[0, \infty)$ are integrable on every finite subinterval from $[0, \infty)$ and that for all $t \in [0, \infty)$ either(6)

$$\text{sign}\,\tilde{\mathrm{E}}(t) = +1 \quad \text{or} \quad \text{sign}\,\tilde{\mathrm{E}}(t) = -1. \tag{3.9}$$

Additionaly we shall assume that β is bounded away from zero; it is equivalent to the requirement

$$\lim_{t\to\infty} \inf|\beta(t)| \neq 0. \tag{3.10}$$

The above assumptions enable the formulation of the following theorem, which is due to that of Bailey and Chen.

Theorem 2. 1. *Suppose that* $\text{sign}\,\mathrm{a}(0) = -\text{sign}\beta(t)$. *If* λ *is bounded above (below) or tends to a non-negative (non-positive respectively) finite or infinite limit, then the same is true for any solution* $\mathrm{a}(t) > 0$ $\big(\mathrm{a}(t) < 0$ respectively$\big)$, $t \geqslant 0$.

2. *Suppose that* $\text{sign}\,\mathrm{a}(0) = \text{sign}\,\beta(t)$. *Let*

$$\omega \equiv \frac{1}{\int\limits_0^\infty |\beta(t)| e^{-\int\limits_0^t \mu(\tau)d\tau} dt}. \tag{3.11}$$

a) *If* $|\mathrm{a}(0)| > \omega$, *then there exists a unique finite time* $t_\infty > 0$ *such that*

$$\int\limits_0^{t_\infty} \beta(t) e^{-\int\limits_0^t \mu(\tau)d\tau} dt = \frac{1}{\mathrm{a}(0)} \tag{3.12}$$

and

$$\lim_{t\to t_\infty} |\mathrm{a}(t)| = \infty.$$

b) *If* $\int\limits_0^\infty |\beta(t)|dt = \infty$ *and* $|\mathrm{a}(0)| < \omega$, *then* $\lim\limits_{t\to\infty} \inf|\mathrm{a}(t)| = 0$.

In view of the results above on the global behaviour of the amplitude $\mathrm{a}(t)$, the number ω, defined by (3.11), is called the critical initial amplitude. The magnitude of the initial amplitude $\mathrm{a}(0)$ of the acceleration wave compared with the critical initial amplitude ω decides whether the acceleration wave amplitude grows without bound over a finite time interval or remains bounded at all times. The former conclusion, of course, suggests that a shock wave is produced.

(6) Condition (3.9) states: for each fixed α_0 the stress-strain law $T = \mathcal{T}(E, \alpha_0)$ is either concave from above or concave from below.

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Let us notice that in the special case, when μ and β are non-zero constants, i. e. μ_0 and β_0 respectively, it follows from (3.2) and (3.11) that

$$(3.13)\qquad \omega = \frac{\mu_0}{|\beta_0|} \text{ if } \mu_0 > 0 \quad \text{and} \quad \omega = 0 \text{ if } \mu_0 < 0.$$

The expression (3.13) is equal to $|\lambda_0|$; in view of the paper [2] it means that for constant coefficients of the governing equation the critical initial amplitude turns into the critical initial amplitude for the acceleration waves propagating into the material which has been at a homogeneous equilibrium state.

4. A wave entering a homogeneous state of a material of grade two

We consider here an acceleration wave which since time $t = 0$ has been propagating into a region which is in a homogeneous strain with the vanishing extrinsic body force, that is $b = 0$ in (2.9) and for all $t \geqslant 0$ and $X > Y(t)$

$$(4.1)\qquad \partial_X E(X, t) = 0.$$

Additionally, we assume that the initial distribution of the internal variables α_0 is homogeneous, too.

For such a case the initial value problem (2.6) for the internal variables has homogeneous solution only, for $t \geqslant 0$ and $X > Y(t)$. Therefore the distribution of the stress is also homogeneous (cf. (2.5)).

It appears that in the case under consideration the problem of finding the deformation which fulfils the condition above reduces to the solution of the following initial value problem [cf. (2.6) and (2.9)]

$$(4.2)\qquad \dot{v} = 0, \quad \dot{E} = \partial_X v, \quad \dot{\alpha} = A(E, \alpha)$$

with given $v(0) = v_0$, $E(0) = E_0$, $\alpha(0) = \alpha_0$ such that $\partial_X E = 0$, $\partial_X \alpha(0) = 0$. Here, the dependence of v, E and α on X is not denoted explicitly. In the equations (4.2) v denotes the velocity function of the particles during the motion with the displacement function u. Hence we can write

$$(4.3)\qquad v = \dot{u}, \quad E = \partial_X u,$$

where $u(X, t) = \chi(X, t) - X$.

The solution of (4.2) in the class of the homogeneous strain functions E is of the form

$$(4.4)\qquad E(t) = E(0) + Vt, \quad v(t) \equiv V(X - X_0) = v(0), \quad V = \text{const} > 0$$

with $\alpha(t)$ as the solution of the equation

$$(4.5)\qquad \dot{\alpha}(t) = A(E(0) + Vt, \alpha(t)), \quad \alpha(0) = \alpha_0.$$

The solution of (4.2) is unique under the condition that the solution of (4.5) is unique([7]).

Now, we can pass to the problem of the propagation of the wave.

([7]) Cf. sec. 2, where the conditions of uniqueness and existence were formulated.

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The assumption about the material ahead the wave enables us to formulate the following condition: for all $t \geqslant 0$ and $X > Y(t)$, i.e. in + region,

$$(4.6) \qquad E(X, t) = E_0 + Vt, \quad \dot{E}(X, t) = V, \quad \partial_X E(X, t) = 0, \quad \partial_x \alpha(X, t) = 0.$$

The variables E and α determine the state of a material. Hence we can say that in + region the material is in a homogeneous state.

We see that for such a wave the differential equation (2.12) cannot be solved explicitly for general materials. The knowledge of the derivatives of $\mathscr{T}$ and A is necessary(8). For this reason we consider the special case of material. We assume:

a) the material [the constitutive equation (2.5)] is of grade two in the strain E and of grade one in one internal state variable α;

b) the function A in (2.6) is linear in both variables.

These requirements mean that $\mathscr{T}(E, \alpha)$ is a polynominal of degree 2 in the variable E and of degree one in the variable α, but $A(E, \alpha)$ is a polynomial of degree one in both variables. Hence we have

$$(4.7) \qquad \begin{aligned} \mathscr{T}(E, \alpha) &\equiv b_1 E + b_2 \alpha + b_3 E\alpha + b_4 E^2 + b_0, \\ A(E, \alpha) &\equiv c_1 E + c_2 \alpha + c_0, \end{aligned}$$

where b_i, c_j, $i = 0, 1, \dots 4$, $j = 0, 1, 2$ are some physical constants.

For that case of the material we shall compute the velocity U and the coefficients μ and β of an acceleration wave propagating into the region in a homogeneous (strain) state.

The derivatives needed have the forms

$$(4.8) \qquad \begin{aligned} &\partial_E \mathscr{T}(E, \alpha) = b_1 + b_3 \alpha + 2b_4 E, \quad \partial_\alpha \mathscr{T}(E, \alpha) = b_2 + b_3 E, \\ &\partial_E^2 \mathscr{T}(E, \alpha) = 2b_4, \quad \partial_E A(E, \alpha) = c_1, \quad \partial_\alpha \partial_E \mathscr{T}(E, \alpha) = b_3. \end{aligned}$$

From the reality of the wave we have

$$(4.9) \qquad \partial_E \mathscr{T}(E, \alpha) = b_1 + b_3 \alpha + 2b_4 E > 0 \quad \text{for each } (E, \alpha).$$

We know that the strain E varies the interval $(-1, \infty)$ but α can be taken from the real axis(9). Hence we have the first condition on the b's:

$$(4.10) \qquad b_3 = 0.$$

From the inequality

$$b_1 + 2b_4 E > 0 \quad \text{for each} \quad E \in (-1, \infty)$$

we have

$$(4.11) \qquad b_1 > 2b_4 > 0.$$

Let us notice that the conditions (4.10), (4.11) are sufficient only.

For this reason instead of (4.8) we have

$$(4.12) \qquad \partial_E \mathscr{T}(E, \alpha) = b_1 + 2b_4 E, \quad \partial_\alpha \mathscr{T}(E, \alpha) = b_2, \quad \partial_\alpha \partial_E \mathscr{T}(E, \alpha) = 0$$

(8) Cf. (2.21) and $(2.13)_2$.

(9) Here, we assume none of the restrictions of the range of α.

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and additionally

$$\partial_E^2 \mathcal{T}(E, \alpha) = 2b_4 > 0. \tag{4.13}$$

Since the wave enters the region in a homogeneous state [which is an equilibrium([10]) one if V vanishes in (4.6) and $A(E_0, \alpha_0) = 0$], thus

$$U(t) = \sqrt{\frac{b_1 + 2b_4(E_0 + Vt)}{\varrho}}. \tag{4.14}$$

For the coefficients $\mu(t)$ and $\beta(t)$ we have the expressions [cf. (2.21), $(2.13)_2$ and (4.6)]

$$\begin{aligned} \mu(t) &= -\frac{1}{2U^2(t)\varrho}\left\{\frac{3}{2}\partial_E^2\mathcal{T}\dot{E}^+ + \partial_\alpha\mathcal{T}\partial_E A\right\} = -\frac{3b_4V + b_2c_1}{2(b_1 + 2b_4(E_0 + Vt))}, \\ \beta(t) &= -\frac{\partial_E^2\mathcal{T}}{2U(t)\partial_E\mathcal{T}} = \frac{-b_4\sqrt{\varrho}}{\sqrt{(b_1 + 2b_4(E_0 + Vt))^3}} \end{aligned} \tag{4.15}$$

and for $\lambda(t)$

$$\lambda(t) = \frac{\mu(t)}{\beta(t)} = \frac{3b_4V + b_2c_1}{2b_4}\sqrt{\frac{b_1 + 2b_4(E_0 + Vt)}{\varrho}}. \tag{4.16}$$

In view of (4.15) we see that the coefficients $\mu(t)$ and $\beta(t)$ are integrable functions on every finite subinterval of $[0, \infty)$. Additionally, the coefficient β fulfils

$$\operatorname{sign}\beta(t) = -1. \tag{4.17}$$

Unfortunately, the condition (3.10) does not hold in our case, because

$$\lim_{t\to\infty} |\beta(t)| = 0. \tag{4.18}$$

Since all the assumptions of Theorem 2 are not satisfied, therefore we cannot apply the results of it. But in view of [1], the condition (3.10) is needed only in the proof of point 1 of that theorem. Hence the point 2a) is true here. We shall formulate it in terms of the constants b's and c's together with other cases of the global behaviour of the amplitude $\mathsf{a}(t)$.

In that formulation the general form of the solution of the governing equation (2.12) will be helpful. Introducing the change of variable

$$\mathsf{d}(t) = \frac{1}{\mathsf{a}(t)}$$

the Eq. (2.12) reduces to

$$\frac{d}{dt}\mathsf{d}(t) = \mu\mathsf{d}(t) - \beta.$$

Since μ and β are integrable on every finite subinterval of $[0, \infty)$, the above equation has the solution

$$\mathsf{d}(t) = e^{\int_0^t \mu(\tau)d\tau}\left\{\mathsf{d}(0) - \int_0^t \beta(\tau)e^{-\int_0^\tau \mu(s)ds}d\tau\right\}.$$

([10]) In [2, 3] the propagation of acceleration waves in a material in a homogeneous equilibrium state was considered.

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Hence, for $\mathrm{a}(t)$ in (2.12) we have the solution

$$\mathrm{a}(t) = \frac{e^{-\int_0^t \mu(\tau)d\tau}}{\frac{1}{\mathrm{a}(0)} - \int_0^t \beta(\tau)e^{-\int_0^\tau \mu(s)ds}\,d\tau}, \tag{4.19}$$

where $\mathrm{a}(0)$ is the initial value of a.

In order to determine the critical initial amplitude ω, we compute

$$\begin{aligned} -\int_0^\tau \mu(\tau)d\tau &= \frac{3b_4V+b_2c_1}{4b_4V}\ln\frac{b_1+2b_4(E_0+Vt)}{b_1+2b_4E_0}, \\ e^{-\int_0^t \mu(\tau)d\tau} &= \left(\frac{\varrho U^2(t)}{\varrho U^2(0)}\right)^{\frac{3b_4V+b_2c_1}{4b_4V}}, \end{aligned} \tag{4.20}$$

where in view of (4.14) the following denotation was used

$$\varrho U^2(t) = b_1+2b_4(E_0+Vt), \quad \varrho U^2(0) = b_1+2b_4E_0.$$

Hence we have

$$\begin{aligned} \int_0^t \beta(\tau)e^{-\int_0^\tau \mu(s)ds}dt &= -b_4\sqrt{\varrho}\left(\varrho U^2(0)\right)^{-\frac{3b_4V+b_2c_1}{4b\,V}}\int_0^t \left(b_1+2b_4(E_0+V\tau)\right)^{\frac{3b_4V+b_2c_1}{4b_4V}}\times \\ &\times \left(b_1+2b_4(E_0+V\tau)\right)^{-\frac{3}{2}}d\tau = -\frac{2b_4\sqrt{\varrho}}{b_4V+b_2c_1}\left\{\left(\varrho U^2(0)\right)^{-\gamma}\left(\varrho U^2(t)\right)^{\eta} - \left(\varrho U^2(0)\right)^{-\frac{1}{2}}\right\}, \end{aligned} \tag{4.21}$$

where

$$\gamma \equiv \frac{3b_4V+b_2c_1}{4b_4V}, \quad \eta \equiv \frac{b_4V+b_2c_1}{4b_4V}. \tag{4.22}$$

For the initial critical amplitude we have the following expressions [cf. (3.11) and (4.21)]

$$\begin{aligned} \omega &= 0 \quad \text{if} \quad \eta > 0, \text{ i.e. } b_4V+b_2c_1 > 0, \\ \omega &= -\frac{b_4V+b_2c_1}{2b_4}U(0) = -2\eta VU(0) \quad \text{if}^{(11)} \quad \eta < 0, \text{ i.e. } b_4V+b_2c_1 < 0. \end{aligned} \tag{4.23}$$

For the material under consideration we have [cf. (3.2), (4.11)]

$$\tilde{\mathrm{E}}(t) = 2b_4 > 0.$$

We shall consider two distinct circumstances:

$$\operatorname{sign}\mathrm{a}(0) = \operatorname{sign}\tilde{\mathrm{E}}(t) = +1 \quad \text{and} \quad \operatorname{sign}\mathrm{a}(0) = -\operatorname{sign}\tilde{\mathrm{E}}(t) = -1$$

For the first case we formulate

[11] In applications the condition $b_2c_1 < 0$ seems natural. Hence $b_4V+b_2c_1 < 0$ is possible.

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THEOREM 3. *Let* $a(0) > 0$. *The amplitude* $a(t)$ *of an acceleration wave propagating into the material of grade two described by the assumptions* (4.7), (4.10), (4.11), *being in the + region in homogeneous state* (4.6), *tends to*:

a) *a finite limit provided* $b_2 c_1 < -3b_4 V$;

b) *an infinite limit, provided* $b_2 c_1 > -b_4 V$ or $-3b_4 V < b_2 c_1 < -b_4 V$.

Proof. Since $\beta(t) < 0$ and $a(0) > 0$ the denominator of (4.19) is always positive. At first we consider the case of a finite limit of the denominator. It is possible if, and only if, $\eta < 0$. Then, in the limit, we have

$$\lim_{t\to\infty}\left(\frac{1}{a(0)} - \int_0^t \beta(\tau) e^{-\int_0^\tau \mu(s)ds} d\tau\right) = \frac{1}{a(0)} - \frac{2b_4}{(b_4 V + b_2 c_1)\, U(0)} = \frac{1}{a(0)} - \frac{1}{2\eta V U(0)}.$$

For this reason the behaviour in time of the solution (4.19) depends only on the numerator. In view of $(4.20)_1$ we have

$$\lim_{t\to\infty} e^{-\int_0^t \mu(\tau)d\tau} = \lim\left(\frac{\varrho U^2(t)}{\varrho U^2(0)}\right)^{\gamma} = \begin{cases} \infty & \text{if} \quad \gamma > 0, \\ 0 & \text{if} \quad \gamma < 0. \end{cases} \tag{4.24}$$

But these conditions are equivalent to

$$\gamma > 0 \text{ iff } 3b_4 V + b_2 c_1 > 0 \quad \text{and} \quad \gamma < 0 \text{ iff } 3b_4 V + b_2 c_1 < 0.$$

Hence the above together with $\eta < 0$ give

$$\lim_{t\to\infty} a(t) = \infty \quad \text{if} \quad -3b_4 V < b_2 c_1 < -b_4 V,$$
$$\lim_{t\to\infty} a(t) = 0 \quad \text{if} \quad b_2 c_1 < -3b_4 V.$$

Now the unbounded denominator will be considered. In this case η must be positive according to (4.23). It follows the positive value of γ. Since the all terms on the right side of (4.19) are finite except for $(\varrho U^2(t))^{\gamma}$ in the numerator and $(\varrho U^2(t))^{\eta}$ in the denominator (cf. 4.21), therefore the limit value of $a(t)$ depends on the value at infinity of the ratio $\dfrac{(\varrho U^2(t))^{\gamma}}{(\varrho U^2(t))^{\eta}} = (U(t))^{2(\gamma-\eta)}$. But $2(\gamma - \eta) = 1$ and hence

$$\lim_{t\to\infty} a(t) = \infty \quad \text{if} \quad \eta > 0, \text{ i.e. } b_2 c_1 > -b_4 V.$$

For the case of the negative initial amplitude $a(0)$ we formulate

THEOREM 4. *Let* $a(0) < 0$. *In the case considered* (cf. Theorem 3) *we have the following global behaviour of the amplitude* $a(t)$:

a) *If* $a(0) < -\omega$, *then there exists a finite time* t_∞ *given by*

$$t_\infty = \frac{1}{2b_4 V}\left\{(\varrho U^2(0))^{\frac{\gamma}{\eta}}\left(\frac{1}{\sqrt{\varrho}\, U(0)} - \frac{2\eta V}{a(0)\sqrt{\varrho}}\right)^{\frac{1}{\eta}} - (\varrho U^2(0))\right\}, \tag{4.25}$$

such that

$$\lim_{t\to t_\infty} a(t) = -\infty.$$

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b) If $\mathfrak{a}(0) > -\omega$, then

$$\lim_{t\to\infty}\mathfrak{a}(t) = -\infty \quad \text{if} \quad -3b_4V < b_2c_1 < -b_4V,$$

$$\lim_{t\to\infty}\mathfrak{a}(t) = 0 \quad \text{if} \quad b_2c_1 < -3b_4V.$$

Proof. a) From the equality (cf. Theorem 2, point 2a)

$$\int_0^{t_\infty} \beta(t)e^{-\int_0^t \mu(\tau)d\tau} dt = \frac{1}{\mathfrak{a}(0)},$$

we have (4.25). Obviously, $\lim_{t\to t_\infty}\mathfrak{a}(t) = -\infty$. b) In order that $\mathfrak{a}(0) > -\omega$ and $\mathfrak{a}(0) < 0$ be possible it must be assumed that $\omega > 0$, which is to say that the integral

$$\int_0^t \beta(\tau)e^{-\int_0^\tau \mu(s)ds} d\tau$$

tends to a finite limit as t goes to infinity. But it is a case of $(4.23)_2$ in which b_4V+ $+b_2c_1 < 0$. For this reason the denominator of (4.19) is always negative and has a finite limit. It implies that the limit value of $\mathfrak{a}(t)$ is finite if $\gamma < 0$ and is infinite if $\gamma > 0$ (cf. proof of Theorem 3). These conditions together with negative η give the complete proof.

Let us notice that the part b) of Theorem 4 in comparison with the point 2b) of Theorem 2 gives a new result in the investigation. This is partially understood by the fact that the condition $\int_0^\infty |\beta(t)|dt = \infty$ is not satisfied in our case, because

$$\int_0^\infty |\beta(t)|dt = \int_0^\infty b_4\sqrt{\varrho}(b_1+2b_4(E_0+Vt))^{-\frac{3}{2}}dt = \frac{1}{VU(0)}. \tag{4.26}$$

Appendix

Remark 1. The material trajectory $\sum$ of the acceleration wave propagating into a material of grade two described by (4.7), (4.10), (4.11), being in a homogeneous state (4.6), is the set (the curve)

$$\sum = \left\{(Y(t), t) \in \mathscr{B}\times[0, \infty)\colon\ Y(t) = Y_0 + \frac{\varrho}{3b_4V}U^3(t)\right\}, \tag{A.1}$$

where $U(t) = \sqrt{\frac{b_1+2b_4(E_0+Vt)}{\varrho}}$ and $Y(0) = Y_0 - \frac{\varrho}{3b_4V}U^3(0)$ is the material point at which the wave is to be found at time $t = 0$.

The proof of the remark above is a simple consequence of the definition of the intrinsic velocity $U(t)$ (given in Sec. 2) and the relation (4.14) for $U(t)$.

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Remark 2. The amplitude of the acceleration wave in that case has the form

$$\text{(A.2)} \qquad \mathrm{a}(t) = \frac{\left(\dfrac{\varrho U^2(t)}{\varrho U^2(0)}\right)^{\gamma}}{\dfrac{1}{\mathrm{a}(0)} + \dfrac{\sqrt{\varrho}}{2\eta V}\left\{(\varrho U^2(0))^{-\gamma}(\varrho U^2(t))^{\eta} - (\varrho U^2(0))^{-\frac{1}{2}}\right\}},$$

where

$$\varrho U^2(0) = b_1 + 2b_4 E_0, \quad \gamma \equiv \frac{3b_4 V + b_2 c_1}{4b_4 V}, \quad \eta \equiv \frac{b_4 V + b_2 c_1}{4b_4 V}$$

and $\mathrm{a}(0)$ is the initial value of the amplitude. The proof of this proposition follows directly from (4.19)–(4.21).

Remark 3. If the initial amplitude $\mathrm{a}(0)$ is equal to the initial critical amplitude with the opposite sign $-\omega$, then the solution (4.19) has the form

$$\text{(A.3)} \qquad \mathrm{a}(t) = 2\eta V U(t)$$

with the property

$$\text{(A.4)} \qquad \lim_{t\to\infty} \mathrm{a}(t) = -\infty.$$

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INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received March 8, 1974.

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