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## A REMARK ON DIFFERENTIAL EQUATIONS.

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CONSIDER a differential equation $f(x, y, p) = 0$, of the first order, but of the degree $n$, where $f$ is a rational and integral function of $(x, y, p)$ not rationally decomposable into factors: the integral equation contains an arbitrary constant $c$, and represents therefore a system of curves, for any one of which curves the differential equation is satisfied: the differential equation is assumed to be such that the curves are algebraical curves. The curves in question may be considered as undecomposable curves; in fact, if the curve $U^\alpha V^\beta W^\gamma \ldots = 0$ (composed of the undecomposable curves $U = 0$, $V = 0$, $W = 0, \ldots$) satisfies the differential equation, then either the curves $U = 0$, $V = 0$, $W = 0, \ldots$ each satisfy the differential equation, and instead of the curve $U^\alpha V^\beta W^\gamma \ldots = 0$ we have thus the undecomposable curves $U = 0$, $V = 0$, $W = 0, \ldots$ each satisfying the differential equation; or if any of these curves, for instance $W = 0$, &c., do not satisfy the differential equation, then $W^\gamma$, &c. are mere extraneous factors which may and ought to be rejected, and instead of the original curve $U^\alpha V^\beta W^\gamma \ldots = 0$, we have the undecomposable curves $U = 0$, $V = 0$ satisfying the differential equation. Assuming, as above, the existence of an algebraical solution, this may always be expressed in the form $\phi(x, y, c) = 0$, where $\phi$ is a rational and integral function of $(x, y, c)$, of the degree $n$ as regards the arbitrary constant $c$: this appears by the consideration that for given values $(x_0, y_0)$ of $(x, y)$ the differential equation and the integral equation must each of them give the same number of values of $p$. It is to be observed that $\phi$ regarded as a function of $(x, y, c)$ cannot be rationally decomposable into factors; for if the equation were $\phi = \Phi\Psi \ldots = 0$, $\Phi$, $\Psi$, &c. being each of them rational and integral functions of $(x, y, c)$, then the differential equation would be satisfied by at least one of the equations $\Phi = 0$, $\Psi = 0, \ldots$ that is, by an equation of a degree less than $n$ in the arbitrary constant $c$.

But the equation $\phi(x, y, c) = 0$ is not of necessity the equation of an undecomposable curve, and the undecomposable curve which constitutes the proper solution of the differential equation cannot always be represented by an equation of the form in question. For although $\phi$ regarded as a function of $(x, y, c)$ is not rationally decomposable into factors, yet it may very well happen that $\phi$ regarded as a function of $(x, y)$ is rationally decomposable into factors (geometrically the sections by the planes $z = c$ of the undecomposable surface $\phi(x, y, z) = 0$ may each of them be composed of two or more distinct curves); and assuming that the function $\phi$ is thus decomposed into its prime factors, then each factor equated to 0 gives an undecomposable curve satisfying the differential equation, and constituting the proper solution thereof.

It may be observed that, by the foregoing process of decomposition, we sometimes reduce the original equation $\phi(x, y, c) = 0$ into a like equation $\phi_1(x, y, c_1) = 0$ of a more simple form. Thus, for instance, if we have $\phi(x, y, c) = U^2 - c = 0$, $U$ being a rational and integral function of $(x, y)$, then instead of $\phi = U^2 - c = 0$ we have the equations $U + \sqrt{c} = 0$, $U - \sqrt{c} = 0$, each of which is an equation of the form $U - c_1 = 0$, or we pass from the original equation $\phi(x, y, c) = U^2 - c = 0$ to the simplified equation

$$\phi_1(x, y, c_1) = U - c_1 = 0.$$

Again, to take a somewhat more complicated instance, if the given integral equation be

$$\phi(x, y, c) = U^4 + c^2V^4 + (c+1)^2 W^2 - 2cU^2V^2 - 2(c+1)U^2W^2 - 2c(c+1)V^2W^2 = 0,$$

then the equation $U + V\sqrt{c} + W\sqrt{c+1} = 0$, writing therein $\sqrt{c} = \dfrac{2c_1}{c_1^2 - 1}$, and therefore $\sqrt{c+1} = \dfrac{c_1^2+1}{c_1^2-1}$, becomes

$$U(c_1^2 - 1) + V.2c_1 + W(c_1^2 + 1) = 0;$$

so that we pass from the original equation $\phi(x, y, c) = 0$ to the simplified equation

$$\phi_1(x, y, c_1) = U(c_1^2 - 1) + V.2c_1 + W(c_1^2 + 1) = 0.$$

But observe that the possibility of the rationalization depends on the form of the radicals $\sqrt{c}$ and $\sqrt{c+1}$; if we had had $\sqrt{c}$ and $\sqrt{c^2+1}$ (or $c$ and $\sqrt{c^4+1}$), the rationalization could not have been effected.

Returning to the case of an integral equation $\phi(x, y, c) = 0$, where $\phi$ regarded as a function of $(x, y)$ is decomposable into factors, then equating to zero any one of the prime factors of $\phi$, we obtain an integral equation $\psi(x, y, c_1, c_2, \ldots c_k) = 0$, where $c_1, c_2 \ldots c_k$ are irrational functions (not of necessity representable by radicals, and without any superior limit to the number of these functions) of $c$: here $\psi$ regarded as a function of $(x, y)$ is of course undecomposable, and the equation $\psi(x, y, c_1, c_2, \ldots c_k) = 0$ belongs to the undecomposable curve which is the proper solution of the differential equation. The result may be stated under a quasi-geometrical form; viz. regarding $c_1, c_2, \ldots c_k$ as the coordinates of a point in $k$-dimensional space, then as these are functions of the single parameter $c$, the point to which they belong is an arbitrary point on a certain curve or $(k-1)$fold locus $C$ in the $k$-dimensional space. And this curve must be such that to given values of $(x, y)$ there shall correspond $n$ points on the curve; that is, treating $(x, y)$ as constants, the surface or onefold locus $\psi(x, y, c_1, c_2 \ldots c_k)=0$, and the curve or $(k-1)$fold locus $C$, shall meet in $n$ points. The conclusion stated in the foregoing quasi-geometrical form is, that the solution of the differential equation may be exhibited in the form $\psi(x, y, c_1, c_2 \ldots c_k)=0$; viz. $\psi$ is a rational and integral function of $(x, y, c_1, c_2 \ldots c_k)$, where $(c_1, c_2 \ldots c_k)$ are the coordinates of an arbitrary or variable point on a curve or $(k-1)$fold locus $C$ in a $k$-dimensional space, which curve meets the surface or onefold locus $\psi(x, y, c_1, c_2 \ldots c_k)$ in $n$ points, and where $\psi$ regarded as a function of $(x, y)$ is not rationally decomposable into factors.

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