Newtonian dynamics of homogeneous strains

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OUR aim in this paper is to formulate the Newtonian mechanics of bodies the deformative behaviour of which is restricted to undergoing homogeneous strains (for example, appropriately reinforced pieces of elastomers, or microobjects, the sizes of which are small when compared with wave-lengths). Equations of motion with taking friction into account are given.

Wyprowadzono newtonowskie równania ruchu dla ciał, których zachowanie deformacyjne ograniczone jest (przez jakieś więzy idealne) do ulegania odkształceniom jednorodnym. Ciałami takimi mogą być np. odpowiednio uzbrojone bloki elastomerów lub ciała mikroskopowe o rozmiarach małych, lub przynajmniej porównywalnych z typową długością fali. Pokazano, w jaki sposób można zastosować teorię do opisu zjawisk dysypacyjnych związanych z tarciem.

Выведены уравнения движения Ньютона для тел, которых деформационное поведение ограничено (вследствие каких-нибудь связей) и подлежит однородным деформациям. Такими телами могут быть например соответственно армированные блоки эластомеров или микроскоповые тела с малыми размерами или по крайней мере сравнимыми с типичной длиной волны. Показано каким образом можно применять теорию для описания диссипативных явлений связанных с трением.

1. Introduction

IN THE PAPER [6] we have formulated analytical mechanics of homogeneously-deformable body. That theory, being based on the variational principle and the Hamilton canonical equations, does not take dissipation phenomena into account; the mecanical energy is a constant of motion. Hence, applicability of the theory is very restricted. In fact, both macroscopic bodies (rubber-blocks, small inclusions in fluids), and microobjects (molecules, small monocrystals) are subject to some friction forces which give rise to energy-dissipation and, consequently, to the damping of vibrations. In macroscopic bodies, the decrease of mechanical energy is due to heat production mainly. Besides, the presupposed homogeneity of deformation is always approximate only, what gives rise to the efflux of energy to "nonhomogeneous" degrees of freedom. In microobjects, some additional phenomena become apparent in the balance of mechanical energy; let us mention only the (electromagnetic) radiation damping and the exchange of energy between nuclei and electrons (the better is satisfied the Born-Oppenheimer approximation, the smaller is the last effect).

In what follows, we are dealing with finite strains; our theory can be applied to describing large, non-linear vibrations. The circumstance that the number of degrees of freedom is finite, is especially advantageous just in non-linear theory. Our topic is strictly related to micromorphic theories. The mechanics of micromorphic media has been studied by ERINGEN and RIVLIN [1, 4]. However, the main aim of those works was to find the equations of micromorphic *continua*. Hence, the resulting theories are inapplicable when

studying essentially *discrete* problems, such as propagation of short waves in discrete micromorphic media (molecular crystals e.g.). To be able to take such phenomena into account one should remain on the level of discrete system of granules rather than to pass to the limit of micromorphic continuum.

Obviously, our theory concerns micromorphic bodies degree 1 [1]. We are going to formulate the Newtonian mechanics of a *single*, homogeneously deformable granule of the body. Passing over to the mechanics of a *system* of such interacting granules presents no difficulty. Independently of the micromorphic applications, our approach provides a convenient framework when investigating the motion of small inclusions in fluids [2].

From the point of view of analytical mechanics, the restriction to homogeneous deformations is equivalent to imposing some holonomic constraints. Hence, to get satisfactory equations of motion, we have to eliminate the reaction forces [5, 7]. The special geometric structure of our problem enables us to achieve this very easily, without using general algorithm. We assume constraints to be ideal, this is our main physical assumption.

Let us sketch briefly general ideas of our derivation of equations of motion. Arbitrary system of material points satisfies the following balance equations for linear and angular momentum:

$$\frac{dP}{dt} = F, \qquad \frac{dJ}{dt} = D,$$

where the angular momentum with respect to the origin O is given as: $J = \sum_i r_i \times p_i$; r_i is the radius-vector of i-th material point with respect to O, and p_i — its linear momentum. Similarly, $D = \sum_i r_i \times F_i$ is a total moment of forces F_i with respect to O. When no external forces are present, the above balance equations become conservation laws. Let us impose holonomic constraints that turn our system of points into rigid body. The balance equations above become then equivalent to the complete system of equations of motion. Let us notice now that the components of angular momentum J are equal to independent components of the anti-symmetric part of the tensor $K = \sum r_i \otimes p_i$ ($K^{ab} = \sum' r_i^a p_i^b$). It is well known that the anti-symmetry of the tensor $K^{[ab]}$, describing angular momentum, reflects geometric pecularities of the rigid rotation.

A system of points which we are going to investigate here is only *affinely-rigid*. Hence, one can suspect that in the corresponding theory the tensor K plays a similar role as J in the mechanics of a rigid body. It is reasonable to expect the equations of motion of a homogeneously deformable body to be equivalent to the balance equations for the total momentum P and the tensor K. In fact, this is the case; however, the balance for K is more complicated than that for J. The tensor K is nonconserved even when there are no external interactions. This is because the kinetic energy depends on the metric tensor. In such a way, the full affine group of kinematical symmetries is broken by the metric geometry of the physical space. Consequently, the group of symmetries is reduced to the orthogonal subgroup in a quite non-dynamical way. Obviously, interactions can reduce it further and, in general, they do.

The notations and fundamental notions used below are essentially the same as in the previous paper [6].

2. A body without translational degrees of freedom

Let us impose constraints which make one point of the body immovable. Hence, similarly as in the previous paper [6], the material and physical spaces will be identified with vector spaces U, V, respectively. We assume them to be endowed with the metric tensors $\eta \in U^* \otimes U^*$, $g \in V^* \otimes V^*$. Raising and lowering of indices in U, V is understood in the sense of η, g, respectively.

Configuration space of our body is identified with $Q = LI(U, V)$ — the manifold of linear isomorphisms of U onto V [6]. The mass-distribution in the body is described by means of the non-negative regular measure μ on U. We do not assume anything more about μ; in particular, the distribution can be both discrete or continuous.

Now, let us consider a medium undergoing arbitrary strains, not necessarily homogeneous ones. Let $V \in x \rightarrow v(x) \in V$ be Eulerian velocity field and m — nonnegative regular measure on V describing the mass-distribution in the physical space.

Affine momentum of the body is defined as a tensor $K \in V \otimes V$ given by:

$$(2.1) \qquad K = \int_V x \otimes v(x)\,dm(x).$$

The skewsymmetric part of K, $K^{[ij]} = \int x^{[i}v^{j]}(x)\,dm(x)$ describes the angular momentum.

Now, let us assume again that the body undergoes homogeneous strains only. When its configuration is $\varphi \in LI(U, V)$ and generalized velocity $\xi \in L(U, V)$, then, making use of (2.1), we obtain:

$$(2.2) \qquad K(\varphi, \xi) = (\varphi \otimes \xi) \cdot J$$

or, in linear coordinates:

$$K^{ij} = \varphi^i{}_A \xi^j{}_B J^{AB},$$

where

$$J = \int_V X \otimes X d\mu(X)$$

is the co-moving tensor of inertia [1, 6].

In a similar way we introduce an *affine momentum of forces*:

$$(2.3) \qquad N = \int_V x \otimes F(x)\,dm(x),$$

where $F(x)$ is the density of forces at $x \in V$, taken per unit mass of the body. When the medium undergoes homogeneous strains only, then:

$$(2.4) \qquad N = \int_V \varphi \cdot X \otimes F(\varphi \cdot X)\,d\mu(X).$$

One can easily show that the rate at which forces F do their work, is given by:

$$(2.5) \qquad \mathscr{P}(\varphi, \xi) = N^{ki} g_{ij} \Omega_l(\varphi, \xi) j_k$$

or shortly:

$$(2.6) \qquad \mathscr{P} = N^k{}_j \Omega_i j^k,$$

where $\Omega_i(\varphi, \xi) = \dot{\xi} \circ \varphi^{-1}$ is *laboratory quasi-velocity* of the body [6] (cf. also [1]; ERINGEN uses the name "*gyration*". All tensor indices above are related to some linear bases $\{e_i\} \subset V$, $\{E_A\} \subset U$.

Differentiating (2.1) with respect to time and applying the second Newton law to all material points, we obtain:

$$(2.7) \qquad \frac{dK}{dt} = (\dot{\xi} \otimes \xi) \cdot J + N_{\text{tot}}$$

or, in linear coordinates:

$$(2.8) \qquad \frac{dK^{ij}}{dt} = \dot{\xi}^i{}_A \xi^j{}_B J^{AB} + N^{ij}_{\text{tot}},$$

where $N_{\text{tot}} = N + N_r$ is the total affine momentum of both given forces (N) and reaction forces (N_r). The mentioned reactions restrict the deformative behaviour of the body to homogeneous strains. This restriction can be achieved by an appropriate reinforcement (for example, very thin rigid rods, randomly distributed in an elastomer block), or, it is assured by some structural peculiarities of the intermolecular forces in microobjects (it seems, the radius of forces should be of the same order as the size of the body). Obviously, the resulting homogeneity of strains is always approximate only.

Now, let us make use of the commonly accepted postulate that the reaction forces do not do any work. Hence, (2.6) implies that $N_r = 0$ and consequently:

$$(2.9) \qquad \frac{dK}{dt} = (\dot{\xi} \otimes \xi) \cdot J + N = 2D_g T + N,$$

where $T = \frac{1}{2} g_{ij} \dot{\xi}^i{}_A \dot{\xi}^j{}_B J^{AB}$ is the kinetic energy of the body [1, 6] and $D_g T$ denotes the derivative at g in the usual sense of differential calculus on vector spaces. In coordinates

$$(2.10) \qquad \frac{dK^{ij}}{dt} = \dot{\xi}^i{}_A \dot{\xi}^j{}_B J^{AB} + N^{ij} = 2 \frac{\partial T}{\partial g_{ij}} + N^{ij}.$$

It follows from (2.9), (2.10) that the symmetric part of K is nonconserved even in the special case of free body, when only reaction forces are present (no real interactions). The corresponding time rate is given by $D_g T$. Hence, the Euclidean geometry of physical space gives rise to the breaking of the full affine symmetry with which we are dealing on the kinematical level.

Differentiating (2.2) with respect to time and comparing the result with (2.9), we obtain:

$$(2.11) \qquad \left(\varphi \otimes \frac{d^2\varphi}{dt^2} \right) \cdot J = N,$$

i.e.:

$$(2.12) \qquad \frac{d^2\varphi^j{}_B}{dt^2} J^{BC} = \varphi^{-1C}{}_i N^{ij}$$

or, equivalently:

$$(2.13) \qquad \frac{d^2\varphi^i{}_A}{dt^2} = \tilde{J}_{AB}\varphi^{-1B}{}_j N^{ji},$$

where $\tilde{J} \in U^* \otimes U^*$ is reciprocal to $J: J_{AB}\ \tilde{J}^{BC} = \delta_A{}^C$ (we *assume* that it does exist).

Hamiltonian equations of motion [6] result as a special case of (2.13). Let us assume the Hamiltonian of the form: $H = \mathscr{T} + V$, where V is a potential depending on φ only, and $\mathscr{T}$ denotes the kinetic term [6]:

$$\mathscr{T}(\varphi, \pi) = \frac{1}{2}\tilde{J}_{AB}\pi^A{}_i\pi^B{}_j g^{ij}.$$

Then, denoting the corresponding Legendre transformation as $\mathscr{L}$, we find:

$$(2.14) \qquad K^{ij} \circ \mathscr{L}^{-1} = \varphi^i{}_A \pi^A{}_k g^{kj} = F_l{}^i{}_k g^{kj} = F_l{}^{ij},$$

where $F_l{}^i{}_k$ are laboratory quasi-momenta defined in [6]. The non-conservation of K^{ij} even in the special case of free body is due to the non-vanishing Poisson bracket $\{F_l{}^i{}_k, \mathscr{T}_g\}$. It is interesting that this Poisson bracket does vanish when instead of $\mathscr{T}_g$, the "non-physical", co-moving kinetic term $\mathscr{T}_\eta$ is used (cf. [6]): $\{F_l{}^i{}_j, \mathscr{T}_\eta\} = 0$. This is because $\mathscr{T}_\eta$ does not depend on g, and, consequently, it is invariant under the full linear group $GL(V)$, generated infinitesimally by $F_l{}^i{}_j$. Hence, $F_l{}^i{}_j$ and, consequently, K^{ij} become then constants of motion. It appears once more that the co-moving kinetic energy T_η, although "non-physical", provides us with a key to understand the structure of the theory.

3. Complete system of equations of motion

Now, we will take into account the translational degrees of freedom. An internal motion will be referred to the centre of the mass. This simplification, although not necessary, facilitates all considerations and formulas (similarly, as in the theory of the gyroscope). We assume still the *material space* to be a vector space U. The centre of the mass will be placed at $0 \in U$:

$$(3.1) \qquad \int\limits_U X d\mu(X) = 0,$$

where the regular, positive measure μ on U describes a distribution of mass in the body (the *co-moving* mass distribution).

We assume the *physical space* M to be endowed with an affine structure; the corresponding linear space of translations will be denoted as V.

The *configuration space* of the body will be identified with the manifold $Q = M \times \times LI(U, V)$.

Any point $(m, \varphi) \in Q$ gives rise to a uniquely defined affine mapping $\Phi_{m,\varphi}: U \to M$ such that:

$$\Phi_{m,\varphi}(0) = m, \qquad \overrightarrow{m\Phi_{m,\varphi}(u)} = \varphi u.$$

A *configuration* (m, φ) should be understood as follows:

(i) the centre of the mass is placed at $m \in M$.

(ii) the radius-vector of an X-th material point, with respect to the centre of the mass, is given by $\varphi \cdot X$. This means that the X-th material point is placed at $\Phi_{m,\varphi}(X) \in M$.

Generalized velocity of the body is described by a vector $(\zeta, \xi) \in V \times L(U, V)$; ζ is a translational velocity (i.e. usual velocity of the centre of the mass) and ξ describes time rates of deformations and rotations ($\xi \circ \varphi^{-1}$ is just what ERINGEN [1] calls "gyration"; we have used the name "laboratory quasi-velocity", because of the reasons explained in [6]).

Let us fix some arbitrary physical point $o \in M$. We assume for a moment that the physical mass distribution in M is described by a positive, regular measure ν, and the Eulerian velocity field is given by an assignment: $M \ni m \to v(m) \in V$. For the sake of generality, we assume for a while the strain state to be quite arbitrary, not necessarily homogeneous one.

Affine momentum of the body with respect to the point $o \in M$, is defined as follows:

$$(3.2) \qquad K(o) = \int_M \overrightarrow{om} \otimes v(m) \, d\nu(m).$$

Now, let us assume again that the body undergoes homogeneous deformations only. Let $(m, \varphi) \in Q$ be its configuration and (ζ, ξ) — generalized velocity. Then, (3.2) implies that:

$$(3.3) \qquad K(o) = (\varphi \otimes \xi) \cdot J + M\overrightarrow{om} \otimes \zeta,$$

where $M = \int_U d\mu(m)$ is the total mass of the body and J — its co-moving tensor of inertia with respect to the centre of the mass. Making use of affine coordinates corresponding to bases $\{E_A\} \subset U$, $\{e_i\} \subset V$ and to the origin $o \in M$, we can rewrite (3.3) as follows:

$$(3.4) \qquad K^{ij}(o) = \varphi^i{}_A \xi^j{}_B J^{AB} + M f^i \zeta^j,$$

where $f = \overrightarrow{om}$.

Hence, the total affine momentum $K(o)$ equals the sum of the *internal affine momentum* $K_{cm} = (\varphi \otimes \xi) \cdot J$ with respect to the centre of the mass and the *affine momentum of the centre of the mass* $K_{orb}(o) = M\overrightarrow{om} \otimes \zeta$ (orbital affine momentum). If the centre of the mass were not placed at $0 \in U$, this would not be the case — then some interference terms would appear. Similarly, (3.1) implies that the kinetic energy is a sum of terms corresponding to the centre of the mass and to the internal motion:

$$(3.5) \qquad T_\theta(\varphi, \xi) = \frac{1}{2}\langle g, (\xi \otimes \xi) \cdot J\rangle + \frac{1}{2}M\langle g, \zeta \otimes \zeta\rangle = \frac{1}{2}g_{ij}\xi^i{}_A \xi^j{}_B J^{AB} + \frac{1}{2}Mg_{ij}\zeta^i\zeta^j.$$

Affine momentum of forces with respect to $o \in M$, is defined as follows:

$$(3.6) \qquad N = \int_M \overrightarrow{om} \otimes F(m) \, d\nu(m),$$

where F is the density of forces per unit mass.

Now, let $s \in M$ be a position of the centre of the mass in the physical space: $\int \overrightarrow{sm} \, dv(m) =$
$= 0$. It enables us to express N as a sum $N = N_{int} + N_{orb}$, where:

$$(3.7) \qquad N_{cm} = \int \overrightarrow{sm} \otimes F(m) \, dv(m),$$

$$(3.8) \qquad N_{orb} = \int \overrightarrow{os} \otimes F(m) \, dv(m) = \overrightarrow{os} \otimes F;$$

$F = \int F(m) \, dv(m)$ is the total force acitng on the body. Similarly, N_{cm} is a total affine momentum of all forces with respect to the centre of the mass, and N_{orb} is an affine momentum of the total force acting on the centre of mass, with respect to the fixed origin $o \in M$.

Proceeding as in §2 and eliminating the reaction forces, we obtain the following balance law for K:

$$(3.9) \qquad \frac{dK}{dt} = (\xi \otimes \xi) \cdot J + M\zeta \otimes \zeta + N.$$

Similarly, one obtains the well-known balance law for the total linear momentum $P = \int v(x) \, dv(x)$:

$$(3.10) \qquad \frac{dP}{dt} = F.$$

Differentiating (3.3) with respect to the time, making use of equations (3.9), (3.10) and substituting $P = M\zeta$, we obtain the following system of equations:

$$(3.11) \qquad \frac{dK_{cm}}{dt} = (\xi \otimes \xi) \cdot J + N_{cm},$$

$$(3.12) \qquad \frac{dP}{dt} = 0$$

or, equivalently:

$$(3.13) \qquad \left(\varphi \otimes \frac{d^2\varphi}{dt^2}\right) \cdot J = N_{cm},$$

$$(3.14) \qquad M \cdot \frac{d^2 f}{dt^2} = F,$$

where $f = \overrightarrow{om}$ is a radius vector of the centre of the mass with respect to the fixed origin $o \in M$. In linear coordinates:

$$(3.15) \qquad \frac{d^2}{dt^2} \varphi^i{}_A = \tilde{J}_{AB} \varphi^{-1B}{}_j N_{cm}^{ji},$$

$$(3.16) \qquad M \frac{d^2 f^i}{dt^2} = F^i.$$

The above balance laws form a complete system of equations of motion of homogeneously deformable body.

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4. Phenomenological description of the viscous friction

Let us rewrite the equations of motion (3.15), (3.16) in the following form:

$$(4.1) \qquad g_{ij} \frac{d^2}{dt^2} \varphi^j{}_B J^{BA} = Q^A{}_i,$$

$$(4.2) \qquad M g_{ij} \frac{d^2 f^j}{dt^2} = Q_i,$$

where $Q^A{}_i$, Q^i are *generalized forces*. Now, let us make the following *dynamical* assumption:

$$(4.3) \qquad Q^A{}_i = -\frac{\partial V}{\partial \varphi^i{}_A} + D^A{}_i,$$

$$(4.4) \qquad Q_i = -\frac{\partial V}{\partial f^i} + D_i,$$

where $V: M \times LI(U, V) \to R$ is the potential energy depending on the configuration only and $D^A{}_i$, D_i depend on generalized velocities (and configurations) in such a way that they vanish when the body is at rest. Hence, generalized forces $D^A{}_i$, D_i are able to describe dissipative and magnetic phenomena. In what follows, we are interested in a dissipation only. Let $E = T + V$ denote the total mechanical energy of the body. We have the following balance law:

$$(4.5) \qquad \frac{dE}{dt} = D^{ji} \Omega_{lij} + D_i \zeta^i,$$

where $D^{ji} = \varphi^j{}_C D^C{}_k g^{ki}$ is an affine momentum of dissipative forces $D^A{}_i$. (Even if magnetic forces are present, their contribution to the energy balance, or rather to the right-hand side of (4.5), vanishes identically).

In many practical problems concerning slow motions one makes assumption that the friction forces are linear in generalized velocities:

$$(4.6) \qquad D^{ji} = -\eta(\Omega, \Omega)^{jim}{}_n \Omega_l{}^n{}_m - \eta(\Omega, \zeta)^{ji}{}_m \zeta^m,$$

$$(4.7) \qquad D^i = -\eta(\zeta, \Omega)^{im}{}_n \Omega_l{}^n{}_m - \eta(\zeta, \zeta)^i{}_k \zeta^k.$$

The components of the viscous friction, $\eta(\Omega, \Omega)$, ... are allowed to depend on the configuration. D^{ji}, D^i are purely dissipative (non-magnetic) when the following symmetries hold:

$$(4.8) \qquad \eta(\Omega, \Omega)_{jimn} = \eta(\Omega, \Omega)_{mnji},$$

$$(4.9) \qquad \eta(\zeta, \zeta)_{ij} = \eta(\zeta, \zeta)_{ji}.$$

"Skewsymmetric" parts do not give rise to the energy balance. Obviously, (4.6), (4.7) describe most general friction forces linear in velocities. To be able to restrict the arbitrariness of the viscous coefficients η, one should go into details of the problem.

For example, assuming that all dissipation forces are due to the internal friction, we obtain the following form of the corresponding dissipation forces and momenta D^i_{int}, D^{ji}_{int}:

$$(4.10) \qquad D^i_{int} = 0,$$

$$(4.11) \qquad D^{ij}_{int} = -\eta_{int}{}^{ijm}{}_n \Omega_l{}^n{}_m,$$

where η_{int} satisfies (4.8) and the following additional symmetry rules:

$$(4.12) \qquad \eta_{\text{int}}{}^{ijmn} = \eta_{\text{int}}{}^{jimn} = \eta_{\text{int}}{}^{ijnm}$$

(i.e. dissipation caused by internal friction depends only on the symmetric part of the velocity gradient).

Assuming isotropy of the body, one obtains [3]:

$$(4.13) \qquad D_{\text{int}}^{ij} = 2\eta\left[\Omega_l^{(ij)} - \frac{1}{3} g^{ij} \Omega_l{}^k{}_k \right] + \zeta \Omega_l{}^k{}_k g^{ij}$$

(obviously, the viscous coefficient ζ has nothing to do with the orbital velocity denoted above by the same symbol).

Dissipation forces and momenta, corresponding to the external, i.e. surface friction, D_{ext}^i, D_{ext}^{ij}, do not satisfy either (4.10) or (4.11). This is because rigid motions, i.e. translations and rotations are subject to the surface damping forces. Surface friction becomes very important when studying the motion of small inclusions in fluids [2]. This subject needs separate treatment. In this paper we will sketch only general ideas of describing homogeneously-deformable inclusions by means of (4.1)–(4.4):

1) Let us assume the configuration of inclusion to be (m, φ) and its generalized velocity: (ζ, ξ). We start with calculating the Eulerian velocity field $x \to V_{(m,\varphi,\zeta,\xi)}(x) \in V$, defined inside the inclusion

$$(4.14) \qquad V_{(m,\varphi,\zeta,\xi)}(x) = \zeta + \xi \circ \varphi^{-1} \cdot \overrightarrow{mx} = \zeta + \Omega_l(\varphi, \xi) \cdot \overrightarrow{mx}.$$

(Obviously, (4.14) is well-defined all over V; however, it is physically interpretable only inside the inclusion).

2) Let $\sum \subset U$ be a (material) surface of inclusion, and $\sigma_{m,\varphi} = \Phi_{m,\varphi}(\sum) \subset V$ — its physical surface in V, corresponding to the configuration (m, φ).

Now, we find a solution (rigorous, or approximate) of hydrodynamic equations describing our fluid. Let $x \to Y_{(m,\varphi,\zeta,\xi)}(x)$ denote the corresponding velocity field of the fluid, defined outside of inclusion and satisfying the following boundary conditions:

$$(4.15) \qquad Y_{(m,\varphi,\zeta,\xi)}|\sigma_{m,\varphi} = V_{(m,\varphi,\zeta,\xi)}|\sigma_{m,\varphi}$$

(i.e. usual boundary conditions of viscours fluid).

3) Having $Y_{(m,\varphi,\zeta,\xi)}$ at our disposal, we calculate the viscous stress tensor $t_{(m,\varphi,\zeta,\xi)}$ and consequently, the forces acting on the surface of inclusion.

4) We calculate the total force and affine momentum of viscous damping:

$$(4.16) \qquad D^i(m, \varphi, \zeta, \xi) = \int_{\sigma_{m,\varphi}} t_{(m,\varphi,\zeta,\xi)}^{ij}(x) d\sigma_j(x),$$

$$(4.17) \qquad D^{ij}(m, \varphi, \zeta, \xi) = \int_{\sigma_{m,\varphi}} x^i t_{(m,\varphi,\zeta,\xi)}^{jk}(x) d\sigma_k(x),$$

where $d\sigma_k(x)$ denotes a surface element.

5) Finally, the viscous forces and momenta, D^i, D^{ij}, should be substituted into equations of motion (3.15), (3.16).

If Y is a rigorous solution of hydrodynamic equations, then, in general, D^i, D^{ij} need not be linear in ζ, ξ. Excepting some special cases (e.g. small "monochromatic" vibrations), damping forces and momenta, D^i, D^{ij}, contain terms nonlocal in time. The corresponding equations of motion become then integro-differential and the system is endowed with a memory [3].

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