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Spatially cognitive media[1]. I. Constitutive theory

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A CONSTITUTIVE theory is developed for media whose mechanical response (e. g., stress) at any one point is determined by their instantaneous global deformation-state. Such media may be said to possess spatial memory, perception, and/or awareness. To obviate the "time" connotation implied by "memory," we employ the generic term *spatially cognitive* or, simply *cognitive*. Appropriate constitutive equations are derived by an application of a *principle of constitutive invariance*, which in turn educes a mathematical rendering of a *global deformation-measure*. The constitutive equations can be simplified by postulating a principle of *fading spatial cognitivity* and, in some cases, the existence of a *spatial influence function*. An expression for the initial-stress distribution within three-dimensional bodies is derived. Contact with classical theories is established.

Opracowana została teoria konstytutywna dla ośrodków, których własności mechaniczne (naprężenie) są określone w każdym punkcie przez ich chwilowy globalny stan deformacji. Mówi się, że ośrodki takie posiadają pamięć przestrzenną, spostrzeganie oraz (lub) świadomość. Aby uniknąć skojarzenia z "czasem" implikowanym przez "pamięć" wprowadzimy *charakterystyczny termin poznawczy przestrzennie* lub po prostu *poznawczy*. Odpowiednie równania konstytutywne zostały wyprowadzone przy użyciu *zasady konstytutywnej niezmienniczości*, która z kolei prowadzi do matematycznego opisu *globalnej miary deformacji*. Równania konstytutywne mogą być uproszczone, jeśli zażąda się spełnienia zasady *zanikania przestrzennej poznawczości*, oraz, w niektórych przypadkach, istnienia *przestrzennej funkcji wpływu*. Wyprowadzono wyrażenia na początkowy rozkład naprężeń w ciałach trójwymiarowych. Wykazano związek z teoriami klasycznymi.

Разработана теория определяющих уравнений для сред, механические свойства (напряжения), которые определены в каждой точке через их мгновенное глобальное состояние деформации. Говорится, что такие среды обладают пространственной памятью, восприятием и (или) сознательностью. Чтобы избегнуть ассоциации с "временем", которое наводит "память", введем характеристический термин *пространственно познавательный* или просто *познавательный*. Соответствующие определяющие уравнения были выведены при использовании *принципа инвариантности определяющих уравнений*, который в свою очередь ведет к математическому описанию *глобальной меры деформации*. Определяющие уравнения могут быть упрощены, если потребуется удовлетворения принципу *исчезания пространственной познавательности* и, в некоторых случаях, существования *пространственной функции влияния*. Выведено выражение для начального распределения напряжений в трехмерных телах. Показана связь с классическими теориями.

1. Introduction

ALL finite bodies, whatever may be their composition, must exhibit some kind of "edge effect". From a molecular viewpoint it is apparent that, all else being "equal", the state

[1] Some facets of the theory herein developed may be found in a 1966 Ph. D. Thesis submitted by Y.S. PAN to the School of Engineering and Applied Science, University of California, Los Angeles (UCLA). The formulation, developed herein, however, is solely due to the present author.

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[3] A seminar on this topic, under the title: *Materials with Spatial Memory*, was presented on 25 November 1966 before the Institute of Fundamental Technological Research, Polish Academy of Sciences.

of stress at a point near one of the bounding surfaces of a finite body must be "different" from that pertaining at one of its "interior" points, because the former "point" is not as symmetrically surrounded by material as is the latter. Hence, even in the absence of external loading, all bodies possess an "initial-stress-state", which must primarily evidence itself within a "boundary layer", one of whose bounding surfaces is the surface of the body. In liquids such a boundary layer is responsible for surface tension effects — of course, its existence has long been recognized. This effect provides an example of the type of phenomena which motivate the present analysis. Such a phenomenon must exist in *all* bodies, be they solid, liquid, or otherwise.

In the case of solid bodies, this effect can be expected to be extremely "small" and restricted to very "thin" layers at the boundary. Nevertheless, it should be derivable from a continuum-mechanical theory. In this presentation we lay a groundwork for such an undertaking. To our knowledge, an analysis with this objective in mind was first given by TOUPIN and GAZIS [1], employing a polar material of grade 2. Naturally, such an analysis could also be attempted for polar materials of grade 3, 4, etc.

Here we adopt an entirely different alternative. After defining our notational scheme, Sect. 2, in Sect. 3 we invoke a *principle of constitutive invariance* [2] to construct an appropriate functional form for the constitutive equation with which each point in the body manifold is equipped [3]. For simplicity, time-memory effects are *not* included in our analysis. Some of our intuitive notions are rendered precise in Sect. 4, where a large variety of types of *elastic spatially cognitive media* are defined via appropriately formulated constitutive equations. A *principle of fading spatial cognitivity* is postulated in Sect. 5. With this principle, together with the postulated existence of a *spatial influence function*, the theory has greater predictive value and applicability. Two types of representations, given by FRÉCHET power series expansions, for the constitutive functionals are derived in Sect. 6. Contact with the now-classical theories is established in Sect. 7. Concluding remarks and some suggestions for further work are set forth in Sect. 8.

Without attempting to give a comprehensive review or comparison of their approaches, which appear to be quite different from that herein presented, we mention other work with similar objectives. EDELEN [4](4) and ERINGEN & EDELEN [5](4) have also considered *non-local* (i.e., here: *spatial-cognitivity*) *effects* in what are called [4] *proto-elastic* bodies. Recently ROGULA [6](4), employing a mode of ingress different from that invoked herein, has studied (in our terminology:) linearly elastic spatially cognitive media.

2. Notation and terminology

In general, we adopt the notation and terminology employed in [7, 8]. Because of its elegance, simplicity, and directness, we adhere to the type-face scheme described in [8, §6]. Explicit note will be made of exceptions at their place of occurrence. In particular, the following frequently used symbols stand for the below-named quantities:

(4) Where further references to the literature may be found. I hope to compile a survey and detailed comparison of these approaches in another presentation.

$\mathscr{B}$ body, composed of *points* endowed with a certain mathematical structure [8, § 15],

B_0 reference image of $\mathscr{B}$, $B_0 \subset \mathscr{E}$,

B_t image of $\mathscr{B}$ at time t, $B_t \subset \mathscr{E}$,

$\mathscr{E}$ three-dimensional Euclidean point space,

$\mathbf{F}$ deformation gradient tensor,

$\mathbf{R}$ local rotation tensor, $\mathbf{R}\mathbf{R}^T = \mathbf{I}$,

$\mathbf{T}$ Cauchy's stress tensor,

$\mathscr{V}$ vector translation-space associated with $\mathscr{E}$,

$\mathbf{X}$ points in B_0, $\mathbf{X} \in \mathscr{E}$, here called *stations*(5); when referred to *station-* (i.e., *material-*) coordinates: $\mathbf{X} = (X^1, X^2, X^3)$,

and

$\mathbf{x}$ points in B_t, $\mathbf{x} \in \mathscr{E}$, called *places*; when referred to *spatial* coordinates: $\mathbf{x} = (x^1, x^2, x^3)$.

The superscripts T and -1 denote "transpose" and "inverse", $\mathbf{I}$ and $\mathbf{0}$ stand for the identity and null tensors respectively. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors in $\mathscr{V}$, $\mathbf{a}\otimes\mathbf{b}$ will be their tensor product, an element of $\mathscr{T}$, the space of all second order tensors (linear transformations $\mathscr{V} \to \mathscr{V}$).

3. Formulation

There are several ways in which the formulation of the relevant constitutive equations may be approached. For instance, we may start from the general theory given in [2], Cf., Eqs. (10.1) or (10.2). Here we prefer to proceed directly.

Suppose that the only quantities to be included in the constitutive equations we wish to construct are:

$$(3.1)\qquad \begin{cases} \mathbf{x}, \mathbf{z} & \text{two } \textit{arbitrary}\text{, not necessarily the same, places in } B_t, \\ \mathbf{X}, \mathbf{Z} & \text{the stations in } B_0 \text{ corresponding to } \mathbf{x}, \mathbf{z}, \\ \mathbf{T} \equiv \mathbf{T}_{\mathbf{x},t} & \text{the stress tensor at the place } \mathbf{x} \in B_t, \text{ at time } t, \text{ and} \\ \mathbf{R} \equiv \mathbf{R}_{\mathbf{x},t} & \text{the local rotation tensor at } \mathbf{x} \in B_t, \text{ at time } t. \end{cases}$$

According to the principle of constitutive invariance [2, § 9], the absolute invariants of the *group* of transformations $G\{\tau, y, \mathscr{R}\}$ formed from the *set* of quantities (3.1):

$$(3.2)\qquad \mathbf{X}, \quad \mathbf{R}^{-1}\mathbf{T}\mathbf{R} \quad \text{and} \quad \mathbf{R}^{-1}(\mathbf{x}-\mathbf{z})$$

must be the only arguments appearing in our constitutive equation; hence, it must have the form:

$$(3.3)\qquad \mathfrak{F}\big(\mathbf{X}, \mathbf{R}^{-1}\mathbf{T}\mathbf{R}, \mathbf{R}^{-1}(\mathbf{x}-\mathbf{z})\big) = \mathbf{0},$$

(5) This is preferred over "material points" since there is nothing "material" about these points The term was suggested by S.K. WANG, one of the senior author's students. In conformance with ordinary usage: A point occupying a given station can, in the course of its motion, occupy various places.

where $\mathfrak{F}$ is a functional(6) *with* respect to its last argument, as indicated by the appearance of the ("dummy") place $\mathbf{z}$.

To obtain explicit results, restrictions must be imposed on the functional $\mathfrak{F}$. For our subsequent investigation we assume that (3.3) is uniquely invertible for $\mathbf{R}^{-1}\mathbf{T}\mathbf{R}$ and that the medium is *homogeneous* and locally *isotropic*, *cf.* [8], in its reference configuration. Then, recognizing that when the dependence on the pair $(\mathbf{x} \in B_t, t \in \mathscr{R})$ is rendered explicit: $\mathbf{T} = \mathbf{T}_{\mathbf{x},t}$ value-wise, (3.3) reduces to:

$$\begin{aligned} &\mathbf{T} = \mathfrak{F}_1(\mathbf{x}-\mathbf{z}), \quad \mathbf{z} \in B_t, \\ &\mathbf{Q}^T{}_1(\mathfrak{F}\mathbf{x}-\mathbf{z})\mathbf{Q} = \mathfrak{F}_1(\mathbf{Q}(\mathbf{x}-\mathbf{z})) \quad \text{for} \quad \textit{all } \mathbf{Q} \in \mathscr{Q}, \end{aligned} \tag{3.4}$$

where $\mathscr{Q}$ is the full orthogonal group for $\mathscr{V}$ (dim $\mathscr{V} = 3$) and, as indicated by the condition $(3.4)_2$, the functional(7) $\mathfrak{F}_1$, is *isotropic*(7) *relative to* $\mathscr{Q}$. Physically, invertibility is equivalent to a principle of determinism of stress [8, § 26]; however, so that we may explore the resulting consequences, we *do not* invoke a "principle of local action" [8, § 26]. Furthermore, since couple stresses may be present, we do not assume that $\mathbf{T}$ is symmetric [9].

Our task is to obtain explicit representations for the constitutive functional $\mathfrak{F}_1$.

4. Some definitions

For each time t, the place $\mathbf{z}$ exhibited in (3.4) is occupied by a point Z in $\mathscr{B}$; in turn, this point occupied the station $\mathbf{Z}$ in a reference configuration B_0 of $\mathscr{B}$. We call $\mathbf{z}$, $\mathbf{Z}$, and Z *characteristic places*, *stations*, and *points*. Basic to almost all of Continuum Mechanics is the supposition that corresponding places, stations, and points are related to each other by homeomorphisms(8); hence, any restriction stated in terms of one of these can be immediately expressed in terms of the others. Since points are primitive elements of the medium [8, §15], we state our definitions in terms of these. Several cases may be distinguished. The medium comprising the body may contain:

I. A countable number of discretely distributed characteristic points Z_i; this number may be:
   1. a positive integer: $Z_1, \ldots, Z_N$, or
   2. countably infinite.

II. An infinite number of continuously distributed characteristic points, which may form:
   1. a finite number of disjoint subsets $\mathscr{P}_1, \ldots, \mathscr{P}_N$ of $\mathscr{B}$, whose union is a proper subset of $\mathscr{B}$, or
   2. the whole body $\mathscr{B}$.

Additionally, we may consider combinations of Cases I and II.

(6) Here, and in the case of (3.4), in the sense of a map (operator) whose range-space is $\mathscr{T}$ and whose domain-space for the last argument is, for each fixed pair $(\mathbf{x} \in B_t,\ t \in \mathscr{R})$, the set $\mathscr{V}_{\mathbf{x},t}$ of $(5.1)_1$.

(7) Thus, known representation theorems [e.g., 8, §§ 10–13] for functions $\mathfrak{f}: \mathscr{V} \to \mathscr{T}$ isotropic relative to $\mathscr{Q}$ *can not* be invoked unless the *operator* $\mathfrak{F}_1$, is subjected to stringent restrictions; *cf.* the example in Sect. 5.

(8) Bi-unique, bi-continuous maps.

Each of these cases distinguishes among different possible classes of materials, whose response to given external loads need not be the same. Nevertheless, the constitutive equations for Cases II should, when suitably restricted, reduce to those of the corresponding Cases I.

Modelled after (3.4), the constitutive equations for these cases may be expressed as follows(9):

$$\text{(4.1)} \qquad \text{I.1} \quad \mathbf{T} = \mathfrak{f}_1(\mathbf{x}-\bar{\mathbf{z}}, \mathbf{x}-\mathbf{z}_1, \ldots, \mathbf{x}-\mathbf{z}_N), \qquad \bar{\mathbf{z}} \in B_t - \bigcup_{i=1}^{N} \mathbf{z}_i,$$

$$\text{(4.2)} \qquad \text{I.2} \quad \mathbf{T} = \mathfrak{f}_2(\mathbf{x}-\bar{\mathbf{z}}, \mathbf{x}-\mathbf{z}_1, \mathbf{x}-\mathbf{z}_2, \ldots), \qquad \bar{\mathbf{z}} \in B_t - \bigcup_{i=1}^{\infty} \mathbf{z}_i,$$

$$\text{(4.3)} \qquad \text{II.1} \quad \mathbf{T} = \mathfrak{F}_2(\mathbf{x}-\bar{\mathbf{z}}, \mathbf{x}-\mathbf{z}), \quad z \in \bigcup_{i=1}^{N} P_i, \quad \bar{\mathbf{z}} \in B_t - \bigcup_{i=1}^{N} P_i,$$

$$\text{(4.4)} \qquad \text{II.2} \quad \mathbf{T} = \mathfrak{F}_3(\mathbf{x}-\bar{\mathbf{z}}), \quad \mathbf{z} \in B_t,$$

where $\mathfrak{f}_1, \mathfrak{f}_2$ are isotropic $\mathscr{T}$-valued functions [functionals] with respect to the vector-valued arguments $\mathbf{x}-\mathbf{z}_i$ $[\mathbf{x}-\bar{\mathbf{z}}]$, $\mathfrak{F}_2, \mathfrak{F}_3$ are isotropic $\mathscr{T}$-valued functionals with respect to the indicated arguments, and P_i are the sets of places corresponding to $\mathscr{P}_i$. We call a medium whose response is governed by (4.4) *spatially cognitive*, to which we restrict the subsequent development. Equations (4.1), (4.2) and (4.3) define media with *interacting characteristic points* and with *interacting characteristic regions.*

5. Fading spatial cognitivity

Of course, as it stands, the constitutive equation (4.4) scarcely has predictive value. In order to render $\mathfrak{F}_3$ with greater explicitness, we explore the consequences of imposing on it a restriction which, in parallel with [8, §38], we call a *qualitative*:

POSTULATE 5.1. *FADING SPATIAL COGNITIVITY. The states of deformation of points occupying distant places in the medium relative to the place* $\mathbf{x}$ *at which the stress is to be determined have less influence on this stress than the deformation states of points occupying places near to* $\mathbf{x}$.

It is reasonable to expect that this postulate holds true, and reflects the physical situation, for most materials. In fact, it is a generalization of the principle of local action [8, §26], upon which are based nearly all linear and nonlinear theories in Continuum Mechanics. Our subsequent task will be that of recasting this postulate in suitable mathematical form.

The operator $\mathfrak{F}_3$ is reduced to a tractable form as a consequence of the following considerations. At the outset, define various subsets of the linear vector space $\mathscr{V}$ associated with the point space $\mathscr{E}$. Let

$$\text{(5.1)} \qquad \mathscr{V}_{\mathbf{x},t} \equiv \{\mathbf{x}-\mathbf{z} | \mathbf{z} \in B_t, \text{fixed}: \ t \in \mathscr{R}, \mathbf{x} \in B_t\}, \quad \mathscr{V}^{*}_{\mathbf{x},t} \equiv \{\mathscr{V}^{\mathbf{Q}}_{\mathbf{x},t} | \mathbf{Q} \in \mathscr{Q}\},$$

(9) Omitting cases in which a particle is multiply "characteristic".

where

$$\mathcal{Q} \text{ is the set of all orthogonal automorphisms of } \mathscr{V} \text{ and}$$

(5.2)

$$\mathscr{V}^{\mathbf{Q}}_{\mathbf{x},t} \equiv \{\mathbf{Q}(\mathbf{x}-\mathbf{z}) | \mathbf{z} \in B_t, \text{ fixed: } t \in \mathscr{R}, \mathbf{x} \in B_t, \mathbf{Q} \in \mathcal{Q}\}.$$

Call $\mathscr{V}_{\mathbf{x},t}$, $\mathscr{V}^{\mathbf{Q}}_{\mathbf{x},t}$ *vector configurations of* B_t *(relative to* $\mathbf{x} \in B_t$*)*. Clearly, $\mathscr{V}^{*}_{\mathbf{x},t}$ is the set of all vector configurations of B_t relative to $\mathbf{x}$, any two of which can be set into coincidence by a *rigid body* (i.e., orthogonal) automorphism of $\mathscr{V}$. The preceding defines an equivalence relation; hence, $\mathscr{V}^{*}_{\mathbf{x},t}$ is an equivalence class of vector-configurations of B_t relative to $\mathbf{x}$. Let $\mathscr{V}^{*}_{\mathbf{x}}$ be the set of all possible vector-configurations of B_t, any two of which are homeomorphic. Form the quotient space $\mathscr{V}_{\mathbf{x}} = \mathscr{V}^{*}_{\mathbf{x}}/\mathscr{V}^{*}_{\mathbf{x},t}$. Then, the elements of $\mathscr{V}_{\mathbf{x}}$ are the equivalence classes $(5.2)_2$.

With the preceding at hand, we may now consider the physical aspects of our problem. By the principle of constitutive invariance, the stress at $\mathbf{x} \in B_t$ at time t, $\mathbf{T}_{\mathbf{x},t}$, must be the same for any two vector-configurations of B_t which are inter-related by a rigid-body transformation. Hence, $\mathbf{T}_{\mathbf{x},t}$ must be given by an operator defined (not on $\mathscr{V}^{*}_{\mathbf{x}}$, but:) on $\mathscr{V}_{\mathbf{x}}$. Since it is isotropic, $\mathfrak{F}_3$ of (4.4) satisfies this condition; i.e., for all $\mathbf{Q} \in \mathcal{Q}$:

$$\mathfrak{F}_3(\mathbf{Q}\mathscr{V}_{\mathbf{x},t}) = \mathbf{Q}^T \mathfrak{F}_3(\mathscr{V}_{\mathbf{x},t})\mathbf{Q} \tag{5.3}$$

which, as it should, shows that $\mathbf{T}_{\mathbf{x},t}$ depends only on the representative of each element in $\mathscr{V}_{\mathbf{x}}$. Alternatively stated, relative to an isotropic response functional, the *set* $\mathscr{V}_{\mathbf{x},t}$ is a valid measure of the global state of deformation of the body.

As a further aid in reducing the form (5.3) of $\mathfrak{F}_3$, to characterize the manner in which the global deformation-measure $\mathscr{V}_{\mathbf{x},t}$ influences the stress at $\mathbf{x} \in B_t$, introduce:

DEFINITION 5.1. ***Spatial influence function***(10). *A rank-two tensor-valued isotropic function*

$$\mathfrak{h} : \mathscr{V} \to \mathscr{T} : \mathbf{v}_{\mathbf{x}}(z) \to \mathfrak{h}\big(\mathbf{v}_{\mathbf{x}}(\mathbf{z})\big), \tag{5.4}$$

where

$$\mathbf{v}_{(\cdot)}(\cdot) : \mathscr{E} \times \mathscr{E} \to \mathscr{V} : (\mathbf{x}, \mathbf{z}) \to \mathbf{v}_{\mathbf{x}}(\mathbf{z}) \equiv \mathbf{x} - \mathbf{z},$$

such that, with $||\cdot||$ *and* $\mathbf{0}$ *denoting the usual Euclidean norm and null rank-two tensor respectively, for each fixed* $\mathbf{x} \in \mathscr{E}$:

$$\operatorname{Lim} \mathfrak{h}\big(\mathbf{v}_{\mathbf{x}}(\mathbf{z})\big) = \mathbf{0} \quad \textit{as} \quad ||\mathbf{v}_{\mathbf{x}}(\mathbf{z})|| \to \infty, \tag{5.5}$$

and

$$\mathfrak{h}\big((\mathbf{v}_{\mathbf{x}}(\mathbf{z})\big) \neq \mathbf{0} \quad \textit{for any finite} \quad ||\mathbf{v}_{\mathbf{x}}(\mathbf{z})|| \geqslant 0.$$

Then, factorize $\mathfrak{F}_3$ such that

$$\mathfrak{F}_3(\mathscr{V}_{\mathbf{x},t}) = \mathfrak{S}\big(\mathfrak{h}(\mathscr{V}_{\mathbf{x},t})\big), \tag{5.6}$$

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(10) Since the prescription of a spatial influence function partially characterizes the material, its definition does (and can) not depend on B_t, which a purely geometric aspect of $\mathscr{B}$. In contrast, as indicated by (5.3), the response of $\mathscr{B}$ at $(\mathbf{x} \in B_t, t \in \mathscr{R})$ does depend on B_t.

where $\mathfrak{S}$ is $\mathscr{T}$-valued and isotropic. The domain of $\mathfrak{S}$ is the collection of *all sets* (*not* points) $\mathfrak{h}(\mathscr{V}_{\mathbf{x},t}) \subset \mathscr{T}$. But, alternatively, for each fixed pair ($\mathbf{x} \in B_t$, $t \in \mathscr{R}$), an assignment of a point-wise one-to-one correspondence between domain and range sets in the appropriate domain and range spaces defines $\mathfrak{h}^{\mathbf{x},t}$, where, also admitting a parametric t-dependence:

$$(5.7) \qquad \mathfrak{h}^{\mathbf{x},t} \equiv (\cdot\,;t) : \mathscr{V}_{\mathbf{x},t} \to \mathscr{T} : \mathbf{x}-\mathbf{z} \to \mathfrak{h}(\mathbf{x}-z;t) \equiv \mathfrak{h}^{\mathbf{x},t}(\mathbf{x}-\mathbf{z}).$$

Invoked in the sequel, another function related to $\mathfrak{h}^{\mathbf{x},t}$ is isolated via the definition:

$$(5.8) \qquad \mathfrak{h}_{\mathbf{x},t} : B_t \to \mathscr{T} : \mathbf{z} \to \mathfrak{h}_{\mathbf{x},t}(\mathbf{z}) \equiv (\mathfrak{h}^{\mathbf{x},t} \circ \mathbf{v}_{\mathbf{x}})(\mathbf{z}) \equiv \Big(\big(\mathfrak{h}(\cdot\,;t)\big) \circ \mathbf{v}_{\mathbf{x}}\Big)(\mathbf{z}).$$

In certain instances where no ambiguity is likely to occur, $\mathfrak{h}_{\mathbf{x}}$ will stand as an abbreviation for $\mathfrak{h}_{\mathbf{x},t}$. Set-wise, let:

$$(5.9) \qquad \mathscr{H}^{*}_{\mathbf{x},t} = \{\mathfrak{h}_{\mathbf{x},t} |\ \text{fixed: } \mathbf{x} \in B_t,\ t \in \mathscr{R}\}.$$

Hence, rendering explicit the dependence on $\mathbf{x}$, t, there exists a map (operator) $\mathfrak{G}_{\mathbf{x},t} : \mathscr{H}^{*}_{\mathbf{x},t} \to \to \mathscr{T}$ such that:

$$(5.10) \qquad \mathbf{T}_{\mathbf{x},t} = \mathfrak{G}^{\mathbf{x},t}(\mathfrak{h}^{\mathbf{x},t}) \equiv \mathfrak{G}_{\mathbf{x},t}(\mathfrak{h}_{\mathbf{x},t}),$$

which is the relation from which there devolves all of our subsequent analysis.

An inner product on $\mathscr{H}^{*}_{\mathbf{x},t}$ is a real ($\mathscr{R}$)-valued bilinear map

$$(5.11) \qquad \langle \cdot, \cdot \rangle : \overset{2}{\times} \mathscr{H}^{*}_{\mathbf{x},t} \to \mathscr{R} : (\mathsf{r}_{\mathbf{x},t}, \mathsf{s}_{\mathbf{x},t}) \to \langle \mathsf{r}_{\mathbf{x},t}, \mathsf{s}_{\mathbf{x},t} \rangle$$

which satisfies the relevant axioms [11, p. 2], where (with $k = 2$ in the present instance) $\overset{k}{\times} \mathscr{H}^{*}_{\mathbf{x},t}$ stands for the k-fold Cartesian product of $\mathscr{H}^{*}_{\mathbf{x},t}$. Let $\mathscr{L}^{2}_{\mathbf{x},t} \equiv \mathscr{L}^{2}(\mathscr{V}_{\mathbf{x},t};t) \subset \mathscr{H}^{*}_{\mathbf{x},t}$ be the sub-set of all functions in $\mathscr{H}^{*}_{\mathbf{x},t}$ which are component-wise LEBESGUE square-integrable. Then, selecting that possibility which appears to best promote the intended application, employing the abbreviation (generically:) $\mathsf{r}_{\mathbf{x}} \equiv \mathsf{r}_{\mathbf{x},t}$, define the map (5.11) by:

$$(5.12) \qquad \langle \mathfrak{r}_{\mathbf{x}}, \mathfrak{s}_{\mathbf{x}} \rangle_t \equiv \int_{B_t} \operatorname{tr}\{[\mathfrak{r}_{\mathbf{x}}(\mathbf{z})\mathsf{w}_{\mathbf{x}}(\mathbf{z})]^T \mathfrak{s}_{\mathbf{x}}(\mathbf{z})\mathsf{w}_{\mathbf{x}}(\mathbf{z})\}\, d\tau, \qquad (\mathfrak{r}_{\mathbf{x}}, \mathfrak{s}_{\mathbf{x}}) \in \overset{2}{\times} \mathscr{L}^{2}_{\mathbf{x},t},$$

where the choice of the isotropic *weight function* is restricted to those $\mathsf{w}_{\mathbf{x}} \equiv \mathsf{w}_{\mathbf{x},t} \in \mathscr{H}^{*}_{\mathbf{x},t}$ for which the integral in (5.12) retains meaning in the LEBESGUE square-integrable sense, "tr" stands for the "trace" operation, and $d\tau$ is the volume element associated with the point $\mathbf{z} \in \mathscr{E}$, the "physical" space. The inner product (5.12) induces the *norm*:

$$(5.13) \qquad \|\mathfrak{s}_{\mathbf{x}}\|^2_t \equiv \langle \mathfrak{s}_{\mathbf{x}}, \mathfrak{s}_{\mathbf{x}} \rangle_t \equiv \int_{B_t} |\mathfrak{s}_{\mathbf{x}}(\mathbf{z})|^2_{\mathsf{w}}\, d\tau$$

which satisfies all of the norm axioms [10, p. 10, 11, p. 2], and where, in parallel with the classical case, $|\mathfrak{s}_{\mathbf{x}}(\mathbf{z})|_{\mathsf{w}}$ may be called the w-*magnitude* of $\mathfrak{s}_{\mathbf{x}}(\mathbf{z})$. Define:

$$(5.14) \qquad \mathscr{H}^{*}_{\mathsf{w}}(\mathbf{x}, t) \equiv \left[\ \mathscr{L}^{2}_{\mathbf{x},t}\ \middle|\ \begin{array}{l}\text{equipped with inner product (5.12), induced norm}\\ \text{(5.13), and completed [11, p. 4] relative to the}\\ \text{topology induced by this norm.}\end{array}\right]$$

Then $\mathscr{H}^{*}_{\mathsf{w}}(\mathbf{x}, t)$ is a HILBERT space.

Henceforth, regard $\mathfrak{G}_{\mathbf{x},t}$ as a $\mathscr{T}$-valued map on $\mathscr{H}^*_{\mathbf{w}}(\mathbf{x}, t)$. Let $\mathscr{G}^*$ denote the set of all such $\mathfrak{G}_{\mathbf{x},t}$. Define a norm on $\mathscr{G}^*$ by([11]):

$$(5.15)\qquad \|\mathfrak{G}_{\mathbf{x},t}\|^2 \equiv \sup_{\mathfrak{F}_{\mathbf{x},t}\in\mathscr{H}^*_{\mathbf{w}}(\mathbf{x},t)} \operatorname{tr}\left\{\left[\mathfrak{G}_{\mathbf{x},t}\left(\frac{\mathfrak{F}_{\mathbf{x},t}}{\|\mathfrak{F}_{\mathbf{x},t}\|}\right)\right]^T \mathfrak{G}_{\mathbf{x},t}\left(\frac{\mathfrak{F}_{\mathbf{x},t}}{\|\mathfrak{F}_{\mathbf{x},t}\|}\right)\right\}$$

Let([12])

$$(5.16)\qquad \mathscr{B}_n = \{\mathfrak{G}_{\mathbf{x},t}\in\mathscr{G}^* \,|\, \|\mathfrak{G}_{\mathbf{x},t}\| < \infty\}.$$

Then, upon completing $\mathscr{B}_n$ relative to the (5.15)-norm induced topology, the resulting $\mathscr{B}^*_n$ is a BANACH space [10, p. 10].

Before proceeding further, to fix ideas, we give an example of possible choices for $\mathfrak{h}$ and $\mathfrak{G}_{\mathbf{x},t}$. Employing curvilinear coordinates, a choice of an objective [8, §17] rank-two-symmetric-tensor valued function on $\mathscr{V}$ satisfying requirements (5.5) is:

$$(5.17)\qquad \mathfrak{h}^i_j = \tilde{\mathfrak{h}}^i_j(\mathbf{v}) \equiv Ae^{-\|\mathbf{v}\|}\delta^i_j + Be^{-2\|\mathbf{v}\|}v^iv_j, \qquad \mathfrak{h}^{ij} \text{ symmetric},$$

where A and B are constants. Next, supposing that symmetric-tensor valued response functionals are admissible, we construct an example of $\mathfrak{G}_{\mathbf{x},t}$ from an objective rank-two-symmetric-tensor valued function $\mathfrak{f}$ on $\mathscr{T}$. An $\mathfrak{f}$ having these properties [8, §12] is:

$$(5.18)\qquad \mathfrak{f}^i_j = \tilde{\mathfrak{f}}^i_j(\tilde{\mathfrak{h}}(\mathbf{v})) = \phi_0\delta^i_j + \phi_1\mathfrak{h}^i_j + \phi_2\mathfrak{h}^i_k\mathfrak{h}^k_j, \qquad \mathfrak{f}^{ij} \text{ symmetric},$$

where $\phi_{0,1,2}$ are values of functions $\bar{\phi}_{0,1,2}$ of the three principal invariants of $\tilde{\mathfrak{h}}$. Then, upon substituting (5.17) in (5.18),

$$(5.19)\qquad \mathfrak{f}^i_j = \bar{\Phi}_0\delta^i_j + \bar{\Phi}_1 v^i v_j,$$

where

$$\bar{\Phi}_0 = \overset{\Delta}{\Phi}_0(\mathbf{v}) \equiv \phi_0 + \phi_1 Ae^{-\|\mathbf{v}\|} + \phi_2 A^2 e^{-2\|\mathbf{v}\|}, \qquad \Phi^*_0 \equiv \overset{\Delta}{\Phi}_0(\mathbf{v}_{\mathbf{x}}(z)),$$

and

$$(5.20)\qquad \bar{\Phi}_1 = \overset{\Delta}{\Phi}_1(\mathbf{v}) \equiv (\phi_1 + 2\phi_2 Ae^{-\|\mathbf{v}\|} + \phi_2 B\|\mathbf{v}\|^2 e^{-2\|\mathbf{v}\|})Be^{-2\|\mathbf{v}\|}, \quad \text{and} \quad \Phi^*_1 = \overset{\Delta}{\Phi}(\mathbf{v}_{\mathbf{x}}(\mathbf{z}))$$

Set

$$(5.21)\qquad T^i_j = [\mathfrak{G}_{\mathbf{x},t}]^i_j \equiv \int_{B_t} (\mathfrak{f}^i_j \circ \mathbf{v}_{\mathbf{x}})(\mathbf{z})\,d\tau = \left(\int_{B_t} \Phi^*_0\,d\tau\right)\delta^i_j + \int_{B_t} \Phi^*_1(\mathbf{v}_{\mathbf{x}}(\mathbf{z}))^i(\mathbf{v}_{\mathbf{x}}(\mathbf{z}))_j\,d\tau.$$

Then $(5.21)_2$ *defines* the functional $\mathfrak{G}_{\mathbf{x},t}$.

With the preceding at hand, we may now formulate *a mathematical* form of Postulate 5.1:

> *POSTULATE* 5.2. *FADING SPATIAL COGNITIVITY* (*Mathematical Rendering*). *For each fixed pair* ($\mathbf{x}\in B_t$, $t\in\mathscr{R}$), *the response functional* $\mathfrak{G}_{\mathbf{x},t}$ *and spatial influence function* $\mathfrak{h}_{\mathbf{x},t}$ *are elements of* $\mathscr{B}^*_n$ *and* $\mathscr{H}^*_{\mathbf{w}}(\mathbf{x}, t)$ *respectively, with* $\mathfrak{h}_{\mathbf{x},t}$ *satisfying* (5.5).

([11]) I am indebted to the referee for propounding an improvement at this juncture in a preceeding version of this presentation. By imposing additional restrictions, a subset $\mathscr{G}^*_{\mathscr{H}}$ of $\mathscr{G}^*$ can be isolated such that $\mathscr{G}^*_{\mathscr{H}}$ is a HILBERT space. However, this additional stringency is not required for the development in the sequel.

([12]) The physical significance of the operator-class $\mathscr{B}_n$ (or $\mathscr{B}^*_n$) is readily perceived: as is physically reasonable, only those operators $\mathfrak{G}_{\mathbf{x},t}$ whose range is the class stress tensors with finite norm are regarded as admissible.

For certain applications it is convenient to introduce a *spatial influence parameter* $\sigma, 0 < \sigma \leqslant 1$, to characterize the *rate of decay* of the spatial influence. Suppose that $\mathfrak{h}$ is dependent on σ in such a way that

$$(5.22) \qquad (\mathfrak{h}_t \circ \mathbf{v}_{\mathbf{x},\sigma})(\mathbf{z}) \equiv \Big(\big(\mathfrak{h}(\cdot\,; t)\big) \circ \mathbf{v}_{\mathbf{x}.\sigma}\Big)(\mathbf{z}) \equiv \mathfrak{h}\left(\frac{\mathbf{x}-\mathbf{z}}{\sigma}; t\right), \; (\mathfrak{h}_t \circ \mathbf{v}_{\mathbf{x}.\sigma})(\mathbf{0}) \neq \mathbf{0}.$$

Thus, as $\sigma \to 0$, the influence of places other than $\mathbf{x}$ will diminish. The parameter σ is a material property; hence, an assignment of σ partially characterizes the medium.

6. Expansions for the constitutive functional (13)

In parallel with [8, §§35–40] we now explore possibilities for obtaining explicit and systematic approximations for the functional $\mathfrak{G}_{\mathbf{x},t}$. It is at this juncture that the conditions of the preceding postulate serve one of their intended roles: that of permitting the application of the theory of FRÉCHET differentials and RIESZ's theorem. First, with a view towards later applications, we derive an expansion in terms of *k-th order spatial influence functions.*

Let $\mathfrak{G}_{\mathbf{x},t}$ be *analytic* (in the sense of [10, p. 81]) about the null element $\boldsymbol{\theta} \in \mathscr{H}^*_{\mathrm{w}}(\mathbf{x}, t)$. Then $\mathfrak{G}_{\mathbf{x},t}$ has a FRÉCHET power series expansion [10, p. 82]:

$$(6.1) \qquad \mathbf{T}_{\mathbf{x},t} = \mathfrak{G}_{\mathbf{x},t}(\mathfrak{h}_t \circ \mathbf{v}_{\mathbf{x},\sigma}) = \sum_{k=0}^{\infty} \frac{1}{k!} \delta^k \mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\theta}; \mathfrak{h}_t \circ \mathbf{v}_{\mathbf{x},\sigma}),$$

where $\delta^k \mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\theta}; \mathfrak{h})$, a homogeneous polynomial of degree k in $\mathfrak{h}$ [10, p. 68], is the k-th variation of $\mathfrak{G}_{\mathbf{x},t}$ about $\boldsymbol{\theta}$ with increment $\mathfrak{h}$. Since it is the value of $\mathbf{T}_{\mathbf{x},t}$ at $\mathbf{x} \in \mathscr{E}$ as

$$\left\|\frac{1}{\sigma}(\mathbf{x}-\mathbf{z})\right\| \to \infty, \; \delta^0 \mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\theta}; \mathfrak{h}_t \circ \mathbf{v}_{\mathbf{x},\sigma}) \equiv \mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\theta}) = \mathbf{0} \in \mathscr{T}_{\mathbf{x},t}.$$

The $k = 1$ term in (6.1) is an isotropic linear operator $\delta^1 \mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\theta}; \cdot) : \mathscr{H}^*_{\mathrm{w}}(\mathbf{x}, t) \to \mathscr{T}_{\mathbf{x},t}$, where $\mathscr{T}_{\mathbf{x},t}$ is the $\mathscr{T}$-fibre over $\mathbf{x} \in B_t$ and, relative to a basis for $\mathscr{T}_{\mathbf{x},t}$, each of its components is a linear functional $[\delta^1 \mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\theta}; \cdot)]_{ij} : \mathscr{H}^*_{\mathrm{w}}(\mathbf{x}, t) \to \mathscr{R}$. Hence, by RIESZ's theorem [10, §16], there exists a unique element $\mathsf{G}_{\mathbf{x},t}(\boldsymbol{\theta}; \cdot) \in \mathscr{T}_{\mathbf{x},t} \otimes \mathscr{H}^*_{\mathrm{w}}(\mathbf{x}, t)$ such that:

$$(6.2) \qquad \delta^1 \mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\theta}; \mathfrak{h}_t \circ \mathbf{v}_{\mathbf{x},\sigma}) = \langle \mathsf{G}_{\mathbf{x},t}(\boldsymbol{\theta}; \cdot), \mathfrak{h}_t \circ \mathbf{v}_{\mathbf{x},\sigma} \rangle .$$

More generally,

$$(6.3) \qquad \delta^k \mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\theta}; \cdot, \ldots, \cdot) : \overset{k}{\times} \mathscr{H}^*_{\mathrm{w}}(\mathbf{x}, t) \to \mathscr{T}_{\mathbf{x},t}, \qquad k = 1, 2, \ldots,$$

is isotropic and k-linear. Then, reasoning as above and generalizing on [11, §21], it has the representation:

$$(6.4) \qquad \delta^k \mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\theta}; {}^1\mathfrak{h}_{\mathbf{x}}, \ldots, {}^k\mathfrak{h}_{\mathbf{x}}) = \langle \mathfrak{L}^{k-1}_{\mathbf{x},t}(\boldsymbol{\theta}; {}^1\mathfrak{h}_{\mathbf{x}}, \ldots, {}^{(k-1)}\mathfrak{h}_{\mathbf{x}}), {}^k\mathfrak{h}_{\mathbf{x}} \rangle$$

(13) Particularly in this Section, since they often occur juxtaposed with summation indices, to distinguish them from the former, *enumerative lables/indices* are set in the *light-face sans-serif* type-face style.

where, with the *lable* $i = 1, 2, \ldots, k$, ${}^i\mathfrak{h}_{\mathbf{x}} \equiv {}^i\mathfrak{h}_t \circ \mathbf{v}_{\mathbf{x},\sigma}$ suppresses the dependence on t and σ, and

$$(6.5) \qquad \mathfrak{L}^{k-1}_{\mathbf{x},t}(\boldsymbol{\theta}; \cdot, \ldots, \cdot) : \overset{k-1}{\times} \mathscr{H}^*_{\mathrm{w}}(\mathbf{x}, t) \to \mathscr{T}_{\mathbf{x},t} \otimes \mathscr{H}^*_{\mathrm{w}}(\mathbf{x}, t)$$

is an isotropic k-linear map. The required representation for a homogeneous polynomial of degree k follows upon symmetrizing (6.4) and evaluating the resulting relation on the diagonal of $\overset{k-1}{\times} \mathscr{H}^*_{\mathrm{w}}(\mathbf{x}, t)$:

$$(6.6) \qquad \delta^k \mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\theta}; \mathfrak{h}_{\mathbf{x}}) = \frac{1}{k!}\Big[\sum_{\pi} \langle \mathfrak{L}^{k-1}_{\mathbf{x},t}(\boldsymbol{\theta}; {}^{\pi(1)}\mathfrak{h}_{\mathbf{x}}, \ldots, {}^{\pi(k-1)}\mathfrak{h}_{\mathbf{x}}), {}^{\pi(k)}\mathfrak{h}_{\mathbf{x}}\rangle\Big]_{{}^i\mathfrak{h}_{\mathbf{x}} = \mathfrak{h}_{\mathbf{x}}}, \qquad i = 1, 2, \ldots, k,$$

where the sum is taken over the $k!$ permutations π of the labels $i = 1, 2, \ldots, k$. Of course (6.2) and (6.6) must coincide for $k = 1$. The expression for $\mathbf{T}_{\mathbf{x},t}$ follows upon substituting (6.2) and (6.6) in (6.1). The consequent equation may be cast in the form:

$$(6.7) \qquad T_{\mathbf{x},t} = \sum_{k=1}^{\infty} \frac{1}{k!}\Big[\frac{1}{k} \sum_{\pi_c} \langle {}^S\mathfrak{L}^{k-1}_{\mathbf{x},t}(\boldsymbol{\theta}; {}^{\pi_c(1)}\mathfrak{h}_{\mathbf{x}}, \ldots, {}^{\pi_c(k-1)}\mathfrak{h}_{\mathbf{x}}), {}^{\pi_c(k)}\mathfrak{h}_{\mathbf{x}}\rangle\Big]_{{}^i\mathfrak{h}_{\mathbf{x}} = \mathfrak{h}_{\mathbf{x}}}, \qquad i = 1, 2, \ldots, k,$$

where the second sum is taken over a cyclic permutation π_c of the labels $1, 2, \ldots, k$, and ${}^S\mathfrak{L}^{k-1}_{\mathbf{x},t}$ is isotropic, $(k-1)$-linear, and completely symmetric in its entries.

Since, subject to the above-mentioned three conditions, the operator ${}^S\mathfrak{L}^{k-1}_{\mathbf{x},t}$ may be arbitrarily assigned(14), greater explicitness can only be achieved by a closer specification of its form. To this end, hypothesize that, for each k, ${}^S\mathfrak{L}^{k}_{\mathbf{x},t}$ is an integral operator. For instance, with weight functions ${}^i\mathbf{W}_{\mathbf{x}}(\cdot, \ldots, \cdot) : \overset{i}{\times} \mathscr{V}_{\mathbf{x},t} \to \mathscr{T}_{\mathbf{x},t}$, $i = 1, 2, \ldots, k$, subject to previously mentioned restrictions, define:

$$(6.8) \qquad {}^S\mathfrak{L}^{k}_{\mathbf{x},t}(\boldsymbol{\theta}; {}^1\mathfrak{h}_{\mathbf{x}}, \ldots, {}^k\mathfrak{h}_{\mathbf{x}}) \equiv \int_{B_t} \ldots \int_{B_t} {}^k\mathbf{W}^T_{\mathbf{x}}(\mathbf{z}_1, \ldots, \mathbf{z}_k) \times \\ \times \Big[[{}^k\mathrm{G}^{s,S}_{\mathbf{x},t}(\boldsymbol{\theta}; \mathbf{z}, \mathbf{z}_1, \ldots, \mathbf{z}_k)]\big({}^1\mathfrak{h}_{\mathbf{x}}(\mathbf{z}_1), \ldots, {}^k\mathfrak{h}_{\mathbf{x}}(\mathbf{z}_k)\big)\Big]^T {}^k\mathbf{W}_{\mathbf{x}}(\mathbf{z}_1, \ldots, \mathbf{z}_k)\, d\tau_1 \ldots d\tau_k.$$

Call the integrand a *k-th order spatial influence function*, where the *kernel*

$$(6.9) \qquad \begin{aligned} &{}^k\mathrm{G}^{s,S}_{\mathbf{x},t}(\boldsymbol{\theta}; \cdot, \ldots, \cdot) \in \mathscr{T}_{\mathbf{x},t} \otimes \mathscr{H}^*_{\mathrm{w}}(\mathbf{x}, t) \otimes (\overset{k}{\otimes} \mathscr{H}^*_{\mathbf{x},t}), \\ &[{}^k\mathrm{G}^{s,S}_{\mathbf{x},t}(\boldsymbol{\theta}, \cdot, \mathbf{z}_1, \ldots, \mathbf{z}_k)] : \overset{k}{\times} \mathscr{T}_{\mathbf{x},t} \to \mathscr{T}_{\mathbf{x},t} \otimes \mathscr{H}^*_{\mathrm{w}}(\mathbf{x}, t), \end{aligned}$$

(14) That such a level of mathematical abstraction *must* be admitted is but a reflection of the dictates of the physical situation — the *class* of spatially cognitive media — towards which we address our inquiry. The degree of concreteness with which the ${}^S\mathfrak{L}^{k-1}_{\mathbf{x},t}$ are specified delineates certain types of sub-classes of such media. In particular, note that at the stage (6.6) no mechanism has been introduced for distinguishing between solid-, intermediate-, or fluid bodies. This type of classification is achieved by regarding as admissible only those subsets of the set of *all* ${}^S\mathfrak{L}^{k-1}_{\mathbf{x},t}$ which possess certain *isotropy* ($\equiv$ material symmetry) *groups*, $k = 1, 2, \ldots$. We do not here delve into these cases since our considerations remain within the proximity of the abstraction-level exemplified by (6.6).

also subject to appropriate integrability conditions, is isotropic, k-linear as denoted by the brackets in $(6.9)_2$, and symmetric with respect to: (s) its $k+1$ arguments: $\mathbf{z}, \mathbf{z}_1, \ldots, \mathbf{z}_k$, and ($S$) its k entries: ${}^1\mathfrak{h}_{\mathbf{x}}, \ldots, {}^k\mathfrak{h}_{\mathbf{x}}$. The component forms of (6.8) are determined relative to a given basis for $\mathscr{T}$; for example, when $k = 2$:

$$(6.10) \qquad {}^S\mathfrak{L}^{2,ij}_{\mathbf{x}}(\boldsymbol{\theta}; {}^1\mathfrak{h}_{\mathbf{x}}, {}^2\mathfrak{h}_{\mathbf{x}})$$
$$= \int_{B_t}\int_{B_t} {}^2\mathrm{G}^{s,S}_{\mathbf{x},t;klstuv}(\boldsymbol{\theta}; \mathbf{z}, \mathbf{z}_1, \mathbf{z}_2)\,{}^1\mathfrak{h}^{st}_{\mathbf{x}}(\mathbf{z}_1)\,{}^2\mathfrak{h}^{uv}_{\mathbf{x}}(\mathbf{z}_2)\,{}^2W^{kl}_{\mathbf{x}}(\mathbf{z}_1, \mathbf{z}_2)\,{}^2W^{lj}_{\mathbf{x}}(\mathbf{z}_1, \mathbf{z}_2)\,d\tau_1\, d\tau_2\,.$$

An aspect of these results is worth noting. If $\mathbf{x}, \mathbf{z}, \mathbf{z}_1, \ldots, \mathbf{z}_k$ are replaced by their corresponding reference stations $\mathbf{X}, \mathbf{Z}, \mathbf{Z}_1, \ldots, \mathbf{Z}_k$, then (6.7) yields an expression for the stress distribution within the body *in the absence of motion*. Of course, that such an observation may be drawn within the explicitness-context of (6.7) is but a reflection of a facet of the abstractly expressed physical content of (5.7). Whatever the starting point, such an observation lends credence to our contention that, in general, most bodies encountered in physical experience *do possess* an initial stress distribution.

We next derive an alternate expansion in terms of referential, rather than spatial, variables. To this end, *define* the set $\mathscr{V}_{\mathbf{X}} \subset \mathscr{V}$, and the functions $\mathbf{V}_{\mathbf{X}}$ and $\mathbf{V}_{\mathbf{X},t;\sigma}$ by:

$$(6.11) \qquad \mathscr{V}_{\mathbf{X}} \equiv \{V \equiv \mathbf{X} - \mathbf{Z} | \mathbf{Z} \in B_0,\ \text{fixed } \mathbf{X} \in B_0\},$$

$$(6.12) \qquad \mathbf{V}_{\mathbf{X}}: B_0 \to \mathscr{V}_{\mathbf{X}} : \mathbf{Z} \to \mathbf{V}_{\mathbf{X}}(\mathbf{Z}) \equiv \mathbf{X} - \mathbf{Z},$$

and

$$(6.13) \qquad \mathbf{V}_{\mathbf{X},t} \equiv \mathbf{V}_{\mathbf{X},t;\sigma}: \mathscr{V}_{\mathbf{X}} \to \mathscr{V} : \mathbf{V} \to \mathbf{V}_{\mathbf{X},t;\sigma}(\mathbf{V}) \equiv \mathbf{V}_{\mathbf{X},t}(\mathbf{V}),$$

with

$$(6.14) \qquad \mathbf{V}_{\mathbf{X},t;\sigma}(\mathbf{V}) \equiv (\mathbf{v}_{\chi_t(\mathbf{X});\sigma} \circ \boldsymbol{\chi}_t)(\mathbf{X} - \mathbf{V}) \equiv \frac{1}{\sigma}\{\boldsymbol{\chi}_t(\mathbf{X}) - \boldsymbol{\chi}_t(\mathbf{X} - \mathbf{V})\},$$

where, in the sequel, $\mathbf{V}_{\mathbf{X},t}$ is to be interpreted as an abbreviation for $\mathbf{V}_{\mathbf{X},t;\sigma}$. The *referential form* $\mathfrak{h}_{\mathbf{x},t}$ of the spatial influence function $\mathfrak{h}(\cdot; t)$ is isolated, via (5.7) and (5.8), by:

$$(6.15) \qquad \mathfrak{h}_{\mathbf{X},t}: \mathscr{V}_{\mathbf{X}} \to \mathscr{T} : \mathbf{V} \to \mathfrak{h}_{\mathbf{X},t}(\mathbf{V}) \equiv (\mathfrak{h}_t \circ \mathbf{V}_{\mathbf{X},t})(\mathbf{V}) = \big((\mathfrak{h}(\cdot; t)) \circ \mathbf{V}_{\mathbf{X},t}\big)(\mathbf{V}).$$

Then, with $\tilde{\mathfrak{G}}_{\mathbf{x},t}$ replacing $\mathfrak{G}_{\mathbf{x},t}$ due to the difference in their domain function-spaces stemming from the difference in the domains of the function classes characterized by $\mathfrak{h}_{\mathbf{x}}$ and $\mathfrak{h}_{\mathbf{X},t}$, (6.1) can be expressed as:

$$(6.16) \qquad \mathbf{T}_{\mathbf{x},t} \equiv \tilde{\mathbf{T}}_{\mathbf{x},t}(\mathbf{X}, t) = \tilde{\mathfrak{G}}_{\mathbf{x},t}(\mathfrak{h}_{\mathbf{X},t}),\quad \tilde{\mathfrak{G}}_{\mathbf{x},t} \in \mathscr{B}^*_N : \mathscr{H}^*_{\mathrm{W}}(\mathbf{X}, t) \to \mathscr{T}_{\mathbf{x},t}$$

where, respectively, $\mathscr{B}^*_N$ and $\mathscr{H}^*_{\mathrm{W}}(\mathbf{X}, t)$ are the referential counterparts of $\mathscr{B}^*_n$ and $\mathscr{H}^*_{\mathrm{W}}(\mathbf{x}, t)$.

If it is analytic about the null element $\boldsymbol{\Theta} \in \mathscr{H}^*_{\mathrm{W}}(\mathbf{X}, t)$, $\mathfrak{G}_{\mathbf{x},t}$ has a FRÉCHET power series expansion, and

$$(6.17) \qquad \tilde{\mathbf{T}}_{\mathbf{x},t}(\mathbf{X}, t) = \sum_{k=1}^{\infty} \frac{1}{k!}\,\delta^k \tilde{\mathfrak{G}}_{\mathbf{x},t}(\boldsymbol{\Theta}; \mathfrak{h}_{\mathbf{X},t}).$$

Aside from a formal change in interpretation from the spatial to the referential description, the representation theory for $\delta^k \mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\Theta}; \mathfrak{h}_{\mathbf{X},t})$ follows upon paralleling the reasoning underlying (6.2)–(6.10); thus, it will not be reproduced here. Instead, another type of representation is considered below.

If([15]) $\mathfrak{h}_{\mathbf{X},t}$ and $\mathbf{V}_{\mathbf{X},t}$ are in C^∞ [cf., (6.13), (6.14), and (6.15)], then they have the following expansions in a neighborhood of $\mathbf{V} = \mathbf{0}$:

$$\mathfrak{h}_{\mathbf{X},t}(\mathbf{V}) = \sum_{m=0}^{\infty} \frac{1}{m!} [(\nabla^m \mathfrak{h}_{\mathbf{X},t})(\mathbf{X})] \mathbf{V}^m \equiv \sum_{m=0}^{\infty} \frac{1}{m!} (\nabla^m_{\mathbf{X}} \mathfrak{h}_{\mathbf{X},t}) \mathbf{V}^m_{\mathbf{X}}(\mathbf{Z}), \tag{6.18}$$

and

$$\mathbf{V}_{\mathbf{X},t}(\mathbf{V}) = \frac{1}{\sigma} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} [(\nabla^n \boldsymbol{\chi}_t)(\mathbf{X})] \mathbf{V}^n \equiv \frac{1}{\sigma} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} {}^n\mathbf{F}\mathbf{V}_{\mathbf{X}}(\mathbf{Z}), \tag{6.19}$$

where $\mathbf{V}^m = \overset{m}{\otimes} \mathbf{V}$; $(\nabla^m \mathfrak{h}_{\mathbf{X},t})(\mathbf{X})$ and $(\nabla^m \boldsymbol{\chi}_t)(\mathbf{X})$ are, respectively, the m-th gradients of $\mathfrak{h}_{\mathbf{X},t}$ and $\boldsymbol{\chi}_t\,(\mathbf{X}-\cdot)$, both evaluated at $\mathbf{V} = \mathbf{0}$; and ${}^m\mathbf{F} = {}^m\tilde{\mathbf{F}}(\mathbf{X}, t)$ is the m-th deformation gradient at $\mathbf{X} \in B_0$. Relative to a given basis for $\mathscr{T}$ and $\mathscr{V}_{\mathbf{X}}$, an example of a component rendering, via the summand in (6.18), is supplied by:

$$\{[(\nabla^m \mathfrak{h}_{\mathbf{X},t})(\mathbf{X})] \mathbf{V}^m\}^{ij} = \left. \frac{\partial^m (\mathfrak{h}^{ij}_t \circ \mathbf{V}_{\mathbf{X},t})}{\partial V^{i_1} \ldots \partial V^{i_m})} \right|_{\mathbf{V}=\mathbf{0}} V^{i_1} \ldots V^{i_m} \tag{6.20}$$

since, by (6.15), $\mathfrak{h}_{\mathbf{X},t} = \mathfrak{h}_t \circ \mathbf{V}_{\mathbf{X},t}$.

In terms of map composition $(\circ)$, define compositions $(\Box, \Diamond)$ of m-th gradients by

$$(\nabla^m_{\mathbf{x}} \mathfrak{h}_t) \Box (\nabla^m_{\mathbf{X}} \dot{\mathbf{V}}_{\mathbf{X},t}) \equiv \nabla^m_{\mathbf{X}} (\mathfrak{h}_t \circ \dot{\mathbf{V}}_{\mathbf{X},t}) \equiv (\nabla^1_{\mathbf{x}} \mathfrak{h}_t, \ldots, \nabla^m_{\mathbf{x}} \mathfrak{h}_t) \Diamond (\nabla^1_{\mathbf{X}} \mathbf{V}_{\mathbf{X},t}, \ldots, \nabla^m_{\mathbf{X}} \mathbf{V}_{\mathbf{X},t}). \tag{6.21}$$

Since $\delta^k \tilde{\mathfrak{G}}_{\mathbf{x},t}(\boldsymbol{\Theta}; \mathfrak{h}_{\mathbf{X},t})$ is k-linear in its argument function, by (6.18), (6.17) may be expressed as:

$$\tilde{\mathbf{T}}_{\mathbf{x},t}(\mathbf{X}, t) = \sum_{k=1}^{\infty} \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \frac{1}{k!\, m_1! \ldots m_k!} \times \tag{6.22}$$
$$\times\, \delta^k \tilde{\mathfrak{G}}_{\mathbf{x},t}\big(\boldsymbol{\Theta}; (\nabla^{m_1}_{\mathbf{X}} \mathfrak{h}_{\mathbf{X},t}) \mathbf{V}^{m_1}_{\mathbf{X}}, \ldots, (\nabla^{m_k}_{\mathbf{X}} \mathfrak{h}_{\mathbf{X},t}) \mathbf{V}^{m_k}_{\mathbf{X}}\big).$$

By (6.14) and $(6.21)_2$, the isotropic, k-linear, completely symmetric operator $\delta^k \mathfrak{G}_{\mathbf{x},t}\,(\boldsymbol{\Theta}; \cdot, \ldots, \cdot)$: $\overset{k}{\times} \mathscr{H}^*_{\mathrm{W}}(\mathbf{X}, t) \to \mathscr{T}_{\mathbf{x},t}$ acts on functions [$\equiv$ elements in $\mathscr{H}^*_{\mathrm{W}}(\mathbf{X}, t)$] of the form:

$$\frac{1}{\sigma^{m_i}} [(\nabla^1_{\mathbf{x}} \mathfrak{h}_t, \ldots, \nabla^{m_i}_{\mathbf{x}} \mathfrak{h}_t) \Diamond (-\nabla^1_{\mathbf{X}} \boldsymbol{\chi}_t, \ldots, (-1)^{m_i} \nabla^{m_i}_{\mathbf{X}} \boldsymbol{\chi}_t)] \mathbf{V}^{m_i}_{\mathbf{X}}, \quad i = 1, 2, \ldots, k. \tag{6.23}$$

With this substitution, (6.22) clearly exhibits the dependence of $\mathbf{T}_{\mathbf{x},t}(\mathbf{X}, t)$ on *all* the deformation gradients at $(\mathbf{X} \in B_0,\ t \in \mathscr{R})$ and spatial gradients of the influence function at $(\mathbf{x} = \boldsymbol{\chi}_t(\mathbf{X}) \in B_t,\ t \in \mathscr{R})$, respectively: $\nabla^m_{\mathbf{X}} \boldsymbol{\chi}_t$ and $\nabla^m_{\mathbf{x}} \mathfrak{h}_t$, $m = 1, 2, \ldots$.

The preceeding development suggests that, for suitably small $\|\mathbf{V}\| > 0$, a (K, m)-th order approximation for $\mathbf{T}_{\mathbf{x},t}(\mathbf{X}, t)$ may be obtained upon truncating the series (6.22) by restricting k and m_i to the ranges: $k = 1, 2, \ldots, K$; $m_i = 1, 2, \ldots, m$; $i = 1, 2, \ldots, k$.

([15]) I.e., differentiable of class C^∞.

If so, then $\tilde{\mathbf{T}}_{\mathbf{x},t}(\mathbf{X}, t)$ of (6.22) depends parametrically on *all* of the first m-gradients, evaluated at $(\mathbf{X}, t)$ and $(\mathbf{x}, t)$, of $\boldsymbol{\chi}_t$ and $\mathfrak{h}_t$. In this case, by (6.21), (6.22), and (6.23), define $\mathfrak{F}^{K,m}_{\sigma;\mathbf{x},t}: \overset{m}{\times} \mathscr{H}^*_{\mathrm{W}}(\mathbf{X}, t) \to \mathscr{T}_{\mathbf{x},t}$ such that:

$$\tilde{\mathbf{T}}_{\mathbf{x},t}(\mathbf{X}, t) \doteq \mathfrak{F}^{K,m}_{\sigma;\mathbf{x},t}(\mathbf{V}^1_{\mathbf{X}}, \ldots, \mathbf{V}^m_{\mathbf{X}}; \nabla^1_{\mathbf{x}}\mathfrak{h}_t, \ldots, \nabla^m_{\mathbf{x}}\mathfrak{h}_t; \nabla^1_{\mathbf{X}}\boldsymbol{\chi}_t, \ldots, \nabla^m_{\mathbf{X}}\boldsymbol{\chi}_t). \tag{6.24}$$

As can be seen from the referential counterparts of (6.8) and (6.9), the reduction of (6.22) or (6.24) to the form:

$$\tilde{\mathbf{T}}_{\mathbf{x},t}(\mathbf{X}, t) = \mathfrak{f}_{\mathbf{x},t}({}^1\tilde{\mathbf{F}}(\mathbf{X}, t), \ldots, {}^m\tilde{\mathbf{F}}(\mathbf{X}, t)), \tag{6.25}$$

the constitutive relation for an elastic medium of *grade* m, can be achieved only by subjecting the operators $\delta^k\tilde{\mathfrak{G}}_{\mathbf{x},t}, \mathfrak{F}^{K,m}_{\sigma;\mathbf{x},t}$, and/or influence functions $\mathfrak{h}^{\sim}_{\mathbf{X},t}$ or $\mathfrak{h}_t$ to rather stringent conditions. For instance, in cases where the referential counterparts of (6.8) and (6.9) are valid, the referential description of the spatial influence function must reduce to certain combinations of *distribution* (e.g., *delta-*) functions.

7. Contact with classical theories

Summarizing the results obtained in the preceeding development, sufficient conditions for a spatially cognitive medium to be [8, §28] *elastic of grade* m (or, of *grade* 1; i.e., *simple*) may be stated as:

THEOREM 7.1. *If, for each fixed* $(\mathbf{X}, t) \in B_0 \times \mathscr{R}$: (1) *the constitutive functional* $\tilde{\mathfrak{G}}_{\mathbf{x},t}$ is analytic about the null element $\boldsymbol{\Theta} \in \mathscr{H}_{\mathrm{W}}(\mathbf{X}, t)$; (2) *respectively, the deformation and spatial-influence functions are differentiable of class* C^m *in the sense that they possess the gradients* $\nabla^k_{\mathbf{X}}\boldsymbol{\chi}_t$ and $\nabla^k_{\mathbf{x}}\mathfrak{h}_t$, $k = 1, \ldots, m$; (3) *the approximation* (6.24) *holds, and* (4) $\tilde{\delta}^k\mathfrak{G}_{\mathbf{x},t}(\boldsymbol{\Theta}; \cdot)$, *or* $\mathfrak{F}^{K,m}_{\sigma;\mathbf{x},t}$, *and/or* $\mathfrak{h}_t$ *are such that* (6.25) *holds, then the spatially cognitive medium is elastic of grade* m.

COROLLARY 7.1. *If* $m = 1$, *then the spatially cognitive medium is simple elastic in the usual sense* [8, §43].

8. Concluding remarks

The preceding development has been cast in a form suitable for spatially cognitive, isotropic, elastic, *solid* media. Naturally, an analysis parallel to that of Sect. 6 for spatially cognitive fluid media awaits development. Also pending is an extension of the theory to encompass spatial-cognitivity and time-memory interactions. Time-memory effects have been extensively explored in the literature [Cf., e.g., 8]. Evidently, there is also the possibility of exploring ways and means whereby the sufficient conditions of Theorem 7.1 may be relaxed and/or stated with greater detail and/or exactitude. Further ramifications will be certainly perceived by the interested reader.

Of course, as has been done in some of the primary literature, the transition from (6.16) to (6.25) can be with mathematical exactitude abruptly executed "*by definition*". As shown by the content of Sect. 6 and the hypothesis of Theorem 7.1, the apparent simplicity of such a transition masks non-trivial questions as to physical interpretation and mathematical complexities and/or subtleties. Naturally, if the main thrust of inquiry is towards media of *grade m* per se, then the preceding considerations are of small consequence. On the other hand, they have significant import if the objects of inquiry are those of establishing mathematical links between some of the apparently unrelated media-types found within the purview of *Continuum Mechanics*.

In a companion paper there is presented the one-dimensional form of the anteriorly developed theory. "Edge effects" are there deduced and exhibited; thus, demonstrating by explicit example the theoretical existence of the "physical" edge-effect phenomena (Cf., Sect. 1) which originally served to motivate the initial stages of this study.

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